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Homogenization of a Cauchy continuum towards a micromorphic continuum

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The micromorphic theory of Eringen and Mindlin, including special cases like strain gradient theory or Cosserat theory, is widely used to model size effects and localization phenomena. The heuristic construction of such theories based on thermodynamic considerations is well-established. However, the identification of corresponding constitutive laws and of the large number of respective constitutive parameters limits the practical application of such theories.

In the present contribution, a closed procedure for the homogenization of a Cauchy continuum at the microscale towards a fully micromorphic continuum is derived including explicit definitions of all involved generalized macroscopic stress and deformation measures. The boundary value problem to be solved on the microscale is formulated either for using static or kinematic boundary conditions. The procedure is demonstrated with an example.

Keywords: micromorphic theory; homogenization; gradient theory; generalized continua

1. Introduction

Classical theories of continuum mechanics can only be applied when the macroscopic wavelengths of relevant field quantities are much larger than the characteristic microstructural dimensions, a limitation manifested already in the lack of an intrinsic length scale in such continuum theories. However, in many engineering problems this condition is not fulfilled, e. g. for micro and nanodevices or if material instabilities lead to a localization of deformations.

In principle, generalized continuum theories can overcome these limitations. Among the generalized theories of continuum mechanics, micromorphic theory of Eringen and Mindlin [8, 9, 10, 31] has an outstanding role since it incorporates many others like the Cosserat theory or strain gradient theory as special cases. For a recent review and a comprehensive classification the reader is referred to [17, 30]. Phenomenologically, it is well-established how such theories can be constructed based on macroscopic thermodynamic considerations and/or the principle of virtual powers. Thereby, additional, generalized stresses occur which appear in additional balance equations [19, 9, 14, 7, 23]. This requires to formulate respective additional constitutive relations and to identify the corresponding constitutive parameters. Mostly, for this purpose classical constitutive laws are generalized heuristically by linear and reversible approaches for the generalized stress measures (i. e. quadratic ansatzes in the thermodynamic potentials) for simplicity, e. g. [14, 7]. In particular in extension of highly non-linear classical constitutive laws, this is questionable and leads to unrealistic predictions as pointed out recently in [24, 15].

Homogenization of the heterogeneous microstructure offers a solution to this problem. Regarding classical continuum mechanics, the homogenization procedure, whereby the macroscopic stresses and strains are defined as the volume averages of their microscopic counterparts and can be prescribed via corresponding boundary conditions at the microscale, is established already for decades [21, 22]. For the so-called couple stress theory (constrained Cosserat theory) the additional boundary conditions of bending-type were specified intuitively as well [1, 11, 4]. However, for the homogenization from a Cauchy continuum at the microscale towards an unconstrained micromorphic continuum at the macroscale the definition of the macroscopic quantities and the formulation of respective boundary conditions at the microscale is not that trivial. Forest and Sab provided explicit integral expressions for the relations between microscopic and macroscopic kinematic quantities of micromorphic theories [13, 16]. However, it turned out to be a problem that the expressions for the generalized deformation measures could not be transformed to surface integrals and thus, in contrast to classical homogenization, not be prescribed by boundary conditions at the microscale. For that reason polynomial expressions were identified according to several strategies that fulfill the integral expressions identically and attempts were made to characterize additional "fluctuation" fields [16, 12, 6, 26]. The drawback of this approach is that no boundary value problem could strictly be formulated on the microscale. Alternatively, Jänicke and Steeb [27] proposed to prescribe Forest's integral definitions of the macroscopic micromorphic deformation measures as minimum constraints to the classical PDE on the microscale. Then, the generalized stress measures are obtained implicitly as respective work-conjugate quantities. However, it had to be postulated that these generalized stresses fulfill the balance equations of micromorphic continua as derived by Eringen. In this context it is to be remarked that Eringen [8, 9] derived those balance equations via a spatial averaging procedure and provided explicit definitions of the generalized stresses (and all flux quantities) in terms of a "surface operator". Unfortunately, at least to the author's knowledge, Eringen did neither give an expression for this surface operator nor for the relation of the macroscopic deformation measures to the microscopic field quantities. The present contribution aims to close this gap by providing explicit expressions for the macroscopic generalized stress measures and their work-conjugate deformation measures and formulating the boundary value problem to be solved at the microscale for obtaining the macroscopic micromorphic constitutive relations.

The present work is organized as follows: section 2 briefly reviews Eringen's micromorphic theory, before the general homogenization procedure is presented in section 3. In particular, thermodynamic considerations in section 3.1 lead to the generalized Hill-Mandel lemma before section 3.2 deals with the micro-macro relations for the generalized measures of stresses and deformations. Based on these findings, the special case of the second gradient theory is considered in section 4. Subsequently, section 5 demonstrates the proposed procedure for a simple uniaxial example. Finally, the present contribution closes with a discussion and some concluding remarks in sections 6 and 7, respectively.

2. Micromorphic continuum

The notation within the present contribution is adopted from Forest and Sab [16], i. e. scalars, vectors and tensors of second, third and forth rank are denoted by a, $\underline{\mathbf{b}}$, $\underset{\sim}{\mathbf{c}}$, $\underset{\sim}{\underline{\mathbf{d}}}$ and $\underset{\approx}{\mathbf{e}}$, respectively. Single, double and threefold contractions are written as $\cdot$, $:$ and $\vdots$, respectively, and are computed from left to right, i. e. $\underset{\sim}{\underline{\mathbf{d}}} \vdots \underset{\sim}{\underline{\mathbf{e}}} = d_{ijk}e_{ijk}$. In particular, $\underset{\sim}{\mathbf{I}}$ and $\underset{\sim}{\epsilon}$ denote the second rank identity tensor and the permutation tensor, respectively. The operator $(\circ)^{\mathrm{T}}$ denotes the complete transposition of all indices of a tensor. For a second rank tensor this is done by the forth rank transposing tensor $\underset{\approx}{\mathbf{I}}_{\mathrm{T}}$ as $\underset{\sim}{\mathbf{c}}^{\mathrm{T}} = \underset{\approx}{\mathbf{I}}_{\mathrm{T}} : \underset{\sim}{\mathbf{c}}$. Analogously, a symmetrization tensor $\underset{\approx}{\mathbf{I}}_{\mathrm{S}}$ is introduced as $\underset{\sim}{\mathbf{c}}^{\mathrm{S}} = \frac{1}{2}(\underset{\sim}{\mathbf{c}} + \underset{\sim}{\mathbf{c}}^{\mathrm{T}}) = \underset{\approx}{\mathbf{I}}_{\mathrm{S}} : \underset{\sim}{\mathbf{c}}$. The symbols $\underline{\mathbf{x}}$ and $\underline{\mathbf{X}}$ refer to the location vector at the microscale and macroscale, respectively. The nabla operator is $\underline{\nabla}$ whose subscript ($\underline{\nabla}_{\underline{\mathbf{X}}}$ or $\underline{\nabla}_{\underline{\mathbf{x}}}$) specifies, whether it is computed with respect to $\underline{\mathbf{X}}$ or $\underline{\mathbf{x}}$. The material time derivate is denoted by a dot $\dot{(\circ)}$.

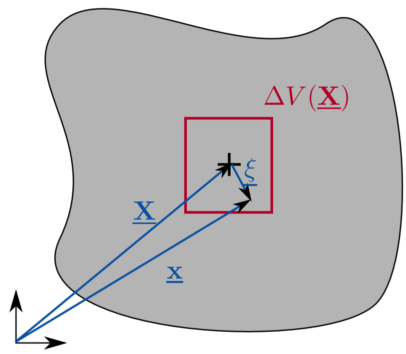


Figure 1: Micromorphic continuum

2.1. Average theorem for general balance laws

Consider a general balance equation in the reference configuration $\underline{\mathbf{x}} \in \Omega$ of type

$$\frac{D}{Dt}\int\limits_{\Omega} \rho\varphi_m \,\mathrm{d}V = \oint\limits_{\partial\Omega} \underline{\mathbf{n}} \cdot \underline{\psi}_a \,\mathrm{d}S + \int\limits_{\Omega} \rho\psi_m \,\mathrm{d}V \tag{1}$$

with φ_m, ψ_m and $\underline{\psi}_a$ and being the densities of storage, sources and flux, respectively, in a continuum of mass density ρ. The global balance (1) can be localized as usual to

$$\rho\overset{\bullet}{\varphi}_m = \underline{\nabla}_{\underline{\mathbf{x}}} \cdot \underline{\psi}_a + \rho\psi_m \text{ in } \Omega\,. \tag{2}$$

According to Eringen [8, 9], and given in detail in appendix A.1, macroscopic counterparts to these balance equations are obtained by dividing the domain Ω into small but *finite* volumes $\Delta V(\underline{\mathbf{X}})$ as sketched in Figure 1 for each of which (1) is valid. Summing them up and approximating the sum of those many elements as integrals over the macroscopic domain $\underline{\mathbf{X}} \in \Omega_{\underline{\mathbf{X}}}$ (with boundary $\partial\Omega_{\underline{\mathbf{X}}}$ whose normal is $\underline{\mathbf{N}}$) yields the macroscopic balance law

$$\frac{D}{Dt}\int\limits_{\Omega} \langle\rho\varphi_m\rangle_V \,\mathrm{d}V = \oint\limits_{\partial\Omega_{\underline{\mathbf{X}}}} \underline{\mathbf{N}} \cdot \left\langle\underline{\psi}_a\right\rangle_S \,\mathrm{d}S + \int\limits_{\Omega_{\underline{\mathbf{X}}}} \langle\rho\psi_m\rangle_V \,\mathrm{d}V \tag{3}$$

with an equivalent local version

$$\langle\rho\varphi_m\rangle_V^{\bullet} = \underline{\nabla}_{\underline{\mathbf{X}}} \cdot \left\langle\underline{\psi}_a\right\rangle_S + \langle\rho\psi_m\rangle_V \,. \tag{4}$$

Therein, the volume averaging operator is denoted as

$$\langle\circ\rangle_V = \frac{1}{\Delta V(\underline{\mathbf{X}})} \int\limits_{\Delta V(\underline{\mathbf{X}})} \circ(\underline{\mathbf{x}}) \,\mathrm{d}V\,. \tag{5}$$

As mentioned in section 1, to the author's knowledge no attempt is documented in the literature yet to provide a direct definition of the surface operator $\langle\circ\rangle_S$.

2.2. Microscopic balance laws of Cauchy continuum

For a geometrically linear analysis to be performed in the following, the Cauchy continuum at the microscale is described by the following balance equations:

$$\text{Mass:} \qquad \dot{\rho} = 0 \tag{6}$$

$$\text{Energy:} \qquad \rho\dot{\Phi} + \frac{1}{2}\rho\,\overset{\bullet}{(\underline{\mathbf{v}} \cdot \underline{\mathbf{v}})} = \underline{\nabla}_{\underline{\mathbf{x}}} \cdot \left(\underset{\sim}{\sigma} \cdot \underline{\mathbf{v}}\right) - \underline{\nabla}_{\underline{\mathbf{x}}} \cdot \underline{\mathbf{q}} + \rho\underline{\mathbf{f}} \cdot \underline{\mathbf{v}} \tag{7}$$

$$\text{Entropy:} \qquad \rho\dot{\eta} + \underline{\nabla}_{\underline{\mathbf{x}}} \cdot \underline{\mathbf{h}} \geq 0 \tag{8}$$

$$\text{Linear momentum:} \qquad \underline{\nabla}_{\underline{\mathbf{x}}} \cdot \underset{\sim}{\sigma} + \rho\underline{\mathbf{f}} - \rho\dot{\underline{\mathbf{v}}} = 0 \tag{9}$$

$$\text{Angular momentum:} \qquad \underset{\approx}{\epsilon} : \underset{\sim}{\sigma} = 0 \tag{10}$$

Therein, Φ and η are the specific intrinsic energy and entropy, respectively. Furthermore, $\underline{\mathbf{v}}$, $\underset{\sim}{\sigma}$ and $\underline{\mathbf{f}}$ denote the velocity, stress and body force, respectively. The symbols $\underline{\mathbf{q}}$ and $\underline{\mathbf{h}}$ refer to the fluxes of heat and entropy, respectively.

2.3. Approximation of microscopic velocity field

For a micromorphic continuum of degree one, the microscopic velocity field $\underline{\mathbf{v}}(\underline{\mathbf{x}})$ is approximated by a polynomial of order one:

$$\tilde{\underline{\mathbf{v}}} = \underline{\mathbf{V}}(\underline{\mathbf{X}}) + \underset{\sim}{\mathbf{L}}_{\chi}(\underline{\mathbf{X}}) \cdot (\underline{\mathbf{x}} - \underline{\mathbf{X}}) \tag{11}$$

with the macroscopic velocity $\underline{\mathbf{V}}$ and a rate of microdeformation $\underset{\sim}{\mathbf{L}}_{\chi}$. Special cases of (11) are the Cosserat (micropolar) continuum with $\underset{\sim}{\mathbf{L}}_{\chi} = -\underline{\mathbf{\Omega}}_{\chi} \cdot \underset{\approx}{\epsilon}$ so that $\underline{\mathbf{\Omega}}_{\chi}(\underline{\mathbf{X}})$ is a microrate of rotation or the microdilatational continuum with $\underset{\sim}{\mathbf{L}}_{\chi} = \frac{1}{3}\dot{\chi}\underset{\sim}{\mathbf{I}}$. In this context, the Cauchy continuum with $\underset{\sim}{\mathbf{L}}_{\chi} = 0$ can be seen as a micromorphic continuum of order zero.

2.4. Macroscopic balance laws

According to section 2.1, the microscopic balance laws of the Cauchy continuum from section 2.2 yield the following macroscopic counterparts:

2.4.1. Mass

Defining the macroscopic mass density as

$$\overline{\rho} = \langle \rho \rangle_V \ , \tag{12}$$

the macroscopic counterpart to (6) reads

$$\dot{\overline{\rho}} = 0 \tag{13}$$

Furthermore, the balances of microinertia

$$\overset{\bullet}{\langle \rho(\underline{\mathbf{x}} - \underline{\mathbf{X}}) \rangle_V} = 0 \, , \qquad \overset{\bullet}{\langle \rho(\underline{\mathbf{x}} - \underline{\mathbf{X}}) \otimes (\underline{\mathbf{x}} - \underline{\mathbf{X}}) \rangle_V} = 0 \tag{14}$$

will be needed. They are obtained by weighting (6) with the relative location $\underline{\mathbf{x}} - \underline{\mathbf{X}}$ and its square, respectively, each of which has the structure of a balance equation (2) so that (4) can be applied yielding (14).

2.4.2. Linear and Angular Momentum

The microscopic balance of linear momentum (9) yields

$$0 = \underline{\nabla}_{\underline{\mathbf{X}}} \cdot \underbrace{\langle \underset{\sim}{\sigma} \rangle_S}_{=:\underset{\sim}{\mathbf{\Sigma}}} + \underbrace{\langle \rho \underline{\mathbf{f}} \rangle_V}_{=:\bar{\rho}\underline{\mathbf{f}}} - \bar{\rho}\dot{\underline{\mathbf{V}}} - \dot{\underset{\sim}{\mathbf{L}}}_\chi \cdot \langle \rho(\underline{\mathbf{x}} - \underline{\mathbf{X}}) \rangle_V \tag{15}$$

and allows to define the macroscopic values $\underset{\sim}{\mathbf{\Sigma}}$ and $\underline{\mathbf{f}}$ of (extrinsic) stress and body force, respectively. In (15), the approximation (11) of the velocity field was inserted for the inertia term. Furthermore, a macroscopic counterpart of the balance of linear momentum weighted with the distance $\underline{\xi} = \underline{\mathbf{x}} - \underline{\mathbf{X}}$ will be needed for the energy balance. With respect to the linear approximation (11) of the velocity field, this might be interpreted as a Galerkin approach. The balance of linear momentum (9) weighted by $\underline{\mathbf{x}}$ can be written as

$$\underline{\nabla}_{\underline{\mathbf{x}}} \cdot \left(\underset{\sim}{\sigma} \otimes \underline{\mathbf{x}} \right) - \underset{\sim}{\sigma}^{\mathrm{T}} + \rho \underline{\mathbf{f}} \otimes \underline{\mathbf{x}} - \rho \dot{\underline{\mathbf{v}}} \otimes \underline{\mathbf{x}} = 0 \tag{16}$$

and thus exhibits also the structure of a balance equation (2) so that the average theorem (4) can be applied to obtain a macroscopic counterpart. Upon subtraction of (15) weighted by the macroscopic location $\underline{\mathbf{X}}$ (and written in a form as (16)) and inserting again (11) for the inertia term, one obtains

$$0 = \underline{\nabla}_{\underline{\mathbf{X}}} \cdot \underbrace{\left\langle \underset{\sim}{\sigma} \otimes (\underline{\mathbf{x}} - \underline{\mathbf{X}}) \right\rangle_S}_{=:\underset{\approx}{\underline{\mathbf{M}}}} + \underbrace{\left\langle \underset{\sim}{\sigma} \right\rangle_S^{\mathrm{T}}}_{=\underset{\sim}{\mathbf{\Sigma}}^{\mathrm{T}}} - \underbrace{\left\langle \underset{\sim}{\sigma}^{\mathrm{T}} \right\rangle_V}_{=:\bar{\underset{\sim}{\sigma}}^{\mathrm{T}}} + \underbrace{\langle \rho \underline{\mathbf{f}} \otimes (\underline{\mathbf{x}} - \underline{\mathbf{X}}) \rangle_V}_{=:\bar{\rho}\underset{\sim}{\mathbf{m}}} - \dot{\underline{\mathbf{V}}} \otimes \left\langle \rho \underline{\xi} \right\rangle_V - \dot{\underset{\sim}{\mathbf{L}}}_\chi \cdot \underbrace{\left\langle \rho \underline{\xi} \otimes \underline{\xi} \right\rangle_V}_{=:\underset{\sim}{\mathbf{G}}_\rho} \tag{17}$$

This equation can be interpreted that the difference between extrinsic (Cauchy) stress $\underset{\sim}{\mathbf{\Sigma}}$ and intrinsic (mean) stress $\bar{\underset{\sim}{\sigma}}$, and higher order body forces $\underset{\sim}{\mathbf{m}}$, are the sources of hyperstresses $\underset{\approx}{\underline{\mathbf{M}}}$ and higher order inertia. The magnitude of the latter depends on the second moment of inertia $\underset{\sim}{\mathbf{G}}_\rho$.

Looking at eqs. (15) and (17) it would be appealing to define the yet unspecified macroscopic location $\underline{\mathbf{X}}$ as centre of gravity $\langle \rho \underline{\mathbf{x}} \rangle_V / \bar{\rho}$ (as done by Eringen and Mindlin in [31, 8] but not in [9]) so that the terms $\left\langle \rho \underline{\xi} \right\rangle_V$ would vanish. However, an alternative but similar definition as geometric centre $\underline{\mathbf{X}} = \langle \underline{\mathbf{x}} \rangle_V$ will be chosen as explained in section 3.2. If the distribution of mass density $\rho(\underline{\mathbf{x}})$ exhibits certain symmetries within the volume element $\Delta V(\underline{\mathbf{X}})$, e. g. a central symmetry as it is mostly the case, then the "cross terms" $\left\langle \rho \underline{\xi} \right\rangle_V$ in the balances of energy and momenta vanish nevertheless.

The balance of angular momentum (10) yields the symmetry of the macroscopic intrinsic stress

$$0 = \underset{\approx}{\epsilon} : \bar{\underset{\sim}{\sigma}} \, . \tag{18}$$

The macroscopic counterpart to (10) weighted with $\underline{\xi}$ does not yield additional information.

Eq. (18) can be also inserted in the skew-symmetric part of (17):

$$0 = \underline{\nabla}_{\underline{\mathbf{X}}} \cdot \underset{\approx}{\underline{\mathbf{M}}} : \underset{\approx}{\epsilon} + \underset{\sim}{\mathbf{\Sigma}}^{\mathrm{T}} : \underset{\approx}{\epsilon} + \bar{\rho} \underset{\sim}{\mathbf{m}} : \underset{\approx}{\epsilon} - \dot{\underline{\mathbf{V}}} \times \left\langle \rho \underline{\xi} \right\rangle_V - \dot{\underset{\sim}{\mathbf{L}}}_\chi \cdot \underset{\sim}{\mathbf{G}}_\rho : \underset{\approx}{\epsilon} \tag{19}$$

The term $\underset{\sim}{\mathbf{\Sigma}}^{\mathrm{T}} : \underset{\approx}{\epsilon}$ on the right-hand side can be transformed to storage and divergence parts as in classical continuum mechanics so that , in absence of body forces and moments, (19) as the skew-symmetric part of (17) has conservation type and can, in analogy to classical continuum mechanics, also be interpreted as macroscopic balance of angular momentum (as usually done in Cosserat theory).

2.4.3. Energy and Entropy

The averaged microscopic energy balance (7) becomes

$$\underbrace{\left\langle \rho \dot{\Phi} \right\rangle_V}_{=:\bar{\rho}\dot{\bar{\Phi}}} + \frac{1}{2} \left\langle \rho \left(\underline{\mathbf{v}} \cdot \underline{\mathbf{v}} \right)^{\bullet} \right\rangle_V = \underline{\nabla}_{\underline{\mathbf{X}}} \cdot \left\langle \underset{\sim}{\sigma} \cdot \underline{\mathbf{v}} \right\rangle_S - \underline{\nabla}_{\underline{\mathbf{X}}} \cdot \underbrace{\left\langle \underline{\mathbf{q}} \right\rangle_S}_{=:\underline{\mathbf{Q}}} + \langle \rho \underline{\mathbf{f}} \cdot \underline{\mathbf{v}} \rangle_V \tag{20}$$

and allows to introduce the macroscopic values $\overline{\Phi}$ and $\underline{\mathbf{Q}}$ of specific intrinsic energy and heat flux, respectively. Replacing the local velocity field $\underline{\mathbf{v}}(\underline{\mathbf{x}})$ by its approximation (11) and inserting the definitions of the macroscopic stress measures yields

$$\begin{aligned}&\overline{\rho}\dot{\overline{\Phi}} + \frac{1}{2}\overline{\rho}\left(\underline{\mathbf{V}}\cdot\underline{\mathbf{V}}\right)^{\bullet} + \frac{1}{2}\underset{\sim}{\mathbf{G}}_{\rho} : \left(\underset{\sim}{\mathbf{L}}_{\chi}{}^{\mathrm{T}}\cdot\underset{\sim}{\mathbf{L}}_{\chi}\right)^{\bullet} + \left(\underline{\mathbf{V}}\cdot\underset{\sim}{\mathbf{L}}_{\chi}\right)^{\bullet}\cdot\langle\rho\underline{\xi}\rangle_{V}\\ &= \underline{\nabla}_{\underline{\mathbf{X}}}\cdot\left(\underset{\sim}{\mathbf{\Sigma}}\cdot\underline{\mathbf{V}} + \underset{\approx}{\underline{\mathbf{M}}} : \underset{\sim}{\mathbf{L}}_{\chi}\right) - \underline{\nabla}_{\underline{\mathbf{X}}}\cdot\underline{\mathbf{Q}} + \overline{\rho}\underline{\overline{\mathbf{f}}}\cdot\underline{\mathbf{V}} + \overline{\rho}\underset{\sim}{\mathbf{m}} : \underset{\sim}{\mathbf{L}}_{\chi}\end{aligned} \tag{21}$$

The balance of internal energy is obtained by applying the product rule and inserting the balances of momenta (15), (17) and (18) as

$$\begin{aligned}\overline{\rho}\dot{\overline{\Phi}} &= \underset{\sim}{\mathbf{\Sigma}} : \underline{\nabla}_{\underline{\mathbf{X}}}\underline{\mathbf{V}} + \left(\bar{\underset{\sim}{\sigma}}^{\mathrm{T}} - \underset{\sim}{\mathbf{\Sigma}}^{\mathrm{T}}\right) : \underset{\sim}{\mathbf{L}}_{\chi} + \underset{\approx}{\underline{\mathbf{M}}} \vdots \underline{\nabla}_{\underline{\mathbf{X}}}\underset{\sim}{\mathbf{L}}_{\chi} - \underline{\nabla}_{\underline{\mathbf{X}}}\cdot\underline{\mathbf{Q}}\\ &= \underset{\sim}{\mathbf{\Sigma}} : \left(\underline{\nabla}_{\underline{\mathbf{X}}}\underline{\mathbf{V}} - \underset{\sim}{\mathbf{L}}_{\chi}{}^{\mathrm{T}}\right) + \bar{\underset{\sim}{\sigma}} : \underset{\sim}{\mathbf{L}}_{\chi}{}^{\mathrm{S}} + \underset{\approx}{\underline{\mathbf{M}}} \vdots \underline{\nabla}_{\underline{\mathbf{X}}}\underset{\sim}{\mathbf{L}}_{\chi} - \underline{\nabla}_{\underline{\mathbf{X}}}\cdot\underline{\mathbf{Q}}\end{aligned} \tag{22}$$

Thus, the work-conjugate deformation rates to the macroscopic stress measures $\underset{\sim}{\mathbf{\Sigma}}$, $\bar{\underset{\sim}{\sigma}}$ and $\underset{\approx}{\underline{\mathbf{M}}}$ are the rates of microstrain, relative deformation and microdeformation gradient, respectively:

$$\dot{\underset{\sim}{\mathbf{E}}} = \underset{\sim}{\mathbf{L}}_{\chi}{}^{\mathrm{S}} \qquad \dot{\underset{\sim}{\mathbf{e}}} = \underline{\nabla}_{\underline{\mathbf{X}}}\underline{\mathbf{V}} - \underset{\sim}{\mathbf{L}}_{\chi}{}^{\mathrm{T}} \qquad \dot{\underset{\approx}{\underline{\mathbf{K}}}} = \underline{\nabla}_{\underline{\mathbf{X}}}\underset{\sim}{\mathbf{L}}_{\chi}\,. \tag{23}$$

The microscopic entropy balance (8) yields its macroscopic counterpart of identical structure

$$\underbrace{\langle\rho\dot{\eta}\rangle_{V}}_{:=\overline{\rho}\dot{S}} + \underline{\nabla}_{\underline{\mathbf{X}}}\cdot\underbrace{\langle\underline{\mathbf{h}}\rangle_{S}}_{:=\underline{\mathbf{H}}} \geq 0 \tag{24}$$

defining the macroscopic values S and $\underline{\mathbf{H}}$ of specific entropy and flux of entropy.

3. Homogenization

3.1. Thermodynamic considerations

Let us assume, that the continuum at the microlevel does not only obey balance equations (6)–(10), but that it is furthermore a Coleman-Noll continuum [5] so that the specific internal energy Φ forms a potential for temperature θ and stress $\underset{\sim}{\sigma}$:

$$\theta = \frac{\partial\Phi}{\partial\eta}, \qquad \underset{\sim}{\sigma} = \rho\frac{\partial\Phi}{\partial\underset{\sim}{\varepsilon}} \tag{25}$$

Consequently, the microscopic energy balance (7) reduces to

$$\rho D - \rho\theta\dot{\eta} = \underline{\nabla}_{\underline{\mathbf{x}}}\cdot\underline{\mathbf{q}}\,, \tag{26}$$

wherein D denotes the specific dissipation due to a change of internal variables. This equation is again of balance type and thus leads to a macroscopic relation

$$\underbrace{\langle\rho D\rangle_{V}}_{=:\overline{\rho}\overline{D}} - \underbrace{\langle\rho\theta\dot{\eta}\rangle_{V}}_{=:\overline{\rho}\Theta\dot{S}} = \underline{\nabla}_{\underline{\mathbf{X}}}\cdot\underline{\mathbf{Q}}\,, \tag{27}$$

which implies the given definition of macroscopic values $\overline{D}$ and Θ of dissipation and temperature, respectively. Eq. (27) can be used to eliminate the heatflux $\underline{\mathbf{Q}}$ from the macroscopic balance of internal energy (22):

$$\overline{\rho}\dot{\overline{\Phi}} = \underset{\sim}{\mathbf{\Sigma}} : \dot{\underset{\sim}{\mathbf{e}}} + \bar{\underset{\sim}{\sigma}} : \dot{\underset{\sim}{\mathbf{E}}} + \underset{\approx}{\underline{\mathbf{M}}} \vdots \dot{\underset{\approx}{\underline{\mathbf{K}}}} + \overline{\rho}\Theta\dot{S} - \overline{\rho}\overline{D} \tag{28}$$

Furthermore, (25) has the consequence that the left-hand side of (28) can be written as

$$\overline{\rho}\dot{\overline{\Phi}} = \left\langle \rho\dot{\Phi} \right\rangle_V = \langle \rho\theta\dot{\eta} \rangle_V + \langle \underset{\sim}{\sigma} : \dot{\underset{\sim}{\varepsilon}} \rangle_V - \langle \rho D \rangle_V \ . \tag{29}$$

Equating this relation to (28) and eliminating identical terms on both sides using (31) and the definitions in (27) leads to a generalized Hill-Mandel lemma

$$\begin{aligned} \langle \underset{\sim}{\sigma} : \dot{\underset{\sim}{\varepsilon}} \rangle_V =& \underset{\sim}{\mathbf{\Sigma}} : \dot{\underset{\sim}{\mathbf{e}}} + \underset{\sim}{\bar{\sigma}} : \dot{\underset{\sim}{\mathbf{E}}} + \underset{\approx}{\underline{\mathbf{M}}} \vdots \dot{\underset{\approx}{\mathbf{K}}} \\ =& \underset{\sim}{\mathbf{\Sigma}} : \underline{\nabla_{\underline{\mathbf{X}}}\mathbf{V}} + \left(\underset{\sim}{\bar{\sigma}}^{\mathrm{T}} - \underset{\sim}{\mathbf{\Sigma}}^{\mathrm{T}} \right) : \underset{\sim}{\mathbf{L}}_\chi + \underset{\approx}{\underline{\mathbf{M}}} \vdots \underline{\nabla_{\underline{\mathbf{X}}}\underset{\sim}{\mathbf{L}}_\chi} \end{aligned} \tag{30}$$

However, in contrast to classical homogenization it is no ad-hoc requirement but a *consequence* of the employed definitions of macroscopic quantities and the fact that the continuum at the microscale is of Coleman-Noll type.

If the macroscopic specific internal energy can be identified as a function $\overline{\Phi}(\underset{\sim}{\mathbf{E}}, \underset{\sim}{\mathbf{e}}, \underset{\approx}{\mathbf{K}}, S, h)$ of macroscopic entropy S and intrinsic variables h as well as deformation measures $\underset{\sim}{\mathbf{E}}$, $\underset{\sim}{\mathbf{e}}$ and $\underset{\approx}{\mathbf{K}}$ then (28) can be fulfilled for all kinematically admissible fields $\underline{\mathbf{V}}(\underline{\mathbf{X}})$ $\underset{\sim}{\mathbf{L}}_\chi(\underline{\mathbf{X}})$ and $S(\underline{\mathbf{X}})$ (and thus also arbitrary values of their gradients) if and only if

$$\underset{\sim}{\bar{\sigma}} = \overline{\rho}\frac{\partial\overline{\Phi}}{\partial\underset{\sim}{\mathbf{E}}}, \qquad \underset{\sim}{\mathbf{\Sigma}} = \overline{\rho}\frac{\partial\overline{\Phi}}{\partial\underset{\sim}{\mathbf{e}}}, \qquad \underset{\approx}{\underline{\mathbf{M}}} = \overline{\rho}\frac{\partial\overline{\Phi}}{\partial\underset{\approx}{\mathbf{K}}}, \qquad \Theta = \frac{\partial\overline{\Phi}}{\partial S} \tag{31}$$

Thus, a homogenization can equivalently be performed with respect to $\overline{\Phi}$ and $\overline{D}$ (and fluxes $\underline{\mathbf{Q}}$ and $\underline{\mathbf{H}}$ of heat and entropy in thermomechanical problems).

3.2. Micro-macro transition

3.2.1. Review of classical homogenization

In classical homogenization, firstly the thermal and mechanical behavior is considered independently of each other. Furthermore, for the mechanical behavior, static conditions are assumed on the microscale and the macroscopic velocity gradient is postulated to correspond to the average of its microscopic counterpart:

$$\underline{\nabla_{\underline{\mathbf{X}}}\mathbf{V}} = \left\langle \underline{\nabla_{\underline{\mathbf{x}}}\mathbf{v}} \right\rangle_V \tag{32}$$

The right hand side can be transformed to a surface integral so that the macroscopic velocity gradient $\underline{\nabla_{\underline{\mathbf{X}}}\mathbf{V}}$ can be prescribed by kinematic boundary conditions. Alternatively, the concept of minimal loading conditions proposed by Jänicke and Steeb [27] is employed here. Let us consider firstly a reversible process. Then, the specific internal energy does not depend on internal variables and the concept of minimal loading conditions can be implemented for classical homogenization by requiring the Lagrange functional

$$\mathcal{L} = \left\langle \rho\dot{\Phi} \right\rangle_V + \underset{\sim}{\lambda}_{\nabla V} : \left(\underline{\nabla_{\underline{\mathbf{X}}}\mathbf{V}} - \left\langle \underline{\nabla_{\underline{\mathbf{x}}}\mathbf{v}} \right\rangle_V \right) \to \text{Min.} \tag{33}$$

to become a minimum. As Φ is a potential for the stress $\underset{\sim}{\sigma}$ according to (25), the corresponding stationarity conditions is

$$0 = \langle \underset{\sim}{\sigma} : \delta\dot{\underset{\sim}{\varepsilon}} \rangle_V - \underset{\sim}{\lambda}_{\nabla V} : \left\langle \underline{\nabla_{\underline{\mathbf{x}}}\delta\mathbf{v}} \right\rangle_V \ . \tag{34}$$

together with (32). The stationarity conditions (34) and (32) can be generalized to hold also for irreversible processes. In this case, it corresponds to the principle of virtual power on the microscale.

In the light of the fact that it holds for all kinematically admissible fields $\delta\underline{\mathbf{v}}(\underline{\mathbf{x}})$, including the real velocity field $\underline{\mathbf{v}}(\underline{\mathbf{x}})$, a comparison with the Hill-Mandel lemma (30) shows, after reinserting the kinematic micro-macro relations (32), that the Lagrange multiplier correspond to the respective work-conjugate macroscopic stress measure

$$\underset{\sim}{\lambda}_{\nabla V} = \underset{\sim}{\mathbf{\Sigma}} \,. \tag{35}$$

With this substitution, the Euler-Lagrange equations to (34) read

$$\underline{\nabla}_{\underline{\mathbf{x}}} \cdot \underset{\sim}{\sigma} = 0 \text{ and } \underset{\sim}{\sigma} = \underset{\sim}{\sigma}^{\mathrm{T}} \qquad \text{in } \Delta V(\underline{\mathbf{X}}) \tag{36}$$

$$\underline{\mathbf{n}} \cdot \underset{\sim}{\sigma} = \underline{\mathbf{n}} \cdot \underset{\sim}{\mathbf{\Sigma}} \qquad \text{on } \partial\Delta V(\underline{\mathbf{X}}) \tag{37}$$

Thus, if no further essential boundary conditions are specified, the concept of minimal loading conditions results in the static approach of homogenization. Consequently, the loading to the microscopic volume element needs to be self-equilibrating. This is the case only if the macroscopic stress is symmetric $\underset{\sim}{\mathbf{\Sigma}} = \underset{\sim}{\mathbf{\Sigma}}^{\mathrm{T}}$. Alternatively to (37), essential boundary conditions can be specified which then need to fulfill the kinematic constraint (32) a priori corresponding to the kinematic approach of classical homogenization (in this case the constraint term with $\underset{\sim}{\lambda}_{\nabla V}$ in (33) is obsolete and can be dropped). In classical homogenization theory, the macroscopic stress is defined as the volume average $\underset{\sim}{\mathbf{\Sigma}} = \left\langle \underset{\sim}{\sigma} \right\rangle_V$ which is consistent with (36) and (37).

3.2.2. Extension to micromorphic theory

For the envisaged micromorphic theory, the macroscopic stress was already defined in (15) by the surface operator $\underset{\sim}{\mathbf{\Sigma}} = \left\langle \underset{\sim}{\sigma} \right\rangle_S$ and the intrinsic stress was introduced as volume average $\bar{\underset{\sim}{\sigma}} = \left\langle \underset{\sim}{\sigma} \right\rangle_V$. For both stress measures to be able to differ, the surface operator needs to be defined in a consistent way to (36) and (37) to be valid for all shapes of ΔV but must not coincide with the volume average $\langle \circ \rangle_V$. This is assured by defining the surface operator[1] as

$$\underset{\sim}{\mathbf{\Sigma}} = \left\langle \underset{\sim}{\sigma} \right\rangle_S := \frac{1}{\Delta V} \int\limits_{\partial\Delta V} \underline{\xi} \otimes \underline{\mathbf{n}} \cdot \underset{\sim}{\sigma} \, \mathrm{d}S \,. \tag{38}$$

It can easily be verified that inserting (37) from classical homogenization to (38) yields an identity for the macroscopic stress $\underset{\sim}{\mathbf{\Sigma}}$. Furthermore, definition (38) ensures that the homogenization procedure is exact in the limit that ΔV becomes infinitesimally small (*asymptotic self-consistency*, see appendix A.2). With the explicit definition (38) of the surface operator $\langle \circ \rangle_S$ available, it is possible to evaluate the additional stress measures of the micromorphic theory. In particular, the hyperstress becomes

$$\underset{\approx}{\underline{\mathbf{M}}} = \left\langle \underset{\sim}{\sigma} \underline{\xi} \right\rangle_S = \frac{1}{\Delta V} \int\limits_{\partial\Delta V} \underline{\xi} \otimes \underline{\mathbf{n}} \cdot \underset{\sim}{\sigma} \otimes \underline{\xi} \, \mathrm{d}S \tag{39}$$

and, applying Gauss theorem, the difference of intrinsic and extrinsic macroscopic stresses according to definition (17) can be written as

$$\underset{\sim}{\mathbf{\Sigma}} - \bar{\underset{\sim}{\sigma}} = \left\langle \underline{\xi} \otimes \left(\underline{\nabla}_{\underline{\mathbf{x}}} \cdot \underset{\sim}{\sigma} \right) \right\rangle_V \,. \tag{40}$$

For the classical theory from section 3.2.1, inserting (36) and (37) leads to $\underset{\approx}{\underline{\mathbf{M}}} = 0$ and $\underset{\sim}{\mathbf{\Sigma}} - \bar{\underset{\sim}{\sigma}} = 0$ under the definition $\left\langle \underline{\xi} \right\rangle_V = 0$, i. e. that the macroscopic location corresponds to the geometric center of the volume element:

$$\underline{\mathbf{X}} = \langle \underline{\mathbf{x}} \rangle_V \,. \tag{41}$$

[1]A similar definition was chosen in [28, 29] for the classical macroscopic stress. However, the hyperstress in [28, 29] differs from (39) by a factor of 1/2.

This implies a definition of the macroscopic velocity as

$$\underline{\mathbf{V}} = \langle \underline{\mathbf{v}} \rangle_V \,. \tag{42}$$

Eq. (42) can be incorporated into the Lagrangian (33) by an additional Lagrangian multiplier $\underline{\lambda}_V$ through which (36) becomes $\underline{\nabla}_{\underline{\mathbf{x}}} \cdot \underset{\sim}{\sigma} = \underline{\lambda}_V$. Due to (41), $\underline{\lambda}_V$ does not contribute to $\bar{\underset{\sim}{\sigma}}$, $\underset{\sim}{\mathbf{\Sigma}}$ or $\underset{\approx}{\mathbf{M}}$ according to (38),(39), (40). A comparison of the corresponding stationarity condition (34) with the Hill-Mandel relation (30) shows that $\underline{\lambda}_V = 0$ so that $\underline{\nabla}_{\underline{\mathbf{x}}} \cdot \underset{\sim}{\sigma} = 0$ remains in classical homogenization even if (42) is enforced.

Thus, consistently definition (38) ensures that the additional stresses $\underset{\approx}{\mathbf{M}}$ and $\underset{\sim}{\mathbf{\Sigma}} - \bar{\underset{\sim}{\sigma}}$ vanish for all shapes of the domain ΔV in the classical case, i. e. if no additional macroscopic deformations are enforced in (33). In this case the additional balance (17) is fulfilled identically and what remains is the classical theory of continuum mechanics.

Furthermore, it has to be remarked that according to (39), the hyperstress $\underset{\approx}{\mathbf{M}}$ is symmetric with respect to its first and last index. Thus, only the part $\bar{\dot{K}}_{ijk} = (\dot{K}_{ijk} + \dot{K}_{kji})/2$ of the gradient of microdeformation $\underset{\approx}{\mathbf{K}}$ contributes to the internal energy which has the same symmetry. Consequently, the Hill-Mandel lemma (30) becomes

$$\left\langle \underset{\sim}{\sigma} : \dot{\underset{\sim}{\varepsilon}} \right\rangle_V = \underset{\sim}{\mathbf{\Sigma}} : \underline{\nabla}_{\underline{\mathbf{X}}} \underline{\mathbf{V}} + \left(\bar{\underset{\sim}{\sigma}}^{\mathrm{T}} - \underset{\sim}{\mathbf{\Sigma}}^{\mathrm{T}} \right) : \underset{\sim}{\mathbf{L}}_{\chi} + \underset{\approx}{\mathbf{M}} \vdots \bar{\dot{\underset{\approx}{\mathbf{K}}}} \,. \tag{43}$$

The next question is how (33) needs to be modified such that the additional stress measures $\underset{\approx}{\mathbf{M}}$ and $\underset{\sim}{\mathbf{\Sigma}} - \bar{\underset{\sim}{\sigma}}$ become nonzero. Most easily, this problem is solved by going in inverse directions through eqs. (33)–(40). Revisiting (39) and (40) in the light of (41), this means that the terms $\underline{\nabla}_{\underline{\mathbf{x}}} \cdot \underset{\sim}{\sigma}$ and $\underset{\sim}{\sigma} \cdot \underline{\mathbf{n}}$ need to be extended by linear terms. Furthermore, the operator in (40) is insensitive to constant terms as mentioned already. Thus, $\underline{\nabla}_{\underline{\mathbf{x}}} \cdot \underset{\sim}{\sigma}$ and $\underset{\sim}{\sigma} \cdot \underline{\mathbf{n}}$ are set to

$$\underline{\nabla}_{\underline{\mathbf{x}}} \cdot \underset{\sim}{\sigma} = \underset{\sim}{\mathbf{A}} \cdot \underline{\xi} + \underline{\mathbf{B}} - \underline{\lambda}_V \qquad \text{in } \Delta V(\underline{\mathbf{X}}) \tag{44}$$

$$\underline{\mathbf{n}} \cdot \underset{\sim}{\sigma} = \underline{\mathbf{n}} \cdot \underset{\sim}{\mathbf{\Sigma}} + \underline{\mathbf{n}} \cdot \underset{\approx}{\mathbf{C}} \cdot \underline{\xi} \qquad \text{on } \partial \Delta V(\underline{\mathbf{X}}) \,. \tag{45}$$

Inserting this ansatz to (39) and (40) allows to express the coefficients in terms of the additional stress measures[2]:

$$\underset{\sim}{\mathbf{A}} = \left(\underset{\sim}{\mathbf{\Sigma}}^{\mathrm{T}} - \bar{\underset{\sim}{\sigma}}^{\mathrm{T}} \right) \cdot \underset{\sim}{\mathbf{G}}^{-1} \tag{46}$$

$$C_{ijk} = \frac{1}{4} \left(M_{ijm} + M_{mji} \right) G^{-1}_{mk} - \frac{1}{2\,(2 + \delta_{pp})} \delta_{ik} M_{mjn} G^{-1}_{mn} \tag{47}$$

Therein, the geometric moment is defined as

$$\underset{\sim}{\mathbf{G}} = \left\langle \underline{\xi} \otimes \underline{\xi} \right\rangle_V \,. \tag{48}$$

Note, that the linear boundary term (47) is not self-equilibrating in (45) for all prescribed hyperstresses $\underset{\approx}{\mathbf{M}}$. Consequently, for the boundary value problem (44),(45) at the microscale to have a solution, the non-equilibrating part needs to be compensated by the volume term

$$\underline{\mathbf{B}} = \left(\underset{\approx}{\mathbf{I}}_{\mathrm{T}} : \underset{\approx}{\mathbf{C}} \right) : \underset{\sim}{\mathbf{I}} = \frac{1}{2 + \underset{\sim}{\mathbf{I}} : \underset{\sim}{\mathbf{I}}} \left[\underset{\approx}{\mathbf{I}}_{\mathrm{T}} : \left(\underset{\approx}{\mathbf{M}} \cdot \underset{\sim}{\mathbf{G}}^{-1} \right) \right] : \underset{\sim}{\mathbf{I}}, \qquad B_j = C_{iji} = \frac{1}{2 + \delta_{pp}} M_{mjn} G^{-1}_{mn} \,. \tag{49}$$

[2]Due to the symmetry of the hyperstress $\underset{\approx}{\mathbf{M}}$, not all components of $\underset{\approx}{\mathbf{C}}$ contribute to $\underset{\approx}{\mathbf{M}}$. Vice versa, (39) does not determine $\underset{\approx}{\mathbf{C}}$ uniquely. Eq. (47) incorporates only those components of $\underset{\approx}{\mathbf{C}}$ which have the respective symmetry and which are thus uniquely defined.

Finally, the balance equation and natural boundary conditions at the microscale become

$$\underline{\nabla}_{\underline{\mathbf{x}}} \cdot \underset{\sim}{\sigma} = \left(\underset{\sim}{\mathbf{\Sigma}}^{\mathrm{T}} - \underset{\sim}{\bar{\sigma}}^{\mathrm{T}}\right) \cdot \underset{\sim}{\mathbf{G}}^{-1} \cdot \underline{\xi} + \frac{1}{2 + \underset{\sim}{\mathbf{I}} : \underset{\sim}{\mathbf{I}}} \left[\underset{\approx}{\mathbf{I}}_{\mathrm{T}} : \left(\underset{\widetilde{\sim}}{\mathbf{M}} \cdot \underset{\sim}{\mathbf{G}}^{-1}\right)\right] : \underset{\sim}{\mathbf{I}} - \underline{\lambda}_V, \quad \underset{\sim}{\sigma}^{\mathrm{T}} = \underset{\sim}{\sigma} \qquad \text{in } \Delta V(\underline{\mathbf{X}}) \quad (50)$$

$$\underline{\mathbf{n}} \cdot \underset{\sim}{\sigma} = \underline{\mathbf{n}} \cdot \underset{\sim}{\mathbf{\Sigma}} + \frac{1}{4}\underline{\mathbf{n}} \cdot \left(\underset{\widetilde{\sim}}{\mathbf{M}} + \underset{\widetilde{\sim}}{\mathbf{M}}^{\mathrm{T}}\right) \cdot \underset{\sim}{\mathbf{G}}^{-1} \cdot \underline{\xi} - \frac{1}{2\left(2 + \underset{\sim}{\mathbf{I}} : \underset{\sim}{\mathbf{I}}\right)} \underline{\mathbf{n}} \cdot \underline{\xi} \left[\underset{\approx}{\mathbf{I}}_{\mathrm{T}} : \left(\underset{\widetilde{\sim}}{\mathbf{M}} \cdot \underset{\sim}{\mathbf{G}}^{-1}\right)\right] : \underset{\sim}{\mathbf{I}} \quad \text{on } \partial\Delta V(\underline{\mathbf{X}}) \quad (51)$$

$$n_i \sigma_{ij} = n_i \Sigma_{ij} + \frac{1}{4} n_i (M_{ijp} + M_{pji}) G^{-1}_{pm} \xi_m - \frac{1}{2(2 + \delta_{kk})} n_i \xi_i M_{njp} G^{-1}_{np}$$

Equations (50) and (51) are the Euler-Lagrange equations of the stationarity condition

$$\begin{aligned} 0 = {}& \langle \underset{\sim}{\sigma} : \delta\dot{\underset{\sim}{\xi}} \rangle_V - \underset{\sim}{\mathbf{\Sigma}} : \left\langle \underline{\nabla}_{\underline{\mathbf{x}}} \delta\underline{\mathbf{v}} \right\rangle_V - \underline{\lambda}_V \cdot \langle \delta\underline{\mathbf{v}} \rangle_V + \left(\underset{\sim}{\mathbf{\Sigma}}^{\mathrm{T}} - \underset{\sim}{\bar{\sigma}}^{\mathrm{T}}\right) : \left(\langle \delta\underline{\mathbf{v}} \otimes \underline{\xi} \rangle_V \cdot \underset{\sim}{\mathbf{G}}^{-1}\right) \\ & - \underset{\widetilde{\sim}}{\mathbf{M}} \vdots \frac{1}{2}\left[\underset{\approx}{\mathbf{I}}_{\mathrm{S}} : \left(\underset{\sim}{\mathbf{G}}^{-1} \cdot \left\langle \underline{\xi} \otimes \underline{\nabla}_{\underline{\mathbf{x}}} \delta\underline{\mathbf{v}} \right\rangle_V - \frac{1}{2 + \underset{\sim}{\mathbf{I}} : \underset{\sim}{\mathbf{I}}} \underset{\sim}{\mathbf{G}}^{-1} \left\langle \underline{\xi} \cdot \underline{\nabla}_{\underline{\mathbf{x}}} \delta\underline{\mathbf{v}} \right\rangle_V\right) : \underset{\approx}{\mathbf{I}}_{\mathrm{T}}\right] \end{aligned} \quad (52)$$

The actual microscopic velocity field $\underline{\mathbf{v}}(\underline{\xi})$ is among the admissible test fields so that (52) holds also if $\delta\underline{\mathbf{v}}(\underline{\xi})$ is replaced by $\underline{\mathbf{v}}(\underline{\xi})$. Then, a comparison with the Hill-Mandel lemma (43) shows that the macroscopic rates of deformation have to be identified as

$$\begin{aligned} \underline{\nabla}_{\underline{\mathbf{X}}} \underline{\mathbf{V}} &= \left\langle \underline{\nabla}_{\underline{\mathbf{x}}} \underline{\mathbf{v}} \right\rangle_V \\ \underset{\sim}{\mathbf{L}}_\chi &= \langle \underline{\mathbf{v}} \otimes \underline{\xi} \rangle_V \cdot \underset{\sim}{\mathbf{G}}^{-1} \\ \dot{\underset{\widetilde{\sim}}{\mathbf{K}}} &= \frac{1}{2}\underset{\approx}{\mathbf{I}}_{\mathrm{S}} : \left(\underset{\sim}{\mathbf{G}}^{-1} \cdot \left\langle \underline{\xi} \otimes \underline{\nabla}_{\underline{\mathbf{x}}} \underline{\mathbf{v}} \right\rangle_V - \frac{1}{2 + \underset{\sim}{\mathbf{I}} : \underset{\sim}{\mathbf{I}}} \underset{\sim}{\mathbf{G}}^{-1} \left\langle \underline{\xi} \cdot \underline{\nabla}_{\underline{\mathbf{x}}} \underline{\mathbf{v}} \right\rangle_V\right) : \underset{\approx}{\mathbf{I}}_{\mathrm{T}} \end{aligned} \quad (53)$$

$$\dot{K}_{ijk} = \frac{1}{2}\left[\frac{1}{2} \langle v_{j,i} \xi_m \rangle_V G^{-1}_{mk} + \frac{1}{2} \langle v_{j,k} \xi_m \rangle_V G^{-1}_{mi} - \frac{1}{2 + \delta_{ll}} \langle v_{j,m} \xi_m \rangle_V G^{-1}_{ik}\right]. \quad (54)$$

Eq. (42) for the macroscopic velocity remains valid together with $\underline{\lambda}_V = 0$. If the stresses $\underset{\sim}{\sigma}$ have a variational potential, then the variational problem to (52) and (53) reads

$$\begin{aligned} \mathcal{L} = {}& \left\langle \rho\dot{\Phi} \right\rangle_V + \underset{\sim}{\mathbf{\Sigma}} : \left(\underline{\nabla}_{\underline{\mathbf{X}}} \underline{\mathbf{V}} - \left\langle \underline{\nabla}_{\underline{\mathbf{x}}} \underline{\mathbf{v}} \right\rangle_V\right) + \underline{\lambda}_V \cdot (\underline{\mathbf{V}} - \langle \underline{\mathbf{v}} \rangle_V) + \left(\underset{\sim}{\bar{\sigma}}^{\mathrm{T}} - \underset{\sim}{\mathbf{\Sigma}}^{\mathrm{T}}\right) : \left(\underset{\sim}{\mathbf{L}}_\chi - \langle \underline{\mathbf{v}} \otimes \underline{\xi} \rangle_V \cdot \underset{\sim}{\mathbf{G}}^{-1}\right) \\ & + \underset{\widetilde{\sim}}{\mathbf{M}} \vdots \left[\dot{\underset{\widetilde{\sim}}{\mathbf{K}}} - \frac{1}{2}\underset{\approx}{\mathbf{I}}_{\mathrm{S}} : \left(\underset{\sim}{\mathbf{G}}^{-1} \cdot \left\langle \underline{\xi} \otimes \underline{\nabla}_{\underline{\mathbf{x}}} \underline{\mathbf{v}} \right\rangle_V - \frac{1}{2 + \underset{\sim}{\mathbf{I}} : \underset{\sim}{\mathbf{I}}} \underset{\sim}{\mathbf{G}}^{-1} \left\langle \underline{\xi} \cdot \underline{\nabla}_{\underline{\mathbf{x}}} \underline{\mathbf{v}} \right\rangle_V\right) : \underset{\approx}{\mathbf{I}}_{\mathrm{T}}\right] \to \text{Min.} \end{aligned} \quad (55)$$

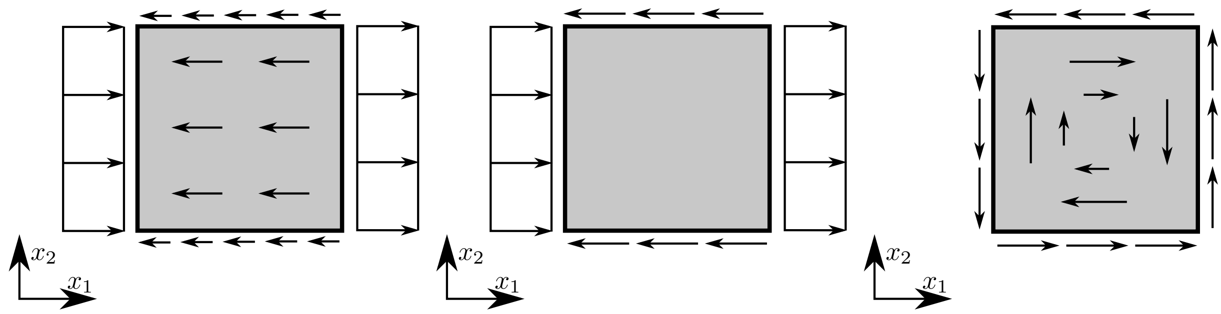
in extension to the classical one (33). Therein, the macroscopic stresses $\underset{\sim}{\mathbf{\Sigma}}$, $\underset{\sim}{\bar{\sigma}} - \underset{\sim}{\mathbf{\Sigma}}$ and $\underset{\widetilde{\sim}}{\mathbf{M}}$ have the role of Lagrange multipliers to enforce the macroscopic measures of deformation (53).

In (53), all deformation measures are insensitive to rigid translations. An affine mapping $\underline{\mathbf{v}} = \underset{\sim}{\mathbf{F}} \cdot \underline{\xi}$ does not affect $\dot{\underset{\widetilde{\sim}}{\mathbf{K}}}$ but leads to identical values $\underline{\nabla}_{\underline{\mathbf{X}}} \underline{\mathbf{V}} = \underset{\sim}{\mathbf{L}}_\chi = \underset{\sim}{\mathbf{F}}$. Consequently, the measures of rate of deformation ${\underset{\sim}{\mathbf{L}}_\chi}^{\mathrm{S}}$, $\dot{\underset{\sim}{\mathbf{e}}} = \underline{\nabla}_{\underline{\mathbf{X}}} \underline{\mathbf{V}} - {\underset{\sim}{\mathbf{L}}_\chi}^{\mathrm{T}}$ and $\dot{\underset{\widetilde{\sim}}{\mathbf{K}}}$ defined in (23) are also insensitive to a rotation with $\underset{\sim}{\mathbf{F}} = \underset{\approx}{\epsilon} \cdot \underline{\mathbf{W}}$ and are thus objective and so are their work-conjugate stresses $\underset{\sim}{\mathbf{\Sigma}}$, $\underset{\sim}{\bar{\sigma}}$ and $\underset{\widetilde{\sim}}{\mathbf{M}}$.

Note that the micro-macro relations for the deformation measures $\underline{\nabla}_{\underline{\mathbf{X}}} \underline{\mathbf{V}}$ and $\underset{\sim}{\mathbf{L}}_\chi$ in (53) are identical to the definitions in [12, 18]. In particular, definitions (42) and (53) for the macroscopic values of $\underline{\mathbf{V}}$ and $\underset{\sim}{\mathbf{L}}_\chi$ (together with (41) for the macroscopic location $\underline{\mathbf{X}}$) are equivalent to a minimum average error

$$\left\langle (\tilde{\underline{\mathbf{v}}} - \underline{\mathbf{v}})^2 \right\rangle_V \to \min \quad (56)$$

between the microscopic velocity field $\underline{\mathbf{v}}$ and its macroscopic approximation $\tilde{\underline{\mathbf{v}}}$ according (11). Thus, the kinematic micro-macro relations (53) are kinematically consistent in a sense that $\tilde{\underline{\mathbf{v}}}$ from (11) leads to



(a) (b) (c)

Figure 2: Loading of the volume element $\Delta V(\underline{\mathbf{X}})$ for several non-classical stresses: (a) $\underset{\approx}{\underline{\mathbf{M}}} = \underline{\mathbf{b}}_1\underline{\mathbf{b}}_1\underline{\mathbf{b}}_1$, (b) $\underset{\approx}{\underline{\mathbf{M}}} = \underline{\mathbf{b}}_1\underline{\mathbf{b}}_1\underline{\mathbf{b}}_1 - \underline{\mathbf{b}}_1\underline{\mathbf{b}}_2\underline{\mathbf{b}}_1$, (c) $\underset{\sim}{\mathbf{\Sigma}} = \underline{\mathbf{b}}_1\underline{\mathbf{b}}_2 - \underline{\mathbf{b}}_2\underline{\mathbf{b}}_1$

identities for $\underline{\mathbf{V}}$ and $\underset{\sim}{\mathbf{L}}_\chi$. However, the expression $(53)_3$ for $\underset{\approx}{\overline{\mathbf{K}}}$ differs to the ones given by Forest [12, 18]. Furthermore, it can be remarked that, in contrast to the derivation of the deformation measures in [12, 18], the present approach does not require that the macroscopic gradients of the geometry entities $\Delta V(\underline{\mathbf{X}})$ and $\underset{\sim}{\mathbf{G}}(\underline{\mathbf{X}})$ vanish.

The loading of the volume element according to (52) for natural boundary conditions is sketched in Figure 2 for several non-classical cases for a rectangular ΔV (whereby the $\underline{\mathbf{b}}_i$ denote to the base vectors of the coordinate system). Figure 2a shows the effect of hyperstress M_{111} for which volume loads occur due to a non-vanishing "spherical" part of $\underset{\approx}{\underline{\mathbf{M}}}$ according to (49). In contrast, for a "deviatoric" hyperstress $M_{111} = -M_{121}$ in Figure 2b the macroscopic stress results only in tractions at the microscopic boundary. In both cases, the tractions at opposite faces are identical, i. e. in the terminology of classical homogenization they are antimetric. Figure 2c shows the loading of ΔV for a skew-symmetric extrinsic stress $\Sigma_{12} = -\Sigma_{21}$. In this case the torque of the tractions is compensated by volume contributions of opposite direction which twist the ΔV.

Kinematic boundary conditions can be specified alternatively to the static ones (51). However, as in classical homogenization they have to fulfill the respective kinematic micro-macro relations ad hoc. For the present micromorphic theory, not all volume averages, which define the macroscopic deformation measures in (53), can be transformed to surface integrals by a partial integration. In particular, the spherical[3] part

$$\dot{\overline{K}}_{ijk} G_{ik} = \frac{1}{2+\delta_{ll}} \left[\left\langle (v_j \xi_k)_{,k} \right\rangle_V - \delta_{kk} \left\langle v_j \right\rangle_V \right] \tag{57}$$

of the symmetric part of the gradient of the rate of microdeformation contains the macroscopic velocity $\underline{\mathbf{V}} = \langle \underline{\mathbf{v}} \rangle_V$ as volume average and can thus not be prescribed as kinematic boundary condition. This is consistent with the fact that the corresponding part of the hyperstress appears as volume term in the microscopic equilibrium condition (50). However, the deviatoric part of $\dot{\underset{\approx}{\overline{\mathbf{K}}}}$

$$\dot{\overline{K}}{}^{\mathrm{d}}_{ijk} := \dot{\overline{K}}_{ijk} - \frac{1}{\delta_{pp}} \dot{\overline{K}}_{ljm} G_{lm} G^{-1}_{ik} = \frac{1}{2} \left[\frac{1}{2} \left\langle (v_j \xi_m)_{,i} \right\rangle_V G^{-1}_{mk} + \frac{1}{2} \left\langle (v_j \xi_m)_{,k} \right\rangle_V G^{-1}_{mi} - \frac{1}{\delta_{pp}} \left\langle (v_j \xi_m)_{,m} \right\rangle_V G^{-1}_{ik} \right] \tag{58}$$

[3]For the practically most relevant shapes of the volume element ΔV of a cube or sphere (square and circle in 2D), the geometric moment $\underset{\sim}{\mathbf{G}}$ is a spherical tensor.

(and the classical velocity gradient $\underline{\nabla}_{\underline{\mathbf{X}}}\underline{\mathbf{V}} = \left\langle \underline{\nabla}_{\underline{\mathbf{x}}}\underline{\mathbf{v}} \right\rangle_V$) can be transformed to pure surface terms by application of Gauss' theorem and thus be prescribed by a polynomial kinematic boundary condition with additional quadratic term with coefficient tensor $\underset{\sim}{\underline{\mathbf{D}}}$ (being symmetric with respect to those indices going with $\underline{\xi}$):

$$\underline{\mathbf{v}} = \underline{\mathbf{V}}_0 + \underline{\xi} \cdot \underline{\nabla}_{\underline{\mathbf{X}}}\underline{\mathbf{V}} + \underline{\xi} \cdot \underset{\sim}{\underline{\mathbf{D}}} \cdot \underline{\xi} \qquad \text{on } \partial\Delta V(\underline{\mathbf{X}}) \tag{59}$$

Inserting (59) to (57) and (58) leads to

$$\overset{\bullet}{\overline{K}}_{ijk} G_{ik} = D_{ljm} G_{lm} - \frac{1}{2+\delta_{ll}} \delta_{kk} \left(V_j - V_{0j}\right) \tag{60}$$

$$\overset{\bullet}{\overline{K}}{}^{\mathrm{d}}_{ijk} = D_{ijk} - \frac{1}{\delta_{pp}} D_{ljm} G_{lm} G^{-1}_{ik} \tag{61}$$

whereby definition (42) of the macroscopic velocity was inserted. The solution to this system of equations is

$$D_{ijk} = \overset{\bullet}{\overline{K}}_{ijk} + \frac{1}{2+\delta_{ll}} \left(V_j - V_{0j}\right) G^{-1}_{ik} \tag{62}$$

Thus, the kinematic boundary condition (59) becomes

$$\underline{\mathbf{v}} = \underline{\mathbf{V}}_0 + \underline{\xi} \cdot \underline{\nabla}_{\underline{\mathbf{X}}}\underline{\mathbf{V}} + \underline{\xi} \cdot \overset{\bullet}{\overline{\underset{\sim}{\underline{\mathbf{K}}}}} \cdot \underline{\xi} + \frac{1}{2 + \underset{\sim}{\mathbf{I}} : \underset{\sim}{\mathbf{I}}} \left(\underline{\mathbf{V}} - \underline{\mathbf{V}}_0\right) \underset{\sim}{\mathbf{G}}^{-1} : \left(\underline{\xi} \otimes \underline{\xi}\right) \qquad \text{on } \partial\Delta V(\underline{\mathbf{X}}) \,. \tag{63}$$

Inserting Eq. (63) to the Hill-Mandel lemma (43) shows that the additional constant term has to be identified as the macroscopic velocity

$$\underline{\mathbf{V}}_0 = \underline{\mathbf{V}} \,.$$

Finally, the kinematic boundary condition (59) for micromorphic media to be specified in addition to the microscopic equilibrium condition (50) becomes

$$\underline{\mathbf{v}} = \underline{\mathbf{V}} + \underline{\xi} \cdot \underline{\nabla}_{\underline{\mathbf{X}}}\underline{\mathbf{V}} + \underline{\xi} \cdot \dot{\underset{\sim}{\underline{\mathbf{K}}}} \cdot \underline{\xi} \qquad \text{on } \partial\Delta V(\underline{\mathbf{X}}) \tag{63b}$$

Note that $\overset{\bullet}{\overline{\underset{\sim}{\underline{\mathbf{K}}}}}$ in (63b) could be replaced equivalently by $\dot{\underset{\sim}{\underline{\mathbf{K}}}}$ as, due to double contraction with $\underline{\xi}$, only those part enters which exhibits the same symmetry as the hyperstress $\underset{\approx}{\underline{\mathbf{M}}}$. The boundary problem at the microscale can also be specified equivalently (in the case the microscopic stresses have a variational potential) by the Lagrangian (55) together with the kinematic boundary condition (63b). However, since the boundary condition (63b) already fulfills the kinematic constraints for $\underline{\nabla}_{\underline{\mathbf{X}}}\underline{\mathbf{V}}$ and $\dot{\underset{\sim}{\underline{\mathbf{K}}}}$, the Lagrangian (55) can be reduced to

$$\mathcal{L} = \left\langle \rho\dot{\Phi} \right\rangle_V + \tilde{\underline{\lambda}}_V : \left(\underline{\mathbf{V}} - \langle \underline{\mathbf{v}} \rangle_V\right) + \left(\bar{\underset{\sim}{\sigma}}^{\mathrm{T}} - \underset{\sim}{\mathbf{\Sigma}}^{\mathrm{T}}\right) : \left(\underset{\sim}{\mathbf{L}}_\chi - \left\langle \underline{\mathbf{v}} \otimes \underline{\xi} \right\rangle_V \cdot \underset{\sim}{\mathbf{G}}^{-1}\right) \to \text{Min.} \tag{64}$$

whereby it has to be remarked that the Lagrange multiplier $\tilde{\underline{\lambda}}_V$ needs to be distinguished from $\underline{\lambda}_V$ for the Euler-Lagrange equations of (64) to be consistent with the microscopic equilibrium conditions (50). The macroscopic stresses $\bar{\underset{\sim}{\sigma}}$, $\underset{\sim}{\mathbf{\Sigma}}$ and $\underset{\approx}{\underline{\mathbf{M}}}$ remain defined by (17), (38), and (39), respectively.

4. Second gradient theory

4.1. Macroscopic theory

The special case of a second gradient theory is obtained if the rate of microdeformation is identified ad-hoc with the macroscopic velocity gradient

$$\underset{\sim}{\mathbf{L}}_\chi := \underline{\mathbf{V}} \otimes \underline{\nabla}_{\underline{\mathbf{X}}} \tag{65}$$

so that the approximation (11) of velocity field $\underline{\mathbf{v}}(\underline{\mathbf{x}})$ becomes

$$\tilde{\underline{\mathbf{v}}} = \underline{\mathbf{V}}(\underline{\mathbf{X}}) + \left(\underline{\mathbf{V}} \otimes \underline{\nabla}_{\underline{\mathbf{X}}}\right) \cdot (\underline{\mathbf{x}} - \underline{\mathbf{X}})\,. \tag{66}$$

Typically, (66) is motivated as a Taylor-expansion [20, 28, 29].

Under the kinematic constraint (65), the extrinsic stress $\underset{\sim}{\mathbf{\Sigma}}$ drops out from the balances of total and internal energy, so that (22) becomes

$$\bar{\rho}\dot{\bar{\Phi}} = \underset{\sim}{\bar{\sigma}} : \dot{\underset{\sim}{\mathbf{E}}} + \underset{\approx}{\underline{\mathbf{M}}} \vdots \dot{\underset{\approx}{\underline{\mathbf{K}}}} - \underline{\nabla}_{\underline{\mathbf{X}}} \cdot \underline{\mathbf{Q}} \tag{67}$$

with the macroscopic rates of deformation being

$$\dot{\underset{\sim}{\mathbf{E}}} = \frac{1}{2}\left(\underline{\nabla}_{\underline{\mathbf{X}}}\underline{\mathbf{V}} + \underline{\mathbf{V}} \otimes \underline{\nabla}_{\underline{\mathbf{X}}}\right) \qquad\qquad \dot{\underset{\approx}{\underline{\mathbf{K}}}} = \underline{\nabla}_{\underline{\mathbf{X}}}\underline{\mathbf{V}} \otimes \underline{\nabla}_{\underline{\mathbf{X}}}\,. \tag{68}$$

Consequently, $\underset{\sim}{\mathbf{\Sigma}}$ can be eliminated from the balances of momenta by inserting (17) as

$$\underset{\sim}{\mathbf{\Sigma}} = \underset{\sim}{\bar{\sigma}} - \underset{\approx}{\underline{\mathbf{M}}}^{\mathrm{T}} \cdot \underline{\nabla}_{\underline{\mathbf{X}}} - \bar{\rho}\underset{\sim}{\underline{\mathbf{m}}}^{\mathrm{T}} + \underset{\sim}{\mathbf{G}}_{\rho} \cdot \left(\underline{\nabla}_{\underline{\mathbf{X}}}\dot{\underline{\mathbf{V}}}\right) \tag{69}$$

into (15) yielding

$$0 = \underline{\nabla}_{\underline{\mathbf{X}}} \cdot \left[\underset{\sim}{\bar{\sigma}} - \underset{\approx}{\underline{\mathbf{M}}}^{\mathrm{T}} \cdot \underline{\nabla}_{\underline{\mathbf{X}}} - \bar{\rho}\underset{\sim}{\underline{\mathbf{m}}}^{\mathrm{T}} + \underset{\sim}{\mathbf{G}}_{\rho} \cdot \left(\underline{\nabla}_{\underline{\mathbf{X}}}\dot{\underline{\mathbf{V}}}\right)\right] + \bar{\rho}\underline{\bar{\mathbf{f}}} - \bar{\rho}\dot{\underline{\mathbf{V}}} \tag{70}$$

whereby it was assumed for simplicity that the volume element at the microlevel exhibits a central symmetry so that $\left\langle \rho\underline{\xi}\right\rangle_V = 0$. The balance of angular momentum $\underset{\sim}{\bar{\sigma}}^{\mathrm{T}} = \underset{\sim}{\bar{\sigma}}$ (eq. (18)) remains valid.

4.2. Micro-macro transition

The Hill-Mandel lemma (43) becomes

$$\left\langle \underset{\sim}{\sigma} : \dot{\underset{\sim}{\varepsilon}}\right\rangle_V = \underset{\sim}{\bar{\sigma}} : \dot{\underset{\sim}{\mathbf{E}}} + \underset{\approx}{\underline{\mathbf{M}}} \vdots \dot{\underset{\approx}{\underline{\mathbf{K}}}}\,. \tag{71}$$

Inserting the kinematic constraint (65) to the microscopic Lagrangian (55) yields

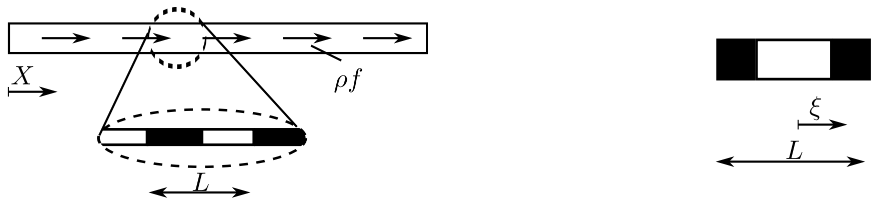
$$\begin{aligned}\mathcal{L} = &\left\langle \rho\dot{\Phi}\right\rangle_V + \underset{\sim}{\mathbf{\Sigma}} : \left(\underset{\sim}{\mathbf{G}}^{-1} \cdot \left\langle \underline{\xi} \otimes \underline{\mathbf{v}}\right\rangle_V - \left\langle \underline{\nabla}_{\underline{\mathbf{x}}}\underline{\mathbf{v}}\right\rangle_V\right) + \underline{\lambda}_V : (\underline{\mathbf{V}} - \langle \underline{\mathbf{v}}\rangle_V) + \underset{\sim}{\bar{\sigma}} : \left(\underline{\nabla}_{\underline{\mathbf{X}}}\underline{\mathbf{V}} - \left\langle \underline{\mathbf{v}} \otimes \underline{\xi}\right\rangle_V \cdot \underset{\sim}{\mathbf{G}}^{-1}\right)^{\mathrm{S}} \\ &+ \underset{\approx}{\underline{\mathbf{M}}} \vdots \left[\dot{\underset{\approx}{\underline{\mathbf{K}}}} - \frac{1}{2}\underset{\approx}{\mathbf{I}}_{\mathrm{S}} : \left(\underset{\sim}{\mathbf{G}}^{-1} \cdot \left\langle \underline{\xi} \otimes \underline{\nabla}_{\underline{\mathbf{x}}}\underline{\mathbf{v}}\right\rangle_V - \frac{1}{2 + \underset{\sim}{\mathbf{I}} : \underset{\sim}{\mathbf{I}}}\underset{\sim}{\mathbf{G}}^{-1}\left\langle \underline{\xi} \cdot \underline{\nabla}_{\underline{\mathbf{x}}}\underline{\mathbf{v}}\right\rangle_V\right) : \underset{\approx}{\mathbf{I}}_{\mathrm{T}}\right] \to \text{Min.}\end{aligned} \tag{72}$$

Thus, the equilibrium conditions at the microscale (50) and (51), or equivalently (52), of the unconstrained micromorphic theory remain valid for the second gradient theory. The only difference to an unconstrained micromorphic theory is that the kinematic micro-macro relation (53) for the microdeformation $\underset{\sim}{\mathbf{L}}_{\chi}$ is to be replaced by

$$\left\langle \underline{\mathbf{v}} \otimes \underline{\nabla}_{\underline{\mathbf{x}}}\right\rangle_V = \left\langle \underline{\mathbf{v}} \otimes \underline{\xi}\right\rangle_V \cdot \underset{\sim}{\mathbf{G}}^{-1} \tag{73}$$

Although the macroscopic extrinsic $\underset{\sim}{\mathbf{\Sigma}}$ stress does not contribute to the intrinsic work (67) in a second gradient theory, it remains the Lagrange multiplier to enforce (73) as microscopic pendant to (65).

In this context, it should be remarked that in general a higher theory of mechanics can be reduced to a special lower order theory either by kinematic constraints or by relaxation of the respective kinetic quantity [31]. The kinematically constrained approach ensures full compatibility with the corresponding higher order theory. In particular, condition (73) ensures that the definition (15) of $\underset{\sim}{\mathbf{\Sigma}}$ remains valid (and thus actually the complete procedure outlined in section 2 including a non-vanishing difference between extrinsic and intrinsic stress $\underset{\sim}{\mathbf{\Sigma}}$ and $\underset{\sim}{\bar{\sigma}}$) which can be prescribed as natural boundary conditions on the macroscale. In the kinetically relaxed approach, (73) is not enforced and thus $\underset{\sim}{\mathbf{\Sigma}}$ (and the associated jump and boundary conditions) can only be defined implicitly on the macrolevel by (70) via (69) as in [28].



(a) Macroscopic problem (b) Microscopic volume element

Figure 3: Homogenization for uniaxial two-phase laminate

5. Example: uniaxial case

The developed homogenization procedure shall be demonstrated for the one-dimensional case with periodic microstructure as shown in Figure 3. In particular, for a volume element $\Delta V = \{\xi \in [-L/2, L/2]\}$ of length L, the geometric moment is computed as $G = L^2/12$ so that equilibrium conditions (50) for the volume element become

$$\sigma'(\xi) = \frac{12}{L^2}(\Sigma - \bar{\sigma})\xi + \frac{4}{L^2}M \text{ for } \xi \in [-L/2, L/2] \tag{74}$$

For the 1D case, equilibrium condition (74) allows to determine the microscopic stress field $\sigma(\xi)$ directly up to a constant of integration. Either static boundary conditions (51)

$$\sigma(\pm L/2) = \Sigma \pm \frac{2}{L}M \tag{75}$$

or kinematic boundary conditions (63b)

$$u(\pm L/2) = \pm\frac{L}{2}\frac{\mathrm{d}U}{\mathrm{d}X} + \frac{L^2}{4}K + U \tag{76}$$

complete the boundary value problem at the microscale. In the former case, the constant of integration is computed from (75) whereas for kinematic boundary conditions (76), the constant can be determined from definition (38) $\Sigma = [\underset{\sim}{\sigma}(L/2) + \underset{\sim}{\sigma}(-L/2)]/2$. For both types of boundary conditions, the relation (39) for hyperstress M is fulfilled identically and the microscopic stress field becomes

$$\sigma(\xi) = \Sigma + \frac{3}{2}(\bar{\sigma} - \Sigma)\left[1 - \frac{\xi^2}{L^2/4}\right] + \frac{4}{L^2}M\xi \tag{77}$$

Let us now consider the macroscopic problem sketched in Figure 3a of a volume loading f under static conditions so that the macroscopic balance equations (18) and (17) read

$$0 = \frac{\mathrm{d}\Sigma}{\mathrm{d}X} + \overline{\rho f} \tag{78}$$

$$0 = \frac{\mathrm{d}M}{\mathrm{d}X} + \Sigma - \bar{\sigma} + \bar{\rho}m \tag{79}$$

In the special case $f = 0$, i. e. uniform macroscopic deformations, it is obvious that, if only trivial natural boundary conditions for the higher order terms are specified, the macroscopic stresses $M = 0$ and $\Sigma - \bar{\sigma} = 0$ fulfill the higher order balance of momentum (79) identically. What remains is (78) together with the boundary value problem (74) on the microscale $\sigma'(\xi) = 0$, $\sigma(\pm L/2) = \Sigma$. This corresponds to the classical homogenization (which is in the 1D case furthermore the exact solution of the problem with resolved microstructure everywhere) whatever the material law $\sigma(\varepsilon)$ is on the microscale and which volume element is chosen (The situation changes when dynamic effect are considered since then non-vanishing hyperstresses M and a difference $stressmacrocomp - \bar{\sigma}$ are in this case necessary in general to compensate the higher order inertia which will lead, realistically, to dispersion of waves, cf. [31]).

For non-vanishing $f \neq 0$, the higher order balance of momentum (79) is statically indeterminate so that the solution depends on the particular micromorphic constitutive law. The latter shall be derived for linear-elastic behavior $\sigma = Yu'$ of all constituents whereby Y denotes Young's modulus. By use of the stress field (77), the microscopic material law can be solved for $u'(\xi)$. Inserting it to the definitions (53) of the macroscopic deformation measures yields for (a centro-symmetric unit cell $Y(-\xi) = Y(\xi)$) the macroscopic constitutive law

$$\begin{aligned} \frac{\mathrm{d}U}{\mathrm{d}X} &= \langle u' \rangle_V = \Sigma \left\langle \frac{1}{Y} \right\rangle_V + (\bar{\sigma} - \Sigma) \left\langle \frac{1}{Y} \frac{3}{2} \left[1 - \frac{4\xi^2}{L^2} \right] \right\rangle_V , \\ \chi &= \frac{1}{G} \langle u\xi \rangle_V = \Sigma \left\langle \frac{1}{Y} \frac{3}{2} \left[1 - \frac{4\xi^2}{L^2} \right] \right\rangle_V + \frac{9}{4} (\bar{\sigma} - \Sigma) \left\langle \frac{1}{Y} \left[1 - \frac{4\xi^2}{L^2} \right]^2 \right\rangle_V , \\ K &= \frac{4}{L^2} \langle u'\xi \rangle_V = M \frac{4}{L^2} \left\langle \frac{1}{Y} \frac{4\xi^2}{L^2} . \right\rangle_V \end{aligned} \tag{80}$$

The simplest case which contains gradient effects is a constant volumetric loading $\rho f = \text{const}$. In this case, the macroscopic volume loads become $\overline{\rho f} = \rho f$ and $m = 0$ according to (18) and (17). Thus, for the constitutive law (80) a particular solution of the macroscopic boundary value problem is $\underset{\sim}{\mathbf{M}} = \text{const}$ and $\Sigma = \bar{\sigma}$ together with the statically determined and classical solution of (78). Consequently, the *microscopic stress field* (77) depends linearly on the location as in the exact solution with discretely resolved microstructure everywhere, a prediction which lies beyond the possibilities of classical homogenization. If macroscopic boundary conditions with respect to M or χ are specified which do not coincide with this particular solution, then additional exponentially decaying terms of $\underset{\sim}{\mathbf{M}}$ and $\bar{\sigma} - \Sigma$ occur at the macroscopic boundaries. According to $(80)_1$, such terms have an effect on the macroscopic displacements $U(X)$, i. e. on the macroscopic stiffness. This is reasonable as for non-homogeneous loading $\rho f = \text{const}$ the distribution of the microscopic stiffness has indeed an effect on the macroscopic stiffness. Of course, this effect becomes negligible as the ratio of macroscopic length and intrinsic length L increases.

For further discussion, the macroscopic constitutive law (80) shall be provided for homogeneous elastic properties i. e. $Y = \text{const.}$:

$$\bar{\sigma} = Y \frac{\mathrm{d}U}{\mathrm{d}X} , \qquad \Sigma - \bar{\sigma} = 5Y \left[\frac{\mathrm{d}U}{\mathrm{d}X} - \chi \right] , \qquad M = \frac{3YL^2}{4} K . \tag{81}$$

For a second gradient theory as described in section 4, the constraint $\chi = \mathrm{d}U/\mathrm{d}X$ implies that on the macrolevel, the higher order deformation measure in $(80)_3$ is identified as $K = \mathrm{d}^2U/\mathrm{d}X^2$. The corresponding constraint (73) on the microlevel is implemented by equating $(80)_1$ and $(80)_2$ which allows to eliminate the corresponding Lagrange multiplier Σ from $(80)_1$. Thus, in the case of homogeneous linear-elastic material at the microscale, the macroscopic relations (78), (79) and (81) become

$$0 = \frac{\mathrm{d}}{\mathrm{d}X} \left[\bar{\sigma} - \frac{\mathrm{d}M}{\mathrm{d}X} \right] , \qquad \bar{\sigma} = Y \frac{\mathrm{d}U}{\mathrm{d}X} , \qquad M = \frac{3YL^2}{4} \frac{\mathrm{d}^2U}{\mathrm{d}X^2} . \tag{82}$$

6. Discussion

In literature, there is a controversial debate whether a homogenization procedure for generalized continuum theory should yield vanishing higher order stresses in the case of homogeneous material at the microscale ΔV or not, compare e. g. [29, 33]. Obviously, for a homogeneous ΔV neither with the present procedure the higher order terms do not vanish in (81), (82) nor was it the case in literature [20, 32, 25]. However, to the author's opinion this behavior is reasonable and the higher order stresses *must not* vanish in this case since, as argued by Mühlich et al. [33], a *locally* homogeneous ΔV can still have an *inhomogeneous neighboring* ΔV whose interaction is still described, in some average sense, by the additional higher order momentum equation (17). This neighbor might also be a macroscopic boundary at which non-vanishing higher order tractions are allowed to be prescribed due to (17). Consequently, from a mathematical point of view, the macroscopic boundary value problem would even be ill-posed if the macroscopic constitutive relation read $\underset{\widetilde{}}{\underline{\mathbf{M}}} = 0$. Whether such boundary conditions are physically reasonable in the case of homogeneous material at the microlevel is of course another question (regarding (81), e. g. at the macroscopic boundary, the intrinsic length L could be correlated to the intrinsic length of an elastic foundation). The explicit definition of the involved generalized stress and deformation quantities derived in the present contribution provides a sound basis for addressing this question.

A similar question is whether for heterogeneous material the obtained macroscopic constitutive relations may depend on the particular choice of the unit cell ΔV. It is obvious in (80), that with the present procedure, even in the uniaxial case the non-classical terms (with $\bar{\sigma} - \Sigma$ and M) depend on location and size of the chosen unit cell. This is inevitably necessary since also potential generalized boundary conditions on χ or M require an interpretation with respect to the location of the corresponding macroscopic boundary relative to material heterogeneities, see e. g. [34]. However, it is recalled that when the non-classical stresses vanish, either through suitable boundary conditions as described above or in sufficient distance to a boundary, then the classical theory is recovered whose solution does *not* depend on the particular choice of the unit cell.

A further question, which was discussed in literature on homogenization of a Cauchy continuum on the microscale to a generalized continuum at the macroscale, was how the additional deformation measures of such theories can be prescribed via boundary conditions at the microscale. The kinematic micro-macro relations derived in the present contribution (as well as those in other papers, e. g. [28, 13, 11]) cannot be completely converted to surface integrals. That is why they are prescribed as integral constraints with corresponding Lagrange multipliers. To the author's opinion, generalized continuum theories like the micromorphic or second gradient theory investigated in the present contribution, should also work in the one-dimensional case, i. e. for rod theory (section 5). For a homogenization of a rod we can prescribe only two boundary conditions at the micro scale. However, in classical theory we have already two macroscopic kinematic quantities, namely strain and displacement. I. e., if we want to incorporate further macroscopic kinematic quantities in a generalized continuum theory, they cannot be linked to the microscale via boundary conditions only but some kind of volume averages must inevitably occur. The implementation of such volume average micro-macro links via Lagrange multipliers is straight forward, both in analytical investigations and numerical implementations. Regarding an FEM implementation, the volume average leads to a few full rows in the otherwise sparse system of equations for whose solution suitable algorithms are favorable.

In classical homogenization, it is required that the volume element ΔV should be "representative". The term was introduced by Hill [21] and associated with the conditions that the volume element "(a) is structurally entirely typical of the whole mixture on average, and (b) contains a sufficient number of inclusions for the apparent overall moduli to be effectively independent of the surface values of traction and displacement, so long as these values are 'macroscopically uniform'. That is, they fluctuate about a mean with a wavelength small compared with the dimensions of the sample, and the effects of such fluctuations become insignificant within a few wavelengths of the surface. The contribution of this surface layer to any average can be made negligible by taking the sample large enough". In classical homogenization, often *periodic boundary conditions* are employed at the microscale to mimic the behavior of an infinite and periodic arrangement of identical volume elements to fulfill condition (b). For this purpose, periodic fluctuations are allowed as deviations from the corresponding kinematic boundary conditions. Regarding micromorphic continua, condition (a) remains reasonable whereas (b) contradicts the intention of most generalized continuum theories (for this reason, the author refrains from using the term "representative" in the context of micromorphic continua). Anyhow, several attempts were made to extend the concept of periodic boundary conditions to the homogenization of generalized continua in order to have the established classical homogenization contained as a strict special case, e. g.[16, 28, 26, 6, 3]. For this purpose, Hill's "macroscopically uniform" field was taken as a polynomial field ([6]: "assigned" field, [26]: "projected" field, [3]: "inserted" field) which fulfills the respective kinematic micro-macro relations ad hoc. Besides a certain arbitrariness in such a definition of a macroscopically uniform field for micromorphic continua, the problem is that the kinematic micro-macro relations for such a generalized homogenization procedure can not completely be transformed to surface integrals. Thus, for performing the homogenization, the split into a macroscopically uniform field and a fluctuation field does not lead to a simplification since volume average kinematic micro-macro relations then still need to be fulfilled, now with respect to the fluctuation field. In the procedure proposed in [3], the macroscopically uniform ("inserted") field actually serves to define indirectly the macroscopic stress measures. In contrast, in the present contribution the macroscopic stress measures are defined directly and explicitly and corresponding work-conjugate deformation measures are derived. The boundary value problem at the microscale is completely defined either using static or kinematic boundary conditions, both including the classical and non-classical terms. Although the non-classical terms will in general prevent the total displacement field from being periodic, it should be possible to modify the proposed kinematic boundary conditions as in [28] to include fluctuations *at the boundary* of ΔV only in such a way that they reduce to classical periodic boundary conditions in the case the non-classical terms are absent.

Generalized continuum models were often used phenomenologically to capture the experimentally observed size effects of porous media, see e. g. [34, 2]. In this context it has to be remarked that with the above-mentioned technique, the respective Lagrange multipliers (corresponding to the stress measures work-conjugate to the enforced macroscopic deformations) act like volume forces at the microscale. Thus, the boundary value problem at the microscale will only have a solution if the material can carry these volume forces. This issue is related to the fact that kinematic micro-macro links in form of volume averages require a uniquely defined displacement field in the complete volume element ΔV (in classical homogenization with static boundary conditions, the same issue applies to material at the boundary). Thus, if porous media shall be homogenized to a micromorphic continuum, the volume element ΔV can encompass the matrix material only. This is plausible insofar as if the corresponding boundary value problem for the microstructure should be solved *without homogenization* (i. e. with discretely resolved microstructure in the complete domain), then the underlying fundamental equations of the Cauchy continuum (section 2.2) would as well apply only to the matrix material of such a porous medium.

Furthermore, it is to be recalled that within the present theory, the hyperstress $\underset{\sim}{\underline{\mathbf{M}}}$, and consequently its work-conjugate deformation measure $\dot{\underset{\sim}{\underline{\bar{\mathbf{K}}}}}$, are symmetric with respect to its first and last index, eqs. (39) and (54), although this symmetry is not required per se in the macroscopic theories of Eringen and Mindlin. However, it should be mentioned that the same symmetry appears in the homogenization theories of Gologanu et al. [20], Kouznetsova et al. [28], Li [29] for strain gradient media which also employ quadratic relations between microscopic tractions and hyperstress (39) or for the gradient term in the kinematic boundary conditions (63b) so that the antimetric parts do not contribute. Within the present framework for the transition to the macroscopic continuum, as outlined in section 2.1, the symmetry is a consequence of the definition of the surface operator $\langle\circ\rangle_S$. It remains open for future investigations whether alternative definitions of $\langle\circ\rangle_S$ can be found which do not induce this symmetry but which are nevertheless compatible with the classical theory of homogenization (section 3.2.1).

7. Summary and outlook

In the present contribution, a consistent theory of homogenization of a classical Cauchy continuum at the microscale towards a micromorphic continuum at the macroscale was presented. Starting point was the average field theory of Eringen whose up to then unspecified surface operator, which defines fluxes like the macroscopic stress, was defined in a consistent way with classical homogenization. Thus having explicit definitions of all macroscopic stress-type quantities of the micromorphic theory, the (strong form of the) boundary value problem of classical homogenization, i. e. equilibrium condition and static boundary conditions, was modified by additional linear terms to yield non-vanishing values of the generalized macroscopic stresses. An equivalent variational formulation was derived in a standard way (being, by the way, well-suited for FE^2 techniques) from which the kinematic micro-macro relations for the macroscopic deformation measures could be derived in explicit form as well. Kinematic boundary conditions with additional quadratic terms could be identified which satisfy *a part* of the kinematic micro-macro relations ad hoc. Those macroscopic kinematic quantities which are not uniquely determined by the boundary conditions, i. e. the microdeformation and the macroscopic displacement and, if static boundary conditions are applied at the microscale, the classical strain and the microdeformation gradient, are determined by integral constraints over the microscopic domain. Subsequently, the special case of a second gradient theory was addressed before the procedure was demonstrated for the uniaxial case. The results of this simplest possible theory of continuum mechanics are taken as illustrative starting point for discussing certain contentious issues of homogenization for generalized continua like the role of boundary conditions and integral constraints at the microscale, the behavior for microscopically homogeneous material or the handling of pores.

In general, homogenization techniques can be applied only if there is a *separation of relevant macroscopic and microscopic length scales*. For classical homogenization, it is known that these scales have to be separated by about one order of magnitude. The homogenization towards generalized continua aims in reducing the necessary level of separation between these scales. Thus, it remains open to demonstrate the developed micromorphic homogenization technique on more elaborate problems in order to quantify the improvements compared to classical theory of homogenization. The results need also to be compared to existing strain gradient homogenization techniques from literature in order to identify those techniques which are suited best for certain purposes. In addition, in the future the present micromorphic homogenization needs to be extended towards large deformations.

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A. Appendix

A.1. Sum approximation of integrals

According to [8, 9], the domain $\underline{\mathbf{x}} \in \Omega$ with its boundary $\partial\Omega$ is divided into a large but finite number of subdomains $\Delta V(\underline{\mathbf{X}})$ so that the flux term of a general global balance law (1) becomes

$$\begin{aligned}
\int_{\Omega} \underline{\nabla}_{\underline{\mathbf{x}}} \cdot \underline{\psi}_a \, \mathrm{d}V &= \oint_{\partial\Omega} \underline{\mathbf{n}} \cdot \underline{\psi}_a \, \mathrm{d}S + \int_{S^{||}} \underline{\mathbf{n}} \cdot \left\lceil \underline{\psi}_a \right\rceil \, \mathrm{d}S \\
&= \sum_K \left[\int_{\Delta A_K} \underline{\mathbf{n}} \cdot \underline{\psi}_a \, \mathrm{d}S + \int_{S^{||}\cap \Delta V_K} \underline{\mathbf{n}} \cdot \left\lceil \underline{\psi}_a \right\rceil \, \mathrm{d}S \right] \quad \text{with } \Delta A_K := \partial\Omega \cap \Delta V_K \\
&= \sum_K \underbrace{\frac{1}{\Delta A_K} \int_{\Delta A_K} \underline{\mathbf{n}} \cdot \underline{\psi}_a \, \mathrm{d}S}_{:=\underline{\mathbf{N}} \cdot \langle \underline{\psi}_a \rangle_S} \Delta A_K + \sum_K \underbrace{\frac{1}{\Delta A_K} \int_{S^{||}\cap \Delta V_K} \underline{\mathbf{n}} \cdot \left\lceil \underline{\psi}_a \right\rceil \, \mathrm{d}S}_{:=\underline{\mathbf{N}} \cdot \left\lceil \langle \underline{\psi}_a \rangle_S \right\rceil} \Delta A_K \\
&\approx \oint_{\partial\Omega_{\underline{\mathbf{X}}}} \underline{\mathbf{N}} \cdot \left\langle \underline{\psi}_a \right\rangle_S \mathrm{d}S + \oint_{S^{||}} \underline{\mathbf{N}} \cdot \left\lceil \left\langle \underline{\psi}_a \right\rangle_S \right\rceil \mathrm{d}S = \int_{\Omega_{\underline{\mathbf{X}}}} \underline{\nabla}_{\underline{\mathbf{X}}} \cdot \left\langle \underline{\psi}_a \right\rangle_S \mathrm{d}V
\end{aligned} \tag{83}$$

Therein, $S^{||}$ refers to a potential surface of discontinuity with a jump $\lceil \circ \rceil$ of the field quantities. An analogous procedure for the source terms yields

$$\int_{\Omega} \rho\varphi_m \mathrm{d}V = \sum_L \int_{\Delta V_L} \rho\varphi_m \mathrm{d}V \qquad = \sum_L \underbrace{\frac{1}{\Delta V_L} \int_{\Delta V_L} \rho\varphi_m \mathrm{d}V}_{:=\langle \rho\varphi_m \rangle_V} \Delta V_L \approx \int_{\Omega_{\underline{\mathbf{X}}}} \langle \rho\varphi_m \rangle_V \, \mathrm{d}V \tag{84}$$

The macroscopic balances (3) are thus obtained by inserting (83) and (84) into the global balance (1).

A.2. Asymptotic self-consistency of homogenization procedures

It is the nature of any homogenization that it yields only an approximate solution to the initial problem (1). The approximation character of (83) can be shown by comparing the volume average of the local balance (2)

$$\langle \rho\dot{\varphi}_m \rangle_V = \left\langle \underline{\nabla}_{\underline{\mathbf{x}}} \cdot \underline{\psi}_a \right\rangle_V + \langle \rho\psi_m \rangle_V \tag{85}$$

with its macroscopic counterpart (4). For both to be equal would require that the residual of homogenization

$$\mathcal{R}_{\Psi} = \left\langle \underline{\nabla}_{\underline{\mathbf{x}}} \cdot \underline{\psi}_a \right\rangle_V - \underline{\nabla}_{\underline{\mathbf{X}}} \cdot \left\langle \underline{\psi}_a \right\rangle_S \tag{86}$$

as difference between (4) and (85) would vanish (Further residuals may occur at the boundaries which are not considered here).

In general, any homogenization procedure should be self-consistent in the sense that it reduces to the theory "plugged in" at the microscopic level in the limit of the volume element ΔV becoming infinitesimally small. This requirement could be termed *asymptotic self-consistency*. Regarding the present approach, this requires that $\mathcal{R}_{\Psi}$ becomes exactly zero for $\Delta V \to 0$. It is found that the classical

Hill-Mandel homogenization with $\left\langle \underline{\psi}_a \right\rangle_S := \left\langle \underline{\psi}_a \right\rangle_V$ is asymptotically self-consistent (as can be verified easily with a Taylor expansion of $\underline{\psi}_a$). Using the Gauss theorem, the definition (38) of the surface operator of the present contribution can be written (as in (40)) as

$$\left\langle \underline{\psi}_a \right\rangle_S = \left\langle \underline{\psi}_a \right\rangle_V + \left\langle \underline{\xi} \underline{\nabla}_{\underline{\mathbf{x}}} \cdot \underline{\psi}_a \right\rangle_V . \tag{87}$$

A Taylor expansion shows that this definition is asymptotically self-consistent as well.