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# An extended Coleman-Noll procedure for generalized continuum theories

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Within rational continuum mechanics, the Coleman-Noll procedure is established to derive requirements to constitutive equations. Aiming in particular at generalized continuum theories, the present contribution demonstrates how this procedure can be extended to yield additionally the underlying balance equations of stress-type quantities. This is demonstrated for micromorphic and strain gradient media as well as for the microforce theory. The relation between the extended Coleman-Noll procedure and the method of virtual powers is pointed out.

**Keywords:** generalized continuum theory, thermodynamics, constitutive equations, micromorphic theory, microforce theory

## 1 Introduction

Lacking an intrinsic length scale, it is nowadays well-known that classical continuum theory cannot predict size effects. In order to be able to describe size effects which are observed e. g. in the contexts of damage and localisation or the formation of boundary layers, numerous so-called generalized continuum theories were developed. A historical review was given e. g. by Maugin [13] who recommends also [10]. The boundary value problem of classical theories of continuum mechanics is typically built from a modular framework consisting of universal balance equations (balances of mass, momentum, moment of momentum, energy and entropy), kinematic relations and constitutive equations, the latter comprising also some considerations on objectivity. However, this procedure requires definitions of local contributions to the balance quantities. In generalized continuum theories, these contributions can hardly be set up ad-hoc as done in classical continuum mechanics. That is why researchers had to look for alternative strategies of formulating the boundary value problems of generalized continuum theories.

The approach by Green and Naghdi [6] was to require the energy to be invariant with respect to superimposed rigid body motions (or a change of observer, respectively). However, this necessary requirement does not provide enough independent equations for certain generalized continuum theories [5, 12].

Today, the method of virtual power or variational formulations[1] are the established ways to derive balance equations and respective boundary conditions, see e. g. [4, 5, 7, 9, 10, 12, 14, 16]. Hellinger [10] traces the method back to Lagrange, pointing out, however, both its universality as well as its purely axiomatic nature.

Remarkably, in many studies (e. g. [12]) the balances of energy and entropy are stressed nevertheless afterwards for constructing the constitutive equations according to the procedures of Coleman-Noll [1]

[1] According to [10], the method of virtual power can be seen as generalization of stationarity conditions of variational problems.

or Müller-Liu [11] which requires the definition not only of *virtual* but additionally of *real* fluxes of energy (and of entropy). Thus, this methodology contains some redundancy.

The present paper aims in extending the Coleman-Noll procedure towards generalized continuum theories in a direct way without the necessity to define virtual powers. It will be shown that the balance equations of stress-type quantities are an outcome of the extended Coleman-Noll procedure.

The notation of the present contribution follows Forest and co-workers [see e.g. 4]. In particular scalars, vectors, second, third and forth order tensors are denoted by $a$, $\underline{\mathbf{b}}$, $\underset{\sim}{\mathbf{c}}$, $\underset{\approx}{\mathbf{d}}$ and $\underset{\approx}{\mathbf{e}}$, respectively. The operator $(\circ)^{\mathrm{T}}$ denotes the complete transposition of all indices of a tensor and $\circ^{\mathrm{S}}$ is the symmetric part of a second order tensor. Single to threefold contractions are indicated by $\cdot$, $:$, $\vdots$, respectively, and are computed from left to right, i. e. $\underset{\sim}{\mathbf{e}} : \underset{\sim}{\mathbf{f}} = e_{ij} f_{ij}$ with $c_{ij}$ referring to the Cartesian coordinates of tensor $\underset{\sim}{\mathbf{c}}$. The tensor product is written in a short notation as $\underline{\mathbf{a}}\,\underline{\mathbf{b}}$. Only if the expressions maybe mistakable the symbol $\otimes$ is inserted equivalently as $\underline{\mathbf{a}} \otimes \underline{\mathbf{b}}$. Furthermore, $\nabla$ denotes the nabla operator and $\dot{()}$ refers to the material time derivative. In particular, nabla acts to those quantity to which it is linked by a product symbol, e. g. $\underline{\nabla} \otimes \underset{\sim}{\mathbf{c}} = \partial c_{ij}/\partial x_k \underline{\mathbf{b}}_k \underline{\mathbf{b}}_i \underline{\mathbf{b}}_j$ and $\underset{\sim}{\mathbf{c}} \cdot \underline{\nabla} = \partial c_{ij}/\partial x_j \underline{\mathbf{b}}_i$ with $\underline{\mathbf{b}}_i$ being the base vectors of a Cartesian frame. Ordered sets are denoted by $\{\circ\}$ and the inner product between two of them is denoted as $\{\circ\} \,.\, \{\circ\}$.

# 2 Extended Coleman-Noll procedure

The central idea of the Coleman-Noll procedure is to require the second law of thermodynamics to be fulfilled *for all admissible thermodynamic processes*. Coleman and Noll [1] define an admissible thermodynamic processes to be compatible "with the constitutive assumptions under consideration" and "with the principles of mechanics and the law of conservation of energy". Under constitutive assumptions they understand the functional dependencies of the involved functions of state. In particular, they postulate the existence of "a caloric equation of state relating the specific internal energy to the 'strain' and the specific entropy" and that the heat flux may additionally depend on the temperature gradient. As principles of mechanics, with which an admissible process has to be compatible with, Coleman and Noll list the principle of objectivity and the law of balance of momentum. In the present work, the latter requirement is dropped and the consequences for the procedure are exploited with a special focus on generalized continuum theories.

In such theories, the "strain" as argument of the specific internal energy is generalized to a set of mechanical (or even more general: non-thermal) constitutive variables {STRAIN} [4]. This set contains *objective* combinations of members of a set of primary kinematic degrees of freedom {DOF} and first gradients thereof (excluding e. g. the displacement $\underline{\mathbf{U}}$) ensuring that the principle of material objectivity is fulfilled. Furthermore, all constitutive functions are allowed to depend on a set of intrinsic variables $\{h\}$ (i. e. quantities, whose gradients do not appear in {STRAIN}). For simplicity, we want to focus on small deformations in the following. For instance, for classical continuum theory {DOF} and {STRAIN} contain only the displacement vector $\{\mathrm{DOF}\} = \{\underline{\mathbf{U}}\}$ and the classical strain tensor $\{\mathrm{STRAIN}\} = \left\{\underset{\sim}{\varepsilon} := \frac{1}{2}\left(\underline{\nabla} \otimes \underline{\mathbf{U}} + \underline{\mathbf{U}} \otimes \underline{\nabla}\right)\right\}$. In the following, the extended Coleman-Noll procedure is performed with unspecified {DOF} and {STRAIN} before particular generalized continuum theories are considered in section 3.

The global first law of thermodynamics

$$\mathcal{P}_{\text{int}} + \dot{\mathcal{K}} - \mathcal{P}_{\text{ext}} = 0 \tag{1}$$

says that in total the sum of all powers vanish. Thereby, the changes of internal and kinetic energy $\mathcal{P}_{\text{int}}$ and $\dot{\mathcal{K}}$, respectively, consist of contributions from the domain only

$$\mathcal{P}_{\text{int}} = \int\limits_{\Omega} \rho \dot{\Phi} \mathrm{d}\Omega \,, \qquad \dot{\mathcal{K}} = \int\limits_{\Omega} \rho \dot{k} \mathrm{d}\Omega \,, \tag{2}$$

whereas external forces contribute both in the domain and at the boundary:

$$\mathcal{P}_{\text{ext}} = \int_{\Omega} \rho \left( p_{\text{m}} + r \right) \mathrm{d}\Omega + \int_{\partial\Omega} \left( \underline{\mathbf{q}}_{\text{m}} - \underline{\mathbf{q}}_{\text{th}} \right) \cdot \underline{\mathbf{n}} \mathrm{d}\Gamma \tag{3}$$

Both, domain and boundary terms, comprise thermal contributions $\rho r$ and $\underline{\mathbf{q}}_{\text{th}}$ as well as mechanical ones $\rho p_{\text{m}}$ and $\underline{\mathbf{q}}_{\text{m}}$, respectively. Furthermore, the boundary terms go linear with the normal $\underline{\mathbf{n}}$ of surface $\partial\Omega$ according to Cauchy's lemma. Eqs. (2)–(3) can be inserted into (1). Transforming the surface integral to a volume integral and requiring this equation to be valid independent of the domain and the conservation of mass leads as usual to the local form

$$\rho \dot{\Phi} + \rho \dot{k} - \rho p_{\text{m}} - \rho r - \underline{\nabla} \cdot \underline{\mathbf{q}}_{\text{m}} + \underline{\nabla} \cdot \underline{\mathbf{q}}_{\text{th}} = 0 \tag{4}$$

and to jump conditions which will not be explored here.

In classical continuum mechanics, the balance equations for the momentum and the moment of momentum are specified a priori and would now be used (in their local forms) to simplify the mechanical term $\rho \dot{k} - \rho p_{\text{m}} - \underline{\nabla} \cdot \underline{\mathbf{q}}_{\text{m}}$ to $-\underset{\sim}{\sigma} : \dot{\underset{\sim}{\varepsilon}}$ (Equation (4) with this transformation is sometimes referred to as *balance of internal energy.*). In the following we will examine the consequences for the Coleman-Noll procedure if we do not have corresponding balance equations available for generalized continuum theories (and thus impose lower restrictions to admissible processes).

We proceed with the second law of thermodynamics in its local form[2]

$$\rho \dot{\eta} + \underline{\nabla} \cdot \left( \frac{\underline{\mathbf{q}}_{\text{th}}}{\theta} \right) - \frac{\rho r}{\theta} \geq 0 \tag{5}$$

with $\eta$ being the specific entropy. Eliminating $\underline{\nabla} \cdot \underline{\mathbf{q}}_{\text{th}}$ from (4) and (5), we obtain:

$$\rho \dot{\eta} \theta - \rho \dot{\Phi} - \rho \dot{k} + \rho p_{\text{m}} + \underline{\nabla} \cdot \underline{\mathbf{q}}_{\text{m}} - \frac{\underline{\mathbf{q}}_{\text{th}} \cdot (\underline{\nabla} \otimes \theta)}{\theta} \geq 0 \tag{6}$$

As mentioned already, all constitutive functions to be specified, in particular the internal energy $\Phi$, may depend on the mechanical constitutive variables {STRAIN}, intrinsic variables $\{h\}$ and on entropy $\eta$. [3] Thus, (6) becomes

$$\rho \left( \theta - \frac{\partial \Phi}{\partial \eta} \right) \dot{\eta} - \rho \frac{\partial \Phi}{\partial \{\text{STRAIN}\}} \cdot \dot{\overline{\{\text{STRAIN}\}}} - \rho \frac{\partial \Phi}{\partial \{h\}} \cdot \dot{\overline{\{h\}}} + \rho p_{\text{m}} + \underline{\nabla} \cdot \underline{\mathbf{q}}_{\text{m}} - \rho \dot{k} - \frac{\underline{\mathbf{q}}_{\text{th}} \cdot (\underline{\nabla} \otimes \theta)}{\theta} \geq 0 \,. \tag{7}$$

This inequality has to be fulfilled for any admissible thermodynamic processes. If the mechanical terms $\rho p_{\text{m}}$, $\underline{\nabla} \cdot \underline{\mathbf{q}}_{\text{m}}$ and $\rho \dot{k}$ were not present, the necessary and sufficient conditions to fulfill (7) would thus read

$$\theta = \frac{\partial \Phi}{\partial \eta} \tag{8}$$

$$- \rho \frac{\partial \Phi}{\partial \{h\}} \cdot \dot{\overline{\{h\}}} - \frac{\underline{\mathbf{q}}_{\text{th}} \cdot (\underline{\nabla} \otimes \theta)}{\theta} \geq 0 \,. \tag{9}$$

[2] It is assumed here that the entropy production is due to heat flow $\underline{\mathbf{q}}_{\text{th}}$ and heat sources $r$ only (no chemical contributions, diffusion or the like). Eringen [2] refers to such processes as *simple thermodynamic processes.* A more general approach was given by Müller [15].

[3] A Legendre transform towards the Helmholtz free energy can be performed if desired. For (6) to be valid for all admissible processes it is then found that the free energy must not depend on the gradient of temperature but only heat flux $\underline{\mathbf{q}}_{\text{th}}$ can (and usually does) depend on $\underline{\nabla} \otimes \theta$, see e. g. [2].

and $\partial\Phi/\partial\{\text{STRAIN}\} = 0$, i. e. $\Phi$ would not be allowed to depend on {STRAIN}. Thus, if we want to construct a continuum theory where the specific internal energy $\Phi$ depends on {STRAIN}, we have to construct the terms $p_\mathrm{m}$, $\underline{\nabla}\cdot\underline{\mathbf{q}}_\mathrm{m}$ and $\rho\dot{k}$ to contain the same terms as $\dot{\{\text{STRAIN}\}}$. Recalling that {STRAIN} contains quantities from {DOF} and first gradients thereof and taking note of the spatial and temporal derivatives in $\underline{\nabla}\cdot\underline{\mathbf{q}}_\mathrm{m}$ and $\rho\dot{k}$, we thus arrive at the following requirements:

- The work powers $\underline{\mathbf{q}}_\mathrm{m}$ and $p_\mathrm{m}$ are linear forms of the rates of the kinematic degrees of freedom {DOF} (The corresponding co-factors of $\underline{\mathbf{q}}_\mathrm{m}$ are stress measures.)

- The kinetic energy $k$ is a quadratic form[4] of the rates of the kinematic degrees of freedom {DOF}

Consequently, the mechanical part of (7) is a linear form in $\dot{\{\text{DOF}\}}$ and its gradients.

At this point the Coleman-Noll procedure is extended: both the rate $\dot{\{\text{DOF}\}}$ and its gradients as well as $\eta$ can be prescribed externally and independently[5]. Under this circumstances, the necessary and sufficient conditions to fulfill the entropy inequality (7) are (9) and that the cofactors of all independent quantities, i. e. of $\dot{\{\text{DOF}\}}$ and its gradients as well as of $\dot{\eta}$, vanish. In particular this condition implies, in addition to (8), that the mechanical part of (7) vanishes:

$$-\rho\frac{\partial\Phi}{\partial\{\text{STRAIN}\}}\cdot\dot{\{\text{STRAIN}\}} + \rho p_\mathrm{m} + \underline{\nabla}\cdot\underline{\mathbf{q}}_\mathrm{m} - \rho\dot{k} = 0\,. \qquad (10)$$

The remaining dissipation inequality (9) has to be fulfilled by adequate formulations of the evolution equations for the intrinsic variables $\{h\}$ and of the heat flux $\underline{\mathbf{q}}_\mathrm{th}$ as usual[6]. In the following, the proposed procedure will be demonstrated for three established generalized continuum theories.

# 3 Examples

## 3.1 Micromorphic Theory

For a (first order) micromorphic theory [3], the so-called microdeformation tensor $\underset{\sim}{\chi}$ is added as primary kinematic variable $\{\text{DOF}\} = \{\underline{\mathbf{U}}, \underset{\sim}{\chi}\}$ and the mechanical constitutive variables $\{\text{STRAIN}\} = \{\underset{\sim}{\varepsilon}, \underset{\sim}{\mathbf{e}}, \underset{\approx}{\mathbf{K}}\}$ can thus be expanded by its gradient $\underset{\approx}{\mathbf{K}} := \underset{\sim}{\chi}\otimes\underline{\nabla}$ and the difference $\underset{\sim}{\mathbf{e}} := \underline{\mathbf{U}}\otimes\underline{\nabla} - \underset{\sim}{\chi}$ to the displacement gradient. Thus, according to the aformentioned considerations we set

$$\underline{\mathbf{q}}_\mathrm{m} = \dot{\underline{\mathbf{U}}}\cdot\underset{\sim}{\mathbf{\Sigma}} + \dot{\underset{\sim}{\chi}} : \underset{\approx}{\mathbf{M}} \qquad (11)$$

$$p_\mathrm{m} = \dot{\underline{\mathbf{U}}}\cdot\underline{\mathbf{f}} + \dot{\underset{\sim}{\chi}} : \underset{\sim}{\mathbf{m}} \qquad (12)$$

$$k = \frac{1}{2}\dot{\underline{\mathbf{U}}}\cdot\dot{\underline{\mathbf{U}}} + \frac{1}{2}\dot{\underset{\sim}{\chi}}^\mathrm{T} : \underset{\approx}{\mathbf{I}} : \dot{\underset{\sim}{\chi}} \qquad (13)$$

whereby possible coupling terms between $\dot{\underline{\mathbf{U}}}$ and $\dot{\underset{\sim}{\chi}}$ in kinetic energy $k$ are omitted here for simplicity[7]. Eq. (13) shows that only the major symmetric part of the microinertia tensor $\underset{\approx}{\mathbf{I}}$ contributes to $k$ which

[4] Actually, it is only necessary that $\dot{k}$ is a linear form in $\dot{\{\text{DOF}\}}$, i. e. also higher order polynomials would be possible. However, in analogy to the Cauchy continuum, which has $k = 1/2\dot{\underline{\mathbf{U}}}\cdot\dot{\underline{\mathbf{U}}}$, quadratic forms are favored. In general, the construction of $k$ requires insight into the physical processes [12].

[5] One might imagine a boundary at which $\dot{\{\text{DOF}\}}$ and thus its gradients along the boundary can be prescribed as essential boundary conditions. A continuum theory should work for all locations and directions of such a boundary, which is why all values of $\dot{\{\text{DOF}\}}$ and its gradient are admissible.

[6] In this context often thermodynamic driving forces $\{Y_h\} = -\frac{\partial\Phi}{\partial\{h\}}$ are defined which are then used in dissipation potentials etc.

[7] The most general case allows also for a "kinetic anisotropy" $\rho k = \frac{1}{2}\dot{\underline{\mathbf{U}}}\cdot\underset{\sim}{\rho}\cdot\dot{\underline{\mathbf{U}}} + \ldots$ with a density tensor $\underset{\sim}{\rho}$, cf. [10].

is why $\underset{\approx}{\mathbf{I}}$ can be assumed to possess a major symmetry a priori. For evaluating (7), we need to compute the divergence of the mechanical work $\underline{\mathbf{q}}_{\mathrm{m}}$. From (11) we obtain

$$\underline{\nabla}\cdot\underline{\mathbf{q}}_{\mathrm{m}} = \left(\underline{\dot{\mathbf{U}}}\otimes\underline{\nabla}\right) : \underset{\sim}{\mathbf{\Sigma}} + \underline{\dot{\mathbf{U}}}\cdot\left(\underset{\sim}{\mathbf{\Sigma}}\cdot\underline{\nabla}\right) + \left(\underset{\sim}{\dot{\chi}}\otimes\underline{\nabla}\right) \,\vdots\, \underset{\sim}{\underline{\mathbf{M}}} + \underset{\sim}{\dot{\chi}} : \left(\underset{\sim}{\underline{\mathbf{M}}}\cdot\underline{\nabla}\right) \tag{14}$$

Inserting (12)–(14) into the second law (7) using the definitions of the strain measures $\{\mathrm{STRAIN}\} = \left\{\underset{\sim}{\varepsilon}, \underset{\sim}{\mathbf{e}}, \underset{\sim}{\underline{\mathbf{K}}}\right\}$ as well as the conservation of microineria $\dot{\underset{\approx}{\mathbf{I}}} = 0$ (based on micromechanical considerations [3]) results after some sorting in

$$\begin{aligned}
&\rho\left(\theta - \frac{\partial\Phi}{\partial\eta}\right)\dot{\eta} - \left[\frac{1}{2}\left(\rho\frac{\partial\Phi}{\partial\underset{\sim}{\varepsilon}} + \left(\rho\frac{\partial\Phi}{\partial\underset{\sim}{\varepsilon}}\right)^{\mathrm{T}}\right) + \rho\frac{\partial\Phi}{\partial\underset{\sim}{\mathbf{e}}} - \underset{\sim}{\mathbf{\Sigma}}\right] : \left(\underline{\dot{\mathbf{U}}}\otimes\underline{\nabla}\right) + \left[\underset{\sim}{\mathbf{\Sigma}}\cdot\underline{\nabla} + \rho\underline{\mathbf{f}} - \rho\underline{\ddot{\mathbf{U}}}\right]\cdot\underline{\dot{\mathbf{U}}} \\
&- \left[\rho\frac{\partial\Phi}{\partial\underset{\sim}{\underline{\mathbf{K}}}} - \underset{\sim}{\underline{\mathbf{M}}}\right] \vdots \left(\underset{\sim}{\dot{\chi}}\otimes\underline{\nabla}\right) + \left[\underset{\sim}{\underline{\mathbf{M}}}\cdot\underline{\nabla} + \rho\frac{\partial\Phi}{\partial\underset{\sim}{\mathbf{e}}} + \rho\underset{\sim}{\mathbf{m}} - \rho\underset{\approx}{\mathbf{I}} : \underset{\sim}{\ddot{\chi}}\right] : \underset{\sim}{\dot{\chi}} - \rho\frac{\partial\Phi}{\partial\{h\}}.\{h\}^{\bullet} - \frac{\underline{\mathbf{q}}_{\mathrm{th}}\cdot(\underline{\nabla}\otimes\theta)}{\theta} \geq 0\,.
\end{aligned} \tag{15}$$

According to the preceding discussion, the rate and gradient of the macroscopic essential field variables $\underline{\mathbf{U}}$, $\underset{\sim}{\chi}$ and $\eta$ can be prescribed externally and independently, which is why their cofactors, i. e. the bracketed terms in (15), have to vanish. So the extracted state laws for the stress, the difference stress and the hyperstress read

$$\underset{\sim}{\sigma} = \rho\frac{\partial\Phi}{\partial\underset{\sim}{\varepsilon}}, \qquad \underset{\sim}{\underline{\mathbf{M}}} = \rho\frac{\partial\Phi}{\partial\underset{\sim}{\underline{\mathbf{K}}}}, \qquad \underset{\sim}{\mathbf{s}} = \rho\frac{\partial\Phi}{\partial\underset{\sim}{\mathbf{e}}}\,, \tag{16}$$

respectively. Furthermore, the balance equations are obtained as

$$\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^{\mathrm{T}} \tag{17}$$

$$\left(\underset{\sim}{\sigma} + \underset{\sim}{\mathbf{s}}\right)\cdot\underline{\nabla} + \rho\underline{\mathbf{f}} - \rho\underline{\ddot{\mathbf{U}}} = 0 \tag{18}$$

$$\underset{\sim}{\mathbf{s}} + \underset{\sim}{\underline{\mathbf{M}}}\cdot\underline{\nabla} + \rho\underset{\sim}{\mathbf{m}} - \rho\underset{\approx}{\mathbf{I}} : \underset{\sim}{\ddot{\chi}} = 0\,. \tag{19}$$

Here, the so-called *intrinsic* stress $\underset{\sim}{\sigma} = \underset{\sim}{\mathbf{\Sigma}} - \underset{\sim}{\mathbf{s}}$ was introduced as work-conjugate to the classical strain $\underset{\sim}{\varepsilon}$ according to conventions in literature on micromorphic theory.

## 3.2 Strain Gradient Theory

In strain gradient theory, the gradient of the classical strain tensor is added to the set $\{\mathrm{STRAIN}\} = \left\{\underset{\sim}{\varepsilon}, \underset{\sim}{\underline{\mathbf{K}}} := \underset{\sim}{\varepsilon}\otimes\underline{\nabla}\right\}$ of mechanical state variables. Correspondingly, the strain tensor $\underset{\sim}{\varepsilon}$ enters the set of primary kinematic field quantities: $\{\mathrm{DOF}\} = \{\underline{\mathbf{U}}, \underset{\sim}{\varepsilon}\}$. Thus, according to the aformentioned considerations we set

$$\underline{\mathbf{q}}_{\mathrm{m}} = \underline{\dot{\mathbf{U}}}\cdot\underset{\sim}{\mathbf{\Sigma}} + \underset{\sim}{\dot{\varepsilon}} : \underset{\sim}{\underline{\mathbf{M}}} \tag{20}$$

$$p_{\mathrm{m}} = \underline{\dot{\mathbf{U}}}\cdot\underline{\mathbf{f}} + \underset{\sim}{\dot{\varepsilon}} : \underset{\sim}{\mathbf{m}} \tag{21}$$

$$k = \frac{1}{2}\underline{\dot{\mathbf{U}}}\cdot\underline{\dot{\mathbf{U}}} + \frac{1}{2}\underset{\sim}{\dot{\varepsilon}}^{\mathrm{T}} : \underset{\approx}{\mathbf{I}} : \underset{\sim}{\dot{\varepsilon}} \tag{22}$$

Since the terms $\underset{\sim}{\mathbf{m}}$, $\underset{\sim}{\underline{\mathbf{M}}}$ and $\underset{\approx}{\mathbf{I}}$ are contracted with the strain rate tensor $\underset{\sim}{\dot{\varepsilon}}$, they inherit its symmetry. Furthermore, the microinertia tensor $\underset{\approx}{\mathbf{I}}$ possesses a major symmetry according to the considerations of the last section. Comparing (20)–(22) with (11)–(13) shows that the resulting entropy relation will be essentially (15) with $\underset{\sim}{\chi}$ replaced by the strain tensor $\underset{\sim}{\varepsilon}$. However, then the term from (15) going with $\underset{\sim}{\chi}$

is not independent from the term going with $\dot{\underline{\mathbf{U}}}\otimes\underline{\nabla}$. Thus, after sorting for the independent quantities one obtains

$$
\begin{aligned}
&\rho\left(\theta-\frac{\partial\Phi}{\partial\eta}\right)\dot{\eta}-\left[\left(\rho\frac{\partial\Phi}{\partial\underset{\sim}{\varepsilon}}\right)^{\mathrm{S}}-\left[\underset{\sim}{\underline{\mathbf{M}}}\cdot\underline{\nabla}+\rho\underset{\sim}{\mathbf{m}}-\rho\underset{\approx}{\mathbf{I}}:\ddot{\underset{\sim}{\varepsilon}}\right]^{\mathrm{S}}-\underset{\sim}{\mathbf{\Sigma}}\right]:\left(\dot{\underline{\mathbf{U}}}\otimes\underline{\nabla}\right)\\
&+\left[\underset{\sim}{\mathbf{\Sigma}}\cdot\underline{\nabla}+\rho\underline{\mathbf{f}}-\rho\ddot{\underline{\mathbf{U}}}\right]\cdot\dot{\underline{\mathbf{U}}}-\left[\rho\frac{\partial\Phi}{\partial\underset{\sim}{\underline{\mathbf{K}}}}-\underset{\sim}{\underline{\mathbf{M}}}\right]\,\vdots\,\left(\dot{\underset{\sim}{\varepsilon}}\otimes\underline{\nabla}\right)-\rho\frac{\partial\Phi}{\partial\{h\}}\cdot\{h\}^{\bullet}-\frac{\underline{\mathbf{q}}_{\mathrm{th}}\cdot(\underline{\nabla}\otimes\theta)}{\theta}\geq 0\,.
\end{aligned}
\tag{23}
$$

Consequently, the extracted state laws and balance equations read:

$$
\underset{\sim}{\sigma}=\rho\frac{\partial\Phi}{\partial\underset{\sim}{\varepsilon}},\qquad\qquad \underset{\sim}{\underline{\mathbf{M}}}=\rho\frac{\partial\Phi}{\partial\underset{\sim}{\underline{\mathbf{K}}}}\tag{24}
$$

$$
\underset{\sim}{\mathbf{\Sigma}}=\underset{\sim}{\sigma}-\left[\underset{\sim}{\underline{\mathbf{M}}}\cdot\underline{\nabla}+\rho\underset{\sim}{\mathbf{m}}-\rho\underset{\approx}{\mathbf{I}}:\ddot{\underset{\sim}{\varepsilon}}\right]\tag{25}
$$

$$
0=\underset{\sim}{\mathbf{\Sigma}}\cdot\underline{\nabla}+\rho\underline{\mathbf{f}}-\rho\ddot{\underline{\mathbf{U}}}\tag{26}
$$

$$
\underset{\sim}{\mathbf{\Sigma}}^{\mathrm{T}}=\underset{\sim}{\mathbf{\Sigma}}\,.\tag{27}
$$

### 3.3 Theory of microforces

Another prominent generalized continuum theory is the theory of microforces proposed by Gurtin [8] as generalization of the Ginzburg-Landau and Cahn-Hilliard equations. Therein, an order parameter $s$ is introduced and the internal energy is allowed to depend also on its gradient. Thus, in the present notation we have $\{\mathrm{DOF}\}=\{s\}$ and $\{\mathrm{STRAIN}\}=\{s,\underline{\nabla}\otimes s\}$. Considering only quasi-static processes $k=0$, we thus set the powers of volume and surface loads to $\underline{\mathbf{q}}_{\mathrm{m}}=\dot{s}\underline{\xi}$ and $p_{\mathrm{m}}=\dot{s}\gamma$ introducing the microstress measure $\underline{\xi}$ as well as an external load $\gamma$ due to far-field interactions. Inserting these quantities into (10), results in

$$
\rho\left(\theta-\frac{\partial\Phi}{\partial\eta}\right)\dot{\eta}+\left(\underline{\nabla}\cdot\underline{\xi}+\rho\gamma-\rho\frac{\partial\Phi}{\partial s}\right)\dot{s}+\left(\underline{\xi}-\rho\frac{\partial\Phi}{\partial\underline{\nabla}\otimes s}\right)\cdot\underline{\nabla}\otimes\dot{s}-\rho\frac{\partial\Phi}{\partial\{h\}}\cdot\{h\}^{\bullet}-\frac{\underline{\mathbf{q}}_{\mathrm{th}}\cdot(\underline{\nabla}\otimes\theta)}{\theta}\geq 0\,.\tag{28}
$$

To be valid for all admissible processes, we thus obtain Gurtin's microforce balance

$$
\underline{\nabla}\cdot\underline{\xi}+\pi+\rho\gamma=0\tag{29}
$$

with constitutive equations

$$
\pi=-\rho\frac{\partial\Phi}{\partial s},\qquad\qquad \underline{\xi}=\rho\frac{\partial\Phi}{\partial\underline{\nabla}\otimes s}\,.\tag{30}
$$

The latter were also obtained by Gurtin [8] using a Coleman-Noll procedure. However, in this work he required an ad-hoc formulation of microforce balance (29). In a later work [9], he derived (29) also by means of the method of virtual powers. Gurtin's dissipative parts of $\pi$ can be incorporates by respective choices of $\Phi$ in terms of internal variables $\{h\}$.

## 4 Concluding remarks

As mentioned already, in the classical Coleman-Noll approach (as well as the extension by Gurtin [8]), the a priori specified balance equations of stress-type quantities would have been used to simplify the mechanical term $\rho\dot{k}-\rho p_{\mathrm{m}}-\underline{\nabla}\cdot\underline{\mathbf{q}}_{\mathrm{m}}$ in (4) which is why the terms going with the rate $\{\mathrm{DOF}\}^{\bullet}$ in (7) and (10) drop out from the beginning[8]. Inversely, it is demonstrated in the present contribution that the balance equations of stress-type quantities are an outcome of an extended Coleman-Noll if they are not specified a priori.

In this context, the general outcome (10) of the procedure can also be rewritten as

$$\rho \frac{\partial \Phi}{\partial \{\text{STRAIN}\}} \cdot \overset{\bullet}{\{\text{STRAIN}\}} + \rho \dot{k} = \rho p_{\text{m}} + \underline{\nabla} \cdot \underline{\mathbf{q}}_{\text{m}} \tag{31}$$

which is a linear form in $\overset{\bullet}{\{\text{DOF}\}}$ and its gradients. If we recall that the extended Coleman-Noll procedure requires (31) to be fulfilled for any admissible sets of $\overset{\bullet}{\{\text{DOF}\}}$ then we may interpret (31) also as the principle of virtual powers in its local form if we identify all *admissible* $\overset{\bullet}{\{\text{DOF}\}}$ as *virtual* changes of $\{\text{DOF}\}$. Vice versa this means that if one starts the derivation of a generalized continuum theory with the principle of virtual powers, one assumes implicitly that the dissipation inequality takes the form (9).

With this analogy, one can adapt the argumentation of Maugin [12] on the interpretation of the balance equations derived by the extended Coleman-Noll procedure. Maugin demonstrated that e. g. the balances of linear and angular momentum are a subset of the derived balance equations if $\{\text{STRAIN}\}$ is objective. In particular, these two balance equations can be identified if a rigid body motion is chosen as admissible $\overset{\bullet}{\{\text{DOF}\}}$ (At this point the relation to the method of Green and Naghdi [6] becomes obvious). Though, performing this step for a particular continuum theory requires to know how $\overset{\bullet}{\{\text{DOF}\}}$ and its gradient transform under superimposed rigid body motions and thus a microscopic interpretation of $\{\text{DOF}\}$. However, at least for vector or tensor-valued members of $\{\text{DOF}\}$ this knowledge is also necessary to construct an objective set $\{\text{STRAIN}\}$.

Thus, it can be concluded that the extended Coleman-Noll procedure proposed in the present contribution does not yield new results. To the author's opinion, however, compared to the classical approach, it provides an elegant and straight-forward way to derive both constitutive equations and balance equations of stress-type quantities for generalized continuum theories within a single step without the axiomatic postulation of the principle of virtual power.


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[8] For the generalized continuum theories in section 3, this would the terms with $\dot{\underline{\mathbf{U}}}$ and $\dot{\underset{\sim}{\chi}}$ for the micromorphic theory in (15), those with $\dot{\underline{\mathbf{U}}}$ and $\dot{\underline{\mathbf{U}}} \otimes \underline{\nabla}$ for the strain gradient theory in (23) and the one with $\dot{s}$ for the microforce theory in (28).

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