TU Bergakademie Freiberg

Quantum Mechanics

Winter term 2014/15

I. Mathematical Basics.

1. Which axioms characterize a complex scalar product?
2. What is the connection between the norm and the scalar product?
3. Compare this with the case three-dimensional, real vectors!
4. Define the adjoint of a linear operator $\hat{A}$!
5. Define the notions of "unitary" and "self-adjoint" operators!
6. Why can a linear operator on a Hilbert space be interpreted as a matrix?
7. What do "self-adjointness" and "unitarity" mean for matrices?
8. What is an orthonormal basis?
9. Which restriction have the basis expansion coefficients of a physical state to fulfill?
10. Let ψ be a state and φ_n a base vector. What is the corresponding basis expansion coefficient!
11. How many bases are there?
12. Why do unitary operators preserve the scalar product?
13. Prove that the eigenvalues of hermitean matrices are real and that the eigenvectors corresponding to different eigenvalues are orthogonal.
14. What says the spectral theorem for self-adjoint operators?
15. Show that with $\hat{A}$ and $\hat{B}$ the operator $\hat{C} := \mathrm{i}[\hat{A}, \hat{B}]$ is self-adjoint as well.
16. Explain the Dirac formalism.
17. What is a Dirac distribution and what is its connection to the position operator?

II. Physical Foundations.

1. Explain experiments which led to the discovery of quantum mechanics!
2. Which mathematical object represents a physical state?
3. What means the normalization condition of physical states?
4. Explain the notions of "state" and "normalization" by the help of one- and two-electron systems!
5. How is a quantum mechanical observable represented?
6. What are the possible measurement results of the observable $\hat{A}$?
7. What is the connection between the normalization condition and the probabilistic interpretation of quantum mechanics?
8. Explain the predictions of quantum mechanics about the measurement of the observable $\hat{A}$ in the state ψ!
9. How is the time evolution in quantum mechanics determined?
10. What is the connection between the time evolution and the stationary Schrödinger equation?
11. What is the time dependence of a stationary state?
12. What is the time dependence of an expectation value in the stationary state?
13. Assume that the measurement of the observable $\hat{A}$ in the state ψ yields the result a. What is the state of the system immediately after the measurement?
14. Why are there two types of time evolution in quantum mechanics?
15. Which dimension has the quantum mechanical state space in general?
16. What is the meaning of the superposition principle?
17. What is an orbital?
18. What is a ground-state?
19. What is the meaning of the degeneracy of an eigenvalue?

20. When are observables compatible and what does that mean physically?
21. Define the parity operator! Determine its eigenfunctions and eigenvalues! When do states have a definite parity?

III. Examples.

1. Determine the commutator of the position and the momentum operator!
2. Show the self-adjointness of position and momentum!
3. Write down Schrödinger's equation for a free particle!
4. When does a plane wave solve the free Schrödinger equation?
5. What is the general solution of the free Schrödinger equation?
6. Determine momentum and position expectation value for a plane wave!
7. When are the solutions of the Schrödinger equation uniquely determined?
8. Write down the general form of position and momentum eigenstates!
9. When are the energy eigenstates momentum eigenstates as well?
10. What is a dispersion relation?
11. Explain the difference between phase and group velocity by the help of the free Schrödinger equation and free electromagnetic waves!
12. What is the content of the so-called de-Broglie-relations and what is their connection to the dispersion relation?
13. Assume that an electron has the state $\psi_1(\boldsymbol{x}_1)$ and assume that yet another electron has the state $\psi_2(\boldsymbol{x}_2)$. What is the corresponding two-electron state?
14. How does one find the quantum system corresponding to a given classical system?
15. Express the general solution to the time-dependent Schrödinger equation in terms of the energy eigenstates!
16. Explain the connection between the stationary Schrödinger equation and discrete spectral lines!

17. Write down the standard Hamiltonian of an electron in an external potential and prove its self-adjointness!
18. Explain the Bose-Fermi alternative and the spin-statistics theorem!
19. Explain the continuity equation!
20. What is the connection of the continuity equation to the normalization condition?
21. Discuss the hydrogen spectrum!

IV. Practical Calculations.

1. What is the stationary Schrödinger equation of the harmonic oscillator?
2. Express the position operator $\hat{x}$ and the momentum operator $\hat{p}$ by means of the ladder operators

$$a = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{m\omega}\, \hat{x} + \frac{\mathrm{i}}{\sqrt{m\omega}}\, \hat{p} \right) , \tag{0.1}$$

$$a^\dagger = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{m\omega}\, \hat{x} - \frac{\mathrm{i}}{\sqrt{m\omega}}\, \hat{p} \right) . \tag{0.2}$$

3. Express the Hamiltonian of the harmonic oscillator with force constant k by means of a and $a^\dagger$ (where $\omega^2 = k/m$).
4. Show that this Hamiltonian is self-adjoint and positive definite!
5. Show that a and $a^\dagger$ are adjoint of each other and determine the value of $[a, a^\dagger]$!
6. Let $|0\rangle$ be the ground-state of the harmonic oscillator and consider the states $|n\rangle$ of the form

$$|n\rangle = c \left(a^\dagger\right)^n |0\rangle \, , \tag{0.3}$$

where c is constant and n a positive integer.

   a) Determine c such that $|n\rangle$ is normalized!
   b) Show that $|n\rangle$ is an energy eigenstate!
   c) Determine the corresponding eigenenergy!
   d) Determine the matrix elements of energy, position and momentum in the basis of the $|n\rangle$!

7. Determine the commutators of the position and momentum operator with the standard one-electron Hamiltonian in an external potential!
8. Explain the consequences of this result for the expectation values of position and momentum!
9. Express the angular momentum operator $\hat{L} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)^{\mathrm{T}}$ in terms of the position and momentum operators!
10. Determine $[\hat{L}^2, \hat{L}_z]$ and $[\hat{L}_x, \hat{L}_y]$!
11. Assume that $\psi(t)$ solves the Schrödinger equation. Show that $\langle\psi(t)|\psi(t)\rangle$ is time independent.
12. Show that the uncertainty of an observable vanishes upon evaluation in an eigenstate!