SIMULATION OF ARTERIAL WALLS: AN ALGEBRAIC INTERFACE TO ITERATIVE SUBSTRUCTURING

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Abstract. Nonoverlapping domain decomposition methods are fast parallel solvers for large equation systems arising from the discretization of partial differential equations. The focus of this talk is the dual iterative substructuring method of the FETI-DP (Finite Element Tearing and Interconnecting Dual-Primal) type and its application to a biomechanical problem. Recently, new, inexact FETI-DP methods have been introduced to further extend the scalability of the FETI-DP method. In this talk, a parallel FETI-DP method is described and applied to the biomechanical problem of elastic deformations of arterial walls. This problem poses various challenges to solver algorithms, i.e. nonlinearity, almost incompressibility, very large material jumps, e.g. when atherosclerotic plaques with calcifications are present, and anisotropy. Our goal is the development of a method which performs robustly within this setting and still shows good scalability. The mechanical modeling is the focus of a companion talk by D. Balzani.
1 INTRODUCTION

In this paper, we will focus on our FETI-DP domain decomposition method that we use to solve the sparse systems of equations obtained from the linearization of our biomechanical problem of the elastic deformation of arterial walls. For a description of FETI-DP methods, see [1], and [2], and the references therein. The FETI-DP method was originally introduced in [3].

Generally, an arterial wall is thought of to consist of three layers, the intima, the media, and the adventitia. It is incompressible, and, as an effect of fiber reinforcement by collagen-, elastin-, and muscle fibers, the mechanical behavior of the arterial wall is anisotropic. For details of the modeling of the arterial wall and its embedded fibers, see the companion paper [4].

If dealt with correctly, the incompressibility does not influence the convergence of the FETI-DP method, see [5]. Therefore, in this paper we will focus on the influence of the anisotropy on the convergence of the iterative domain decomposition algorithm.

Therefore, we will later study the model problem of a perforated disc with embedded fibers. This model problem has the favorable property that the anisotropic behavior of the material can clearly be seen in the shape of the hole in the deformed configuration.

2 FETI DOMAIN DECOMPOSITION

We will briefly describe our implementation of the FETI-DP method that we use in conjunction with FEAP (Finite Element Analysis Program).

The input for our dual-primal FETI domain decomposition algorithm is a stream of local stiffness matrices $K^{e}$ of size $n^{e} \times n^{e}$ and local right hand sides $f^{e}$ together with a local to global d.o.f. index mapping. Using this mapping, a graph can be defined where element stiffness matrices are nodes and two such nodes are connected by an edge if they share more than $n^{e}$ global degrees of freedom. Note that the local stiffness matrices do not have to be of the same size and thus different types of elements can be mixed.

This graph is then decomposed into $N$ subgraphs using a graph partitioner. The subdomains of our domain decomposition algorithm are defined by the subgraphs. This definition of subdomains is purely algebraic. It does not make use of any geometric information and does not rely on element information other than the local to global mapping. Nevertheless, the result of this partitioning can be visualized as in Figure 1.

For each subdomain the local stiffness matrix and right hand side $K^{(i)}, f^{(i)}, i = 1, \ldots, N$ are assembled. Let us denote the global displacement increment on subdomain $\Omega^{(i)}$ by $\Delta D^{(i)}$. We then partition the displacement increment $\Delta D^{(i)}$ by using different index sets. Let us define the multiplicity of a variable as the number of subdomains it belongs to. The inner variables $\Delta D_{I}^{(i)}$ belong to one subdomain, only, and thus have a multiplicity of one. All variables that are shared by more than one subdomain are called interface variables $\Delta D_{I}^{(i)}$.

We therefore have
Note that the local stiffness matrices $K^{(i)}$ are singular in general since they may not have essential boundary conditions. In order to make our local subdomain problems invertible we assemble the local matrices in a subset of the interface variables denoted by $\Delta D^{(i)}_I$ using standard finite element assembly. These assembled variables then form a global displacement vector $\Delta D_I$. Note that if we chose $\Pi = \Gamma$ then we would recover our original global problem.

Let us denote all other interface variables by $\Delta D^{(i)}_\Delta$. Instead of using finite element assembly also in these variables, we introduce Lagrange multipliers to enforce an equality constraint where subdomain degrees of freedom coincide across subdomain boundaries.

If we define

$$K_{II} := \text{diag}(K^{(i)}_I), K_{\Delta I} := \text{diag}(K^{(i)}_{\Delta})$$

then it remains to solve the following saddle point problem,

$$
\begin{bmatrix}
K_{II} & K_{I\Delta} & K_{III} & 0 \\
K_{I\Delta} & K_{\Delta\Delta} & K_{\Delta III} & B^T_J \\
K_{III} & K_{I\Delta} & K_{III} & 0 \\
0 & 0 & B_J & 0
\end{bmatrix}
\begin{bmatrix}
\Delta D_I \\
\Delta D_\Delta \\
\Delta D_{III} \\
\lambda
\end{bmatrix}
= 
\begin{bmatrix}
f_I \\
f_\Delta \\
f_{III} \\
0
\end{bmatrix}
$$

By eliminating all displacement variables $[\Delta D_I^{(i)^T} \Delta D_\Delta^{(i)^T} \Delta D_{III}^{(i)^T}]^T$ using a direct solver we obtain an equation system

$$F \lambda = d$$

Note that this elimination process can be done in two steps. In a first step all variables $[\Delta D_I^{(i)^T} \Delta D_\Delta^{(i)^T}]^T$ are eliminated in a completely parallel step. It can be done on $N$ processors. The second step of eliminating $D_{III}^{(i)}$ then requires global communication between all subdomains.

The FETI-DP method then is defined by using a Krylov subspace method to solve $F\lambda = d$ using a suitable preconditioner.
4 MODEL PROBLEM AND NUMERICAL RESULTS

Our three dimensional computational domain is depicted in the sketch of Figure 2. It is a relatively thin perforated disc. We use our anisotropic material model, see the companion paper [3], that was developed for the simulation of our arterial wall, but relax the incompressibility condition to isolate the influence of the anisotropy. Here, all fibers are oriented in one direction and pass diagonally through Figure 2 as indicated by the direction $d$. The stiffness of the fibers grows exponentially with the stretch of the fibers. The disc is expanded by 20 percent by setting Dirichlet displacement conditions on the boundary as indicated by the arrows depicted in Figure 2. Therefore, the anisotropy becomes exponentially larger throughout the simulation.
In Figure 3 (left) the reference configuration is shown. Here, it can be seen that the structure is relatively flat. The discretization by 4927 second order tetrahedral elements results in 25,050 degrees of freedom.

![Reference configuration](image1)

![Deformed configuration](image2)

Fig. 3: Reference configuration of our domain (left). Deformed configuration of our domain (right).

The number of load steps necessary to solve this problem is quite large. We use 1000 load steps for the expansion by 20 percent. We stop our Newton iteration once the absolute residual falls below 1e-5. The FETI-DP GMRES iterations are stopped once the residual is smaller than 1e-11 or if the residual was reduced by more than 15 magnitudes.

The result of the simulation is depicted in Figure 3 (right). The influence of the fibers can clearly be seen in the shape of the hole. The average number of Newton steps for each load step is only slightly larger than 3 and does not change significantly throughout the simulation.

![Number of FETI-DP iterations](image3)

Fig. 3: The number of FETI-DP iterations grows slowly as the anisotropy becomes larger.

On the other hand, we see from Figure 3 (right) that the number of FETI-DP iterations grows, as the stiffness of the fibers and thus also the anisotropy increases exponentially. The method is still robust and does not break down. Nevertheless, we definitely see an influence of the anisotropy on the behavior of the solver. This results indicates that our method may profit
REFERENCES


