A Hierarchy of Models for Lattices of Boolean Functions

Bernd Steinbach\(^1\) and Christian Posthoff\(^2\)

\(^1\) Freiberg University of Mining and Technology, Institute of Computer Science, D-09596 Freiberg, Germany
\(^2\) The University of The West Indies, St. Augustine Campus, Trinidad & Tobago

Abstract. The utilization of lattices of Boolean functions for the synthesis of circuits combines the benefits of more freedom for optimization with limited calculations on mark functions. We extend the known hierarchy of lattices by a third level.

1 Introduction

Boolean functions specify the output values for all input combinations. Very often not all combinations are needed. Functions for which not all output values are specified are called incompletely specified functions (ISF). ISFs play a central role in the optimization of logic circuits as they represent the degrees of freedom for the assignment of a circuit structure [1].

From another point of view an ISF represents a set of Boolean functions from which an arbitrary one can be selected and realized in a circuit. An ISF with \(|f_\phi|\) don’t cares specifies a set of \(2^{|f_\phi|}\) completely specified Boolean functions. This set of functions satisfies the rules of a Boolean lattice.

2 Three Mark Functions of a First-Level Lattice

A benefit of a lattice is that an exponential number of Boolean functions can be described by two of the three mark functions \(f_q(x), f_r(x),\) and \(f_\phi(x)\). The set of all \(2^n\) input patterns \(x = (x_1, x_2, \ldots, x_n)\) of an incompletely specified function can be divided into three disjoint sets:

- \(x \in \) don’t-care-set \(\Leftrightarrow f_\phi(x_1, \ldots, x_n) = 1\)
  \(\Leftrightarrow\) it is allowed to choose the function value of \(f\) without any restrictions,
- \(x \in \) ON-set \(\Leftrightarrow f_q(x_1, \ldots, x_n) = 1\)
  \(\Leftrightarrow f(x_1, \ldots, x_n) = 1\) \& \((f_\phi(x_1, \ldots, x_n) = 0) \land (f(x_1, \ldots, x_n) = 1)\),
- \(x \in \) OFF-set \(\Leftrightarrow f_r(x_1, \ldots, x_n) = 1\)
  \(\Leftrightarrow (f_\phi(x_1, \ldots, x_n) = 0) \land (f(x_1, \ldots, x_n) = 0)\).

A function \(f(x)\) belongs to a first-level lattice \(L^1\{f_q(x), f_r(x)\}\) if it satisfies the condition:

\[
\overline{f(x)} \land f_q(x) \lor f(x) \land f_r(x) = 0 .
\] (1)
3 Second-Level Lattices of Boolean Functions

Derivative operations of the Boolean Differential calculus are very useful for the synthesis of circuits [2]. Such operations can be executed for lattices of Boolean function based on their mark functions. The results of vectorial derivative operations are lattices that cannot be expressed by (1). A more general definition for such second-level lattices $L^2(f_q(x), f_r(x), f_{id^2}(x))$ was published in [3, 4, 5]:

$$f(x) \land f_q(x) \lor f(x) \land f_r(x) \lor \bigvee_{i=1}^{k} \frac{\partial f(x)}{\partial x_0} = 0 ,$$  \hspace{1cm} (2)

where $f_{id^2}(x)$ (the independency function of the level 2) is the disjunction of certain vectorial derivatives as shown as last term in (2).

4 Third Level in the Hierarchy of Lattices

A lattice $L^2$ as described in (2) can be the result of a vectorial derivative of a lattice $L^1$. Lattices $L^2$ have the benefit that they describe only functions that are independent of certain directions of change. However, there are lattices which do not satisfy this strong requirement for the whole Boolean space, but for the region of the don’t-care-set. Lattices $L^3(f_q(x), f_r(x), f_{id^2}(x), f_{id^3}(x))$ of this new third level in the hierarchy of lattices are defined by:

$$f(x) \land f_q(x) \lor f(x) \land f_r(x) \lor \bigvee_{i=1}^{k} \frac{\partial f(x)}{\partial x_0} \lor f_{p}(x) \land \left( \bigvee_{j=1}^{l} \frac{\partial f(x)}{\partial x_0} \right) = 0 ,$$  \hspace{1cm} (3)

where $f_{id^3}(x)$ (the independency function of the level 3) is the disjunction of certain vectorial derivatives restricted to $f_{p}(x) = 1$ as shown as last term in (3).

References


This article was processed using the \LaTeX macro package with LLNCS style