Theory of hyperbolic equations

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1 Basic tools

1.1 Classification of linear partial differential equations

Let us consider the model partial differential equation

$$D_t^m u + \sum_{k+|\alpha|\le m,\, k\neq m} a_{k,\alpha}(t,x) D_x^\alpha D_t^k u = f(t,x),$$

where $D_t = -i\partial_t$ and $D_{x_k} = -i\partial_{x_k}, k = 1,\cdots,n,\ i^2 = -1$. We introduce the notions *principal part of the differential operator*, it is

$$D_t^m + \sum_{k+|\alpha|=m,\, k\neq m} a_{k,\alpha}(t,x) D_x^\alpha D_t^k,$$

and *part of terms of lower order*, it is

$$\sum_{k+|\alpha|<m} a_{k,\alpha}(t,x) D_x^\alpha D_t^k.$$

To decide which type a given partial differential equation has we study the *principal symbol* of the *principal part*, it is given by

$$\tau^m + \sum_{k+|\alpha|=m,\, k\neq m} a_{k,\alpha}(t,x) \xi^\alpha \tau^k.$$

Definition 1.1 *Let us consider the above differential operator, where the coefficients of the principal part are assumed to be real in a domain $G \subset \mathbb{R}^{n+1}$. Then the operator is called*
• **elliptic in a point** $(t_0,x_0) \in G$ *if the characteristic equation*

$$\tau^m + \sum_{k+|\alpha|=m,\, k\neq m} a_{k,\alpha}(t,x) \xi^\alpha \tau^k = 0$$

has no non-trivial real solutions $(\tau,\xi) \in \mathbb{R}^{n+1}$.
• **elliptic in a domain** G *if the operator is elliptic in every point $(t_0,x_0) \in G$.*
• **hyperbolic in a point** $(t_0,x_0) \in G$ *if the characteristic equation*

$$\tau^m + \sum_{k+|\alpha|=m,\, k\neq m} a_{k,\alpha}(t,x) \xi^\alpha \tau^k = 0$$

has only real roots $\tau_1 = \tau_1(t_0, x_0, \xi), \cdots, \tau_n = \tau_n(t_0, x_0, \xi)$.
• **strictly hyperbolic in a point** $(t_0, x_0) \in G$ *if the characteristic equation*

$$\tau^m + \sum_{k+|\alpha|=m,\, k\neq m} a_{k,\alpha}(t,x)\xi^\alpha \tau^k = 0$$

has only real and distinct roots $\tau_1 = \tau_1(t_0, x_0, \xi), \cdots, \tau_n = \tau_n(t_0, x_0, \xi)$.
• **hyperbolic (strictly hyperbolic) in a domain** G *if the operator is hyperbolic (strictly hyperbolic) in every point* $(t_0, x_0) \in G$.

Exercise 1 Find elliptic, hyperbolic and strictly hyperbolic operators. What type has the *stationary plate operator* $\triangle^2$? Is the heat operator $\partial_t - \triangle$ elliptic or hyperbolic? Find an operator with constant coefficients which is hyperbolic but not strictly hyperbolic!

Question: What type does the *non-stationary plate operator* $\partial_t^2 + \triangle^2$ have?
Answer: In the moment we are not able to give an answer because this operator does not fit into the class of operators from Definition 1.1. This operator is *not hyperbolic.*

To give a better answer let us introduce the notion *p-evolution operator*. Therefore we consider the model non-Kovalevskian partial differential equations

$$D_t^m u + \sum_{j=1}^m A_j(t,x,D_x) D_t^{m-j} u = f(t,x),$$

where $A_j = A_j(t,x,D_x) = \sum_{k=0}^{jp} A_{j,k}(t,x,D_x)$ are linear differential operators of order pj for a fixed integer $p \geq 1$, and $A_{j,k} = A_{j,k}(t,x,D_x)$ are linear differential operators of order k. The *principal part* of the differential operator *in the sense of Petrovsky* is defined by

$$D_t^m + \sum_{j=1}^m A_{j,jp}(t,x,D_x) D_t^{m-j}.$$

Definition 1.2 *The given operator*

$$D_t^m + \sum_{j=1}^m A_j(t,x,D_x) D_t^{m-j}$$

is a **p-evolution operator** *if the principal symbol*

$$\tau^m + \sum_{j=1}^m A_j(t,x,\xi)\tau^{m-j}$$

has only real and distinct roots for all (t,x) *from the domain of definition of coefficients and for all* $\xi \neq 0$.

Exercise 2 Show that the classical Schrödinger operator and the plate operator are 2-evolution operators.

Remark: 1-evolution operators are the strictly hyperbolic operators. The p-evolution operators with $p \geq 2$ represent generalizations of Schrödinger operators.

1.2 Classification of linear systems of partial differential equations

We study in this section linear systems of m first order equations in m unknowns and two independent variables of the form

$$\partial_t u_k + \sum_{j=1}^{m} \Big(a_{kj}(t,x)\partial_x u_j + b_{kj}(t,x)u_j \Big) = f_k(t,x),\ k = 1, \cdots, m.$$

In obvious matrix notation this system has the form ($U = (u_1, \cdots, u_m)^T$)

$$\partial_t U + A(t,x)\partial_x U + B(t,x)U = F(t,x).$$

Just as in the case of a single partial differential equation it turns out that most of the important properties of solutions depend on its *principal part* $\partial_t U + A(t,x)\partial_x U$. Since the principal part is completely characterized by the coefficient matrix A, this matrix plays a fundamental role in the study of these systems. There are two important classes of systems of the above form which are defined in properties of the matrix A.

Definition 1.3 *Let us consider the above differential system, where the coefficients of the matrix A are assumed to be real in a domain $G \subset \mathbb{R}^{n+1}$. Then the system is called*
• **elliptic in a point** $(t_0, x_0) \in G$ *if the matrix $A(t_0, x_0)$ has no real eigenvalues $\lambda_1 = \lambda_1(t_0, x_0), \cdots, \lambda_m = \lambda_m(t_0, x_0)$.*
• **elliptic in a domain** G *if the system is elliptic in every point $(t_0, x_0) \in G$.*
• **hyperbolic in a point** $(t_0, x_0) \in G$ *if the matrix $A(t_0, x_0)$ has real eigenvalues $\lambda_1 = \lambda_1(t_0, x_0), \cdots, \lambda_m = \lambda_m(t_0, x_0)$ and a full set of right eigenvectors.*
• **strictly hyperbolic in a point** $(t_0, x_0) \in G$ *if the matrix $A(t_0, x_0)$ has distinct real eigenvalues $\lambda_1 = \lambda_1(t_0, x_0), \cdots, \lambda_m = \lambda_m(t_0, x_0)$.*
• **hyperbolic (strictly hyperbolic) in a domain** G *if the operator is hyperbolic (strictly hyperbolic) in every point $(t_0, x_0) \in G$.*

A very useful class of hyperbolic systems is that one if the matrix A is *real and symmetric.* This class can enlarged to *general symmetric hyperbolic systems* of the form

$$A_0(t,x)\partial_t U + \sum_{k=1}^{n} A_k(t,x)\partial_{x_k} U + B(t,x)U = F(t,x),$$

where the real matrix A_0 is *positive definite* uniformly with respect to (t,x) from the domain of definition of coefficients, and where the matrices A_k are *real and symmetric.*

Question: Are we able to transform single partial differential equations to systems?

Answer: Every scalar hyperbolic equation of second order can be transformed into a symmetric hyperbolic system. Let

$$\partial_t^2 u - \sum_{j,k=1}^{n} a_{j,k}(t,x)\partial^2_{x_j x_k} u + \sum_{j=1}^{n} b_j(t,x)\partial_{x_j} u + c(t,x)\partial_t u + d(t,x)u = f(t,x),$$

where all coefficients are real-valued and $(a_{j,k}(t,x))_{j,k=1}^n$ is a *symmetric positive definite* $n \times n$ *matrix*, uniformly with respect to t and x. Let $u_1 := \partial_{x_1} u, \cdots, u_n := \partial_{x_n} u,\ u_{n+1} := \partial_t u,\ u_{n+2} := u$. Then we obtain the following system of differential equations for the $N := n+2$ functions $u_1, \cdots, u_{n+2}$:

$$\sum_{k=1}^n a_{j,k}(t,x)\partial_t u_k - \sum_{k=1}^n a_{j,k}(t,x)\partial_{x_k} u_{n+1} = 0, \quad k = 1, \cdots, n,$$
$$\partial_t u_{n+1} - \sum_{j,k=1}^n a_{j,k}(t,x)\partial_{x_k} u_j + \sum_{j=1}^n b_j(t,x)u_j + c(t,x)u_{n+1} + d(t,x)u_{n+2} = f(t,x),$$
$$\partial_t u_{n+2} - u_{n+1} = 0.$$

Setting $U = (u_1, \cdots, u_{n+2})^T$ and $F = (0, \cdots, 0, f, 0)^T$ this system is equivalent to a symmetric hyperbolic system

$$LU = A_0(t,x)\partial_t U + \sum_{k=1}^n A_k(t,x)\partial_{x_k} U + B(t,x)U = F(t,x)\ ,$$

where

$$A_0 := \begin{pmatrix} a_{1,1} & \dots & a_{1,n} & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n} & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix} \qquad B := \begin{pmatrix} 0 & \dots & 0 & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \\ b_1 & \dots & b_n & c & d \\ 0 & \dots & 0 & -1 & 0 \end{pmatrix}$$

$$A_k := \begin{pmatrix} 0 & \dots & 0 & -a_{1,k} & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & \dots & 0 & -a_{n,k} & 0 \\ -a_{1,k} & \dots & -a_{n,k} & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix} \qquad \text{for} \quad k = 1, \cdots, n\ .$$

There are other strategies to reduce Cauchy problems for scalar hyperbolic equations of higher order to Cauchy problems for systems of first order. Let us consider the Cauchy problem

$$D_t^m u + \sum_{k+|\alpha| \le m, k \neq m} a_{k,\alpha}(t,x) D_x^\alpha D_t^k u = f(t,x),$$
$$u(0,x) = u_0(x), D_t u(0,x) = u_1(x), \cdots, D_t^{m-2} u(0,x) = u_{m-2}(x),\ D_t^{m-1} u(0,x) = u_{m-1}(x).$$

If we introduce $U = (\langle D_x \rangle^{m-1} u, \langle D_x \rangle^{m-2} D_t u, \cdots, \langle D_x \rangle D_t^{m-2} u, D_t^{m-1} u)^T$, then we get

$D_tU - A(t,x,D_x)U = F(t,x),\ U(0,x) = U_0(x),$ where

$$-A(t,x,D_x) := \left(\begin{array}{ccc} 0 & -\langle D_x\rangle & \cdots \\ \cdot & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \\ \sum\limits_{|\alpha|\le m} a_{0,\alpha}(t,x)D_x^\alpha \langle D_x\rangle^{1-m} & \sum\limits_{|\alpha|\le m-1} a_{1,\alpha}(t,x)D_x^\alpha \langle D_x\rangle^{2-m} & \cdots \end{array} \right.$$

$$\left. \begin{array}{cc} 0 & 0 \\ \vdots & \vdots \\ -\langle D_x\rangle & 0 \\ 0 & -\langle D_x\rangle \\ \sum\limits_{|\alpha|\le 2} a_{m-2,\alpha}(t,x)D_x^\alpha \langle D_x\rangle^{-1} & \sum\limits_{|\alpha|\le 1} a_{m-1,\alpha}(t,x)D_x^\alpha \end{array} \right).$$

The matrix A has the so-called *Sylvester structure*.

Question: What is the difference between both systems of first order?

Answer: The second system is a *pseudo-differential system of first order*. Here the pseudo-differential operator $\langle D_x\rangle$ comes in. How is $\langle D_x\rangle$ defined? After the reduction to the Sylvester structure the *theory of pseudo-differential operators* should be applied.

1.3 Classification of geometries and formulation of problems

Besides partial differential equations or systems of partial differential equations we need additional conditions to the solution describing a technical process. First we have to distinguish between *stationary processes* or *non-stationary processes*. Moreover, we have to distinguish between different geometries of a domain in which a process takes place. Here we have *interior domains* (transversal vibration of a string which is fixed in two points, bending of a plate, heat conduction in a body), *exterior domains* (potential flow around a cylinder), *whole space* $\mathbb{R}^n$ (potential of a mass point, of a single layer or a double layer), *wave guides* (propagation of sound in an infinite tube), these are domains $\{(x,y) \in \mathbb{R}^{n+m} : (x,y) \in \mathbb{R}^n \times G,\ G \subset \mathbb{R}^m\}$, where G is an interior domain, the *exterior to wave guides* (diffraction of electromagnetic waves around an infinite cylinder), *half space* (reflection of waves on a large plane obstacle).

If we have a stationary (non-stationary) process, then the process takes place in the domain G (cylinder $G \times [0,T]$), where G is one of the above geometries. We are only interested to observe the forward (in time) process. If we observe the process for a long time, then we set $T = \infty$. In relation to the geometry we have to describe different additional conditions.

Stationary processes: We prescribe only *boundary conditions* on the boundary ∂G of the domain G. If we have an interior domain, then there exist boundary conditions of first, second or third kind, these are called boundary conditions of *Dirichlet*, *Neumann* or *Robin*-type.

Exercise 3 How many boundary conditions can we describe for solutions to $\triangle^m u = 0$ in an interior domain?

If we study boundary value problems for the elliptic equation

$$\sum_{k,j=1}^{n} a_{jk}(x)\partial_{x_k x_j} u + \sum_{k=1}^{n} b_k(x)\partial_{x_k} u + c(x)u = f(x)$$

in an interior domain, then we state instead of the Neumann boundary condition *the conormal boundary condition*. This boundary condition is related to the structure of the operator.

If we study boundary value problems in exterior domains, then decay conditions (conditions for the solution if $|x|$ tends to infinity) select among all solutions the practical one. Thus the problem

$$\triangle u = 0, \quad u\Big|_{\partial G} = g(x), \quad u(x) = O(1/|x|) \text{ for } x \to \infty,$$

in $G = \{x \in \mathbb{R}^3 : |x| > 1\}$ is of interest in the potential theory. Both conditions determine the solution in a unique way.

If we try to study this boundary value problem for solutions to the *Helmholtz equation* $\triangle u + k^2 u = 0$, $k^2 > 0$, then we see that boundary condition and decay condition do not determine the solution in a unique way.

Example: Let us study in the exterior domain $G = \{x \in \mathbb{R}^3 : |x| > 1\}$ the boundary value problem

$$\triangle u + (2\pi)^2 u = 0, \quad u\Big|_{\partial G} = 0, \quad u(x) = O(1/|x|) \text{ for } x \to \infty.$$

Show that $u(x) = -C\sin(2\pi|x|)/(4\pi|x|)$ is a family of solutions. Sommerfeld proposed the *Sommerfeld's radiation condition*

$$\begin{aligned} &\text{either} \quad r\partial_r u - i2\pi r u \to 0 \quad \text{or} \quad r\partial_r u + i2\pi r u \to 0 \quad \text{for} \quad r = |x| \to \infty \\ &\text{in general either} \quad r\partial_r u - ikru \to 0 \quad \text{or} \quad r\partial_r u + ikru \to 0 \quad \text{for} \quad r = |x| \to \infty \end{aligned}$$

to select the solution $u \equiv 0$.

Exercise 4 Study the above boundary value problem for the Helmholtz equation. Use the Laplace operator in polar-coordinates

$$\triangle u = \frac{1}{r^2}\partial_r\Big(r^2\partial_r u\Big) + \frac{1}{r^2\sin\theta}\partial_\theta\Big(\sin\theta\,\partial_\theta u\Big) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 u.$$

Interpret the *radiation condition* in the case $k = 0$.

Non-stationary processes: We prescribe *boundary conditions* on the lateral surface $\partial G \times (0,T)$ of the cylinder $G \times (0,T)$ and initial conditions on the bottom $G \times \{t=0\}$. The number of initial conditions corresponds to the order m in $D_t^m u = \cdots$ if we prescribe the initial conditions $u(0,x) = u_0(x)$, $D_t u(0,x) = u_1(x)$, $\cdots$, $D_t^{m-1}u(0,x) = u_{m-1}(x)$ on $G \times \{t=0\}$. The number of boundary conditions on the lateral surface is half of the order of the elliptic part if we consider $D_t^m u - P(t,x,D_x)u = f(t,x)$, where $P = P(t,x,D_x)$ is supposed to be an elliptic operator in $G \times (0,T)$.

Exercise 5 What kind of additional conditions can we prescribe for the non-stationary plate

$$\partial_t^2 u + \triangle^2 u = 0 \text{ in the exterior domain } \{x \in \mathbb{R}^3 : |x| > 1\}?$$

Problems which consist of partial differential equations, initial conditions and boundary conditions are called *initial boundary value problems* or *mixed problems*. Finally, we have to pay attention to the so-called *compatibility conditions* between initial conditions and boundary conditions on the boundary of the bottom $\partial G \times \{t=0\}$.

Exercise 6 Explain the compatibility conditions for the model of a vibrating string which is fixed in two points $x=0$ and $x=1$. Explain the compatibility conditions for the model of a potential inside of a rectangular $R = \{(x,y) \in \mathbb{R}^2 : (x,y) \in [0,a] \times [0,b]\}$.

In this course we are mainly interested in *initial value problems* or *Cauchy problems* in the strip $\mathbb{R}^n \times [0,T]$ or $\mathbb{R}^n \times [0,\infty)$. Let us consider the model partial differential equation

$$D_t^m u + \sum_{k+|\alpha| \le m,\, k \ne m} a_{k,\alpha}(t,x) D_x^\alpha D_t^k u = f(t,x).$$

Then the Cauchy problem means that for the solution to this equation we can prescribe m *Cauchy conditions* on the hyperplane $\{(x,t) \in \mathbb{R}^n \times [0,T] : t = 0\}$ in the following form:

$$u(x,0) = u_0(x),\ D_t u(0,x) = u_1(x), \cdots, D_t^{m-2}u(0,x) = u_{m-2}(x),\ D_t^{m-1}u(0,x) = u_{m-1}(x).$$

A typical question is stated for the solutions to Cauchy problems is the question for *well-posedness of the Cauchy problem.* Here well-posedness means *existence of a solution, uniqueness of the solution and continuous dependence of the solution on the data* u_0, u_1, u_{m-2}, u_{m-1}. But all these desired properties depend on the choice of the function spaces.

Definition 1.4 *The Cauchy problem*

$$D_t^m u + \sum_{k+|\alpha| \le m,\, k \ne m} a_{k,\alpha}(t,x) D_x^\alpha D_t^k u = f(t,x),$$

$$u(x,0) = u_0(x),\ D_t u(0,x) = u_1(x), \cdots, D_t^{m-2}u(0,x) = u_{m-2}(x),\ D_t^{m-1}u(0,x) = u_{m-1}(x)$$

is **well-posed** *if we can fix function spaces* $A_0, A_1, \cdots, A_{m-2}, A_{m-1}$ *for the data* $u_0, u_1, \cdots, u_{m-2}, u_{m-1}$*,* B *and* D *for the right-hand side* $f = f(t,x)$*,* $B_0, B_1, \cdots, B_{m-2}, B_{m-1}$

and $M_0, M_1, \cdots, M_{m-2}, M_{m-1}$ for the solution u in such a way, that to given data and right-hand side

$$u_0 \in A_0,\ u_1 \in A_1, \cdots, u_{m-2} \in A_{m-2},\ u_{m-1} \in A_{m-1}, \quad f \in M([0,T], B)$$

there exists a uniquely determined solution

$$u \in M_0([0,T], B_0) \cap M_1([0,T], B_1) \cap \cdots \cap M_{m-2}([0,T], B_{m-2}) \cap M_{m-1}([0,T], B_{m-1}).$$

Moreover, the solution depends continuously on the data and on the right-hand side, that is, if we introduce small perturbations of the data and the right-hand side in the topologies of $A_0, A_1, \cdots, A_{m-2}, A_{m-1}$ and of $M([0,T], B)$, then we get only a small perturbation of the solution in the topology of $M_0([0,T], B_0) \cap M_1([0,T], B_1) \cap \cdots \cap M_{m-2}([0,T], B_{m-2}) \cap M_{m-1}([0,T], B_{m-1})$.

Question: What are the typical function spaces to prove well-posedness?
Answer: The choice of the function spaces depends on the one hand on the coefficients of the partial differential operator and on the other hand on the structure of the partial differential operator. Let us suppose that the coefficients are smooth enough, thus they have no bad influence on the choice of the function spaces. If we take the p-evolution operators from Definition 1.2, then typical function spaces to prove well-posedness are

- in the case $p = 1$: $A_k = H^{s-k}$, $B_k = H^{s-k}$, $B = H^{s-m+1}$, $M_k = C^k$, $M = C$ or $A_k = C^{m_k}$, $B_k = C^{n_k}$, $B = C^{n_m}$, $M_k = C^{r_k}$, $M = C^{r_m}$, where $k = 0, \cdots, m-1$.
- in the case $p > 1$: $A_k = H^{s-pk}$, $B_k = H^{s-pk}$, $B = H^{s-p(m-1)}$, $M_k = C^k$, $M = C$.

Examples:

1. Thus we can study for the hyperbolic Cauchy problem

$$D_t^m u + \sum_{k+|\alpha| \le m, k \ne m} a_{k,\alpha}(t,x) D_x^\alpha D_t^k u = f(t,x),$$

$$u(0,x) = u_0(x), D_t u(0,x) = u_1(x), \cdots, D_t^{m-2} u(0,x) = u_{m-2}(x), D_t^{m-1} u(0,x) = u_{m-1}(x)$$

   the existence of solutions

$$u \in C([0,T], H^{m-1}) \cap C^1([0,T], H^{m-2}) \cap \cdots \cap C^{m-2}([0,T], H^1) \cap C^{m-1}([0,T], L^2)$$

   to given data and right-hand side

$$u_0 \in H^{m-1},\ u_1 \in H^{m-2}, \cdots, u_{m-2} \in H^1,\ u_{m-1} \in L^2, \quad f \in C([0,T], L^2).$$

2. Thus we can study for the hyperbolic Cauchy problem

$$D_t^m u + \sum_{k+|\alpha| \le m, k \ne m} a_{k,\alpha}(t,x) D_x^\alpha D_t^k u = f(t,x),$$

$$u(0,x) = u_0(x), D_t u(0,x) = u_1(x), \cdots, D_t^{m-2} u(0,x) = u_{m-2}(x), D_t^{m-1} u(0,x) = u_{m-1}(x)$$

the existence of solutions

$$u \in C^{r_0}([0,T], C^{n_0}) \cap C^{r_1}([0,T], C^{n_1}) \cap C^{r_{m-2}}([0,T], C^{n_{m-2}}) \cdots \cap \cap C^{r_{m-1}}([0,T], C^{n_{m-1}})$$

to given data and right-hand side

$$u_0 \in C^{m_0},\ u_1 \in C^{m_1}, \cdots, u_{m-2} \in C^{m_{m-2}},\ u_{m-1} \in C^{m_{m-1}}, \quad f \in C^{r_m}([0,T], C^{n_m}).$$

3. Thus we can study for the Cauchy problem to the p-evolution equation

$$D_t^m u + \sum_{j=1}^{m} \sum_{|\alpha| \le pj,} a_{j,\alpha}(t,x) D_x^\alpha D_t^{m-j} u = f(t,x),$$

$$u(0,x) = u_0(x), D_t u(0,x) = u_1(x), \cdots, D_t^{m-2} u(0,x) = u_{m-2}(x), D_t^{m-1} u(0,x) = u_{m-1}(x)$$

the existence of solutions

$$u \in C([0,T], H^{p(m-1)}) \cap C^1([0,T], H^{p(m-2)}) \cap \cdots \cap C^{m-2}([0,T], H^p) \cap C^{m-1}([0,T], L^2)$$

to given data and right-hand side

$$u_0 \in H^{p(m-1)},\ u_1 \in H^{p(m-2)}, \cdots, u_{m-2} \in H^p,\ u_{m-1} \in L^2, \quad f \in C([0,T], L^2).$$

Question: Why it is impossible to choose in the case $p > 1$ the function spaces $A_k = C^{m_k}$, $B_k = C^{n_k}$, $B = C^{n_m}$, $M_k = C^{r_k}$, $M = C^{r_m}$?

Answer: In the case $p > 1$ the solutions do not possess the property of *finite propagation speed of perturbations* (compare with Section 3.2.2.2).

Remark: Without new difficulties we can consider for p-evolution operators instead of the *forward Cauchy problem* (only $t \ge 0$ is of interest) the *backward Cauchy problem* (only $t \le 0$ is of interest). This is different to parabolic Cauchy problems which can be studied only in one time direction.

Exercise 7 In which time directions can we study the Cauchy problems

$$u_t - \triangle u = f(t,x),\ u(0,x) = u_0(x), \quad u_t + \triangle u = f(t,x),\ u(0,x) = u_0(x)?$$

1.4 Classification of solutions

Up to now we did not explain what does it mean solution for models we are interested in. Let us consider the partial differential equation (not necessary of Kovalevskian type)

$$L(t,x,D_t,D_x) := D_t^m u + \sum_{k+|\alpha| \le r,\, k<m} a_{k,\alpha}(t,x) D_x^\alpha D_t^k u = f(t,x).$$

Classical solutions are all functions satisfying this equation in the *classical sense*, that is, taking a function, forming all partial derivatives appearing in the equation and setting these into the equation gives an identity. The notion of *classical solution* is to restrictive. If some of the coefficients, or the right-hand side, or the initial and boundary data are not smooth enough we cannot expect classical solutions. Even a non-smooth boundary (with corners, cusps and so on) has an influence on regularity properties of the solution.

Definition 1.5 (Sobolev solution)
Let $G \subset \mathbb{R}^{n+1}$ be a domain and let $u \in H^{k,p}(G)$ be a given function. Then u is called **a Sobolev solution** *from $H^{k,p}(G)$ of $L(t,x,D_t,D_x)u = f(t,x)$ if for all test functions $\phi \in C_0^{\max(r,m)}(G)$ the following integral identity holds:*

$$\int_G u(t,x)L^*(t,x,D_t,D_x)\phi(t,x)d(t,x) = \int_G f(t,x)\phi(t,x)d(t,x),$$

where $L^*(t,x,D_t,D_x)$ denotes the adjoint operator to $L(t,x,D_t,D_x)$. Here we have to assume that all coefficients $a_{k,\alpha}$ belong to $C^{\max(r,m)}(G)$.

Exercise 8 What does it mean Sobolev solution of $\triangle u = f$? Determine all Sobolev solutions of $d_t u = f$, where $f(t) = 0$ if $t \leq 0$ and $f(t) = 1$ for $t > 0$.

Definition 1.6 (Distributional solution or solution in the distributional sense)
Let $G \subset \mathbb{R}^{n+1}$ be a domain and let $u \in D'(G)$ be a distribution. Then u is called **a distributional solution** *from $D'(G)$ of $L(t,x,D_t,D_x)u = f(t,x)$ if for all test functions $\phi \in C_0^\infty(G)$ the following identity holds:*

$$u(t,x)(L^*(t,x,D_t,D_x)\phi(t,x)) = f(t,x)(\phi(t,x)),$$

where $u(\phi)$ denotes the action of the distribution u on the test function ϕ. Here we have to assume that all coefficients $a_{k,\alpha}$ belong to $C^\infty(G)$.

Exercise 9 Find all distributional solutions of $d_t u = \delta_0$ and of $- \triangle u = \delta_0$, where δ_0 denotes the *Dirac's delta distribution.*

Sometimes one is interested in Sobolev solutions having additional properties. Let us consider the wave equation $u_{tt} - \triangle u = 0$. Then one can ask for solutions having an energy $E(u)(t)$, that is, for almost all $t \in (0,T)$ we have $u(t,\cdot) \in H^1$ and $u_t(t,\cdot) \in L^2$. Such solutions are called *energy solutions.*

Exercise 10 Let us assume that we have an energy solution $u \in C([0,T],H^s) \cap C^1([0,T],H^{s-1})$ with $s \geq 1$. For which s do we have a classical solution?

In generalization to this notion we can study for the hyperbolic Cauchy problem

$$D_t^m u + \sum_{k+|\alpha|\leq m, k\neq m} a_{k,\alpha}(t,x)D_x^\alpha D_t^k u = f(t,x),$$

$$u(0,x) = u_0(x), D_t u(0,x) = u_1(x), \cdots, D_t^{m-2}u(0,x) = u_{m-2}(x),\ D_t^{m-1}u(0,x) = u_{m-1}(x)$$

the existence of energy solutions $u \in C([0,T],H^{m-1}) \cap C^1([0,T],H^{m-2}) \cap \cdots \cap C^{m-2}([0,T],H^1) \cap C^{m-1}([0,T],L^2)$.

We will not discuss in detail what about boundary or initial conditions in a *weak sense*. We only mention that one has to recall to the notion of *traces*. In the case of energy solutions for hyperbolic Cauchy problems we have a regular behavior with respect to the time variable t. Thus Cauchy conditions on the hyperplane $t = 0$ are understood in the *classical sense, that the restriction* of $D_t^k u(t,x)$ on $t = 0$ exists in a given function space.

Finally, we have to remark that the different notions of solutions can be transferred in a direct way to systems of differential equations.

Definition 1.7 (Sobolev solution)
Let $G \subset \mathbb{R}^{n+1}$ be a domain and let $U \in H^{k,p}(G)$ be a given vector function. Then U is called a **Sobolev solution** *from $H^{k,p}(G)$ of*

$$A_0(t,x)\partial_t U + \sum_{k=1}^{n} A_k(t,x)\partial_{x_k} U + B(t,x)U = F(t,x),$$

if for all test vector functions $\Phi \in C_0^1(G)$ the following integral identity holds:

$$\int_G U(t,x) \cdot \Big(-\partial_t(A_0^T\Phi) - \partial_{x_k}(A_k^T\Phi) + B^T\Phi\Big)(t,x)d(t,x) = \int_G F(t,x) \cdot \Phi(t,x)d(t,x).$$

where A_0^T denotes the transposed matrix to A_0.

2 The Cauchy-Kovalevsky theorem

2.1 Classical version

The Cauchy-Kovalevsky theorem is one of the oldest results in the theory of differential equations. We explain the result for the Cauchy problem

$$\partial_t^m u + \sum_{k+|\alpha|\le m,\, k\neq m} a_{k,\alpha}(t,x)\partial_x^\alpha \partial_t^k u = f(t,x),$$

$$u(x,0) = u_0(x),\ \partial_t u(0,x) = u_1(x), \cdots, \partial_t^{m-2}u(0,x) = u_{m-2}(x),\ \partial_t^{m-1}u(0,x) = u_{m-1}(x),$$

where the coefficients $a_{k,\alpha} = a_{k,\alpha}(t,x)$, the right-hand side $f = f(t,x)$ and the data are defined in a cylinder $G \times [-T,T]$ containing as an interior point the origin in $\mathbb{R}^{n+1}$.

Theorem 2.1 *Let us assume that the coefficients $a_{k,\alpha}$ and the right-hand side f are analytic in a neighborhood U of the origin in $\mathbb{R}^{n+1}$. Moreover, we suppose that the data $u_0,\ u_1, \cdots, u_{m-2},\ u_{m-1}$ are analytic in $U \cap \{t = 0\}$. Then there exists a neighborhood $W \subset U$ of the origin and a unique analytic solution of the Cauchy problem in W.*

Remark: This theorem is independent of the type of the differential operator. Its proof bases on the assumption of analyticity which allows local representations of coefficients, right-hand side and data into power series.

It is impossible to generalize the Cauchy-Kovalevsky to non-analytic classes. If we are interested in the Cauchy problem

$$\partial_t^2 u + \partial_x^2 u = 0,\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x)$$

with data $\varphi, \psi \in C^\infty$, then the *interior regularity for elliptic equations* shows that there is in general *no solution*. A very important contribution to the Cauchy-Kovalevsky theory has been given by Lewy's example.

Lewy's Example: There exists an infinite differentiable function $F = F(t,x,y)$ such that the differential equation $\partial_x u + i\partial_y u - 2i(x+iy)\partial_t u = F(t,x,y)$ has no Sobolev solution in any neighborhood of any point $(t_0, x_0, y_0) \in \mathbb{R}^3$.

Exercise 11 Show that the Cauchy-Kovalevsky theorem does in general not hold for the Cauchy problem for non-Kovalevskian equations. For this reason study the forward Cauchy problem

$$\partial_t u = \partial_x^2 u,\ u(0,x) = \frac{1}{1-x} \text{ for } |x| < 1.$$

2.2 Abstract version

There exists an abstract version of the Cauchy-Kovalevsky theorem. To formulate this theorem we introduce a so-called scale of Banach spaces $\{B_s, \|\cdot\|_s\}$ with $s \in [0,1]$. This scale has the following properties:

- $B_s \subset B_{s'}$ for $0 \le s' \le s \le 1$,
- $\|u\|_{s'} \le \|u\|_s$ for all $u \in B_s$ and for $0 \le s' \le s \le 1$.

There exist operators which are unbounded in some function space B_s but which are bounded in the scale of Banach spaces $\{B_s\}_{s\in[0,1]}$. Let A be a linear operator acting in $\{B_s\}_{s\in[0,1]}$ in the following way:

- the operator A maps B_s into $B_{s'}$ for all $0 \le s' < s \le 1$,
- the norm of the operator $A \in L(B_s \to B_{s'})$ can be estimated by

$$\frac{a}{s-s'}, \text{ that is, } \|Au\|_{s'} \le \frac{a}{s-s'}\|u\|_s \text{ for } 0 \le s' < s \le 1.$$

Then the norm of A in $L(B_s \to B_{s'})$ is defined by

$$\|A\|_{L(B_s\to B_{s'})} = \sup_{0\le s'<s\le 1}\ \sup_{u\in\{B_s\}} (s-s')\|Au\|_{s'}\|u\|_s^{-1}.$$

Finally, we have to assume that the family $\{A(t)\}$, $t \in [0,T]$, of operators from $L(B_s \to B_{s'})$ depends continuously on t, that is, to every ε, t_0, s, s' there exists a $\delta = \delta(\varepsilon, t_0, s, s')$ such that

$$\|A(t) - A(t_0)\|_{L(B_s \to B_{s'})} \leq \varepsilon \text{ if } |t - t_0| < \delta.$$

Theorem 2.2 (abstract Cauchy Kovalevsky theorem)
Let us consider the abstract Cauchy problem $d_t u = A(t)u + f(t)$, $u(0) = u_0$, *under the following assumptions:*
- $u_0 \in B_1$,
- $f \in C([0,T], B_1)$,
- *the family* $\{A(t)\}$, $t \in [0,T]$, *of operators from* $L(B_s \to B_{s'})$ *satisfies the above described conditions.*

Then there exists a unique solution $u \in C((-T(s), T(s)), B_s)$ *for* $s \in (0,1)$ *with* $T(s) = \min(T, K(1-s))$ *and with a suitable positive constant* K.

Example: Let us consider the complex system of first order

$$\partial_t U - \sum_{k=1}^{n} A_k(t,z)\partial_{z_k} U - B(t,z)U = F(t,z),\ U(0,z) = U_0(z),$$

where $z = (z_1, z_2, \cdots, z_{n-1}, z_n)$ and the matrix functions A_k and B are holomorphic in the polycylinder $K = K_1 \times K_2 \times \cdots \times K_{n-1} \times K_n$ and continuous on its closure $\overline{K}$, where K_j denotes the unit disk with respect to the variable z_j. Finally, the matrices A_k and B and the vector function F are continuous with respect to t, that is, $A_k, B, F \in C([0,T], H(K) \cap C(\overline{K}))$. With a small positive constant a_0 let us introduce the family of polycylinders $K_s = K_{1,s} \times K_{2,s} \times \cdots \times K_{n-1,s} \times K_{n,s}$, where $K_{j,s}$ denotes the disk with radius $1 - a_0(1-s)$ for $s \in [0,1]$ with respect to the variable z_j. Now we are able to introduce $B_s := H(K_s) \cap C(\overline{K}_s)$, that is the space of functions which are holomorphic in the polycylinder K_s and continuous on the closure $\overline{K}_s$. Applying Cauchy's integral formula one can show that

$$A(t)_{L(B_s \to B_{s'})} \leq \frac{a}{s - s'}$$

with a positive constant a. Consequently, all assumptions of Theorem 2.2 are satisfied and one has a unique solution $u \in C((-T(s), T(s)), B_s)$ for $s \in (0,1)$ with $T(s) = \min(T, K(1-s))$ and with a suitable positive constant K.

We can transform the above complex system of first order to a real system of first order. Thus, in general, one can expect for type independent real systems of first order with analytic coefficients only solutions possessing a *conical evolution*. Here *conical evolution* means that as larger the domain of existence K_s is as smaller the life span $(-T(s), T(s))$ is.

Exercise 12 Show the scale type estimate in $\{B_s\}_{s\in[0,1]}$

$$\|Au\|_{s'} \leq \frac{a}{s-s'}\|u\|_s \text{ for } A(t) := \sum_{k=1}^{n} A_k(t,z)\partial_{z_k}.$$

Question: What is the difference between Theorems 2.1 and 2.2?
Answer: In Theorem 2.1 we suppose analyticity of coefficients in t, in Theorem 2.2 we suppose only continuity of coefficients in t. What property does the solution from Theorem 2.2 have with respect to t?

2.3 Holmgren's uniqueness theorem

Holmgren was interested in the Cauchy-Kovalevsky theorem. He asked if the Cauchy problem from Theorem 2.1 *has non-analytic solutions.*

Theorem 2.3 *In the set of m times differentiable solutions the analytic one from Theorem 2.1 is the only one.*

The Holmgren theorem is not true for non-Kovalevskian equations. Let us consider the Cauchy problem

$$\partial_t u - \partial_x^2 u = 0,\ u(0,x) = 0.$$

Then there exists the solution $u \equiv 0$. Let us try to construct a second solution. For this reason we choose the function $\psi(t) = \exp(-t^{-2})$ for $t > 0$ and $\psi(t) = 0$ for $t \leq 0$. Then $u(t,x) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} \psi^{(k)}(t)$ is a second solution which is infinitely times differentiable.

Exercise 13 Study the properties of the solution u!

The statement of the Theorem of Holmgren can be used to conclude the following *uniqueness result.*

Corollary 2.1 *Let us consider the Cauchy problem*

$$\partial_t^m u + \sum_{k+|\alpha|\leq m,\, k\neq m} a_{k,\alpha}(t,x)\partial_x^\alpha \partial_t^k u = f(t,x),$$

$$u(x,0) = u_0(x),\ \partial_t u(0,x) = u_1(x), \cdots, \partial_t^{m-2}u(0,x) = u_{m-2}(x),\ \partial_t^{m-1}u(0,x) = u_{m-1}(x),$$

where the coefficients $a_{k,\alpha} = a_{k,\alpha}(t,x)$ are analytic in a neighborhood of the origin in $\mathbb{R}^{n+1}$. Then there exists at most a unique m times differentiable solution in a neighborhood of the origin even if the data and right-hand side are supposed to be non-analytic.

2.4 Do we get any benefit from additional hyperbolicity?

In [7] the authors studied the Cauchy problem

$$u_{tt} - \sum_{k,l=1}^{n} \partial_{x_k}(a_{kl}(t,x)\partial_{x_l}u) = 0 \quad \text{on } (0,T)\times G\,, \tag{2.1}$$

$$u(0,x) = \varphi(x)\ ,\ u_t(0,x) = \psi(x) \quad \text{on } G\ , \tag{2.2}$$

where G is an arbitrary open set of $\mathbb{R}^n$ and $T > 0$. The coefficients a_{kl} of the elliptic operator in self-adjoint form $\sum_{k,l=1}^{n} \partial_{x_k}(a_{kl}(t,x)\partial_{x_l}\cdot)$ satisfy the next analyticity assumption:

For any compact set K of G and for any multi-index β there exist a constant A_K and a function $\Lambda_K = \Lambda_K(t)$ belonging to $L^1(0,T)$ such that

$$\Big|\sum_{k,l=1}^{n} \partial_x^\beta a_{kl}(t,x)\Big| \le \Lambda_K(t) A_K^{|\beta|}|\beta|!\ .$$

Moreover, the strict hyperbolicity condition

$$\lambda_0|\xi|^2 \le \sum_{k,l=1}^{n} a_{kl}(t,x)\xi_k\xi_l \le \Lambda(t)|\xi|^2$$

is satisfied with $\lambda_0 > 0$ and $\sqrt{\Lambda(t)} \in L^1(0,T)$.

Under these assumptions the following result was proved:

Theorem 2.4 *If the data φ and ψ are real analytic on G, then there exists a unique solution $u = u(t,x)$ on the conoid $\Gamma_G^T \subset \mathbb{R}^{n+1}$. The conoid is defined by*

$$\Gamma_G^T = \Big\{(t,x):\ \text{dist}\ (x, \mathbb{R}^n\setminus G) > \int_0^T \sqrt{\Lambda(s)}ds\ ,\ t\in[0,T]\Big\}\ .$$

The solution is C^1 in t and real analytic in x.

Proof: The proof uses the self-adjoint form of the elliptic operator, the abstract Cauchy-Kovalevsky theorem, scale type estimates, and the Fourier-Laplace transformation after developing the coefficients into power series. □

Exercise 14 What is the difference between the statement of Theorem 2.4 and the statement we obtain after the application of the abstract Cauchy-Kovalevsky theorem to the above Cauchy problem? Which kind of benefit do we have from the hyperbolicity assumption?

Exercise 15 Why can we not expect solutions in the cylinder $G \times (0, T)$?

We know from the results of [5] for the Cauchy problem

$$u_{tt} - a(t)u_{xx} = 0\ ,\ \ u(0,x) = \varphi(x)\ ,\ \ u_t(0,x) = \psi(x)\,,$$

that assumptions as $a \in L^p(0,T),\ p > 1$, or even $a \in C[0,T]$, don't allow to weaken the analyticity assumption for the data φ, ψ to get well-posedness results.

Theorem 2.5 *For any class $\mathcal{E}\{M_h\}$ of infinitely differentiable functions which strictly contains the space $\mathcal{A}$ of real analytic functions on $\mathbb{R}$ there exists a coefficient $a = a(t) \in C[0,T],\ a(t) \geq \lambda > 0$, such that the above Cauchy problem is not well-posed in $\mathcal{E}\{M_h\}$.*

By $\mathcal{E}\{M_h\}$ we denote the following function space: $\mathcal{E}\{M_h\} = \{\varphi \in C^\infty(\mathbb{R})$: for any compact set $K \subset\subset \mathbb{R}$, there exist C_K and Λ_K such that $|d_x^h\varphi(x)| \leq C_K \Lambda_K^h M_h$ for all $x \in K,\ h \geq 0\}$. The constants M_h satisfy $M_h^2 \leq M_{h-1}M_{h+1}(h \geq 1), M_{h+1} \leq A^h M_h,\ \sqrt[h]{\frac{M_h}{h!}} \leq B\sqrt[k]{\frac{M_k}{k!}}$ for $h \leq k$ and for some constants A and B.

Example: Let us take $M_h = h!^s$ with $s \geq 1$. Then all the above properties are satisfied. The spaces of infinitely differentiable functions with $M_h = h!^s$ are called *Gevrey spaces*. They are denoted by Γ^s. If $s = 1$, then we have the space of real analytic functions, that is, $\mathcal{A} = \Gamma^1$. But if $s > 1$, then we obtain spaces containing functions with compact support. Thus Γ^s strictly contains $\mathcal{A}$ if $s > 1$. The scale of Gevrey spaces $\{\Gamma^s\}_{s\in(1,\infty)}$ describes functions lying between $\mathcal{A}$ and C^∞. There exists a reach literature about the study of partial differential equations in Gevrey spaces.

3 Cauchy problem for hyperbolic equations

We begin our considerations with the famous Lax-Mizohata theorem. Therefore let us consider the Cauchy problem for the evolution equation of Kovalevskian type

$$D_t^m u + \sum_{k+|\alpha|\leq m,\, k\neq m} a_{k,\alpha}(t,x) D_x^\alpha D_t^k u = f(t,x).$$

We assume that the coefficients, the right-hand side and the data are C^∞ in a neighborhood of the origin. Then Lax (1957) proved for special cases (the characteristic roots of the principal symbol are simple) and later Mizohata (1961) for the general case the following theorem.

Lax-Mizohata theorem *If the Cauchy problem for the above evolution equation of Kovalevskian type is near he origin locally uniquely solvable in C^∞, then for all $\xi \in \mathbb{R}^n$ all the characteristic roots of the principal symbol at the origin, $\lambda_l(0,0,\xi)$, $l = 1, \cdots, m$, should be real.*

In this way we are able to understand Definition 1.1 because the C^∞ well-posedness of the Cauchy problem is somewhat we expect for hyperbolic equations.

Question: Is the converse statement true?

Answer: **No!** Here we cite the following result from [19]:

The Cauchy problem

$$u_{tt} - t^{2l} u_{xx} + a t^k u_x = 0,\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x),$$

is C^∞ well-posed if and only if $k \geq l-1$. If $k < l-1$, then one has to study the well-posedness in Gevrey spaces (see the example at the end of Chapter 2).

Thus one essential question is that for the C^∞ well-posedness of hyperbolic Cauchy problems. If we have the property of *finite propagation speed* of perturbations, then we can reduce the question for C^∞ well-posedness to that for H^∞ well-posedness of Cauchy problems in $\mathbb{R}^n \times [0,T]$, or for H^s well-posedness with an eventually *loss of derivatives* (see among other things Section 8.1).

Another important question is that for the *propagation of singularities*. Let us suppose that data $\varphi,\ \psi \in H^\infty$ besides a neighborhood of a point $x_0 \in \mathbb{R}^n$. How does this "singular behavior" propagate from $t = 0$ into the set $\{(x,t) : t > 0\}$? How do we describe such a singular behavior? Some comments about this topic are given in Section 3.2.7.

Finally, an important topic of modern research is that one for *energy estimates* or for $L^p - L^q$ *decay estimates or Strichartz type estimates* for solutions of wave equations with variable coefficients (see also Section 8.2). The meaning of such decay estimates is explained in Chapters 4 and 7.

3.1 Wave equations with terms of lower order

In opposite to hyperbolic equations of higher order where a classification of terms of lower order is not clear we can characterize terms of lower order in wave equations. Let us begin with the classical wave equation

$$u_{tt} - \triangle u = 0.$$

This equation describes the propagation of waves. It appears in numerous models as for the vibrating string or membrane, the propagation of sound, the longitudinal vibrations of an elastic rod or beam, surface water waves, the propagation of electric signals or for the description of electric or magnetic fields.

Klein (1927) and Gordon (1926) derived the following Klein-Gordon equation describing a charged particle in an electro-magnetic field:

$$u_{tt} - \triangle u + m^2 u = 0.$$

The term m^2u is called *mass or potential.*

A well-known equation is the *telegraph equation*

$$u_{tt} - u_{xx} + au_t + bu = 0,$$

where a and b are constants. This equation arises in the study of propagation of electric signals in a cable of transmission line, in the propagation of pressure waves in the study of pulsatile blood flow in arteries and in one-dimensional random motion of bugs along a hedge. Here bu is a *mass term*, and au_t is a *damping term* or *dissipation term.* A higher-dimensional generalization is

$$u_{tt} - \triangle u + au_t + bu = 0.$$

Finally, we mention wave equations with a *convection term* or a *transport term* $\sum_{k=1}^{n} a_k(t,x)\partial_{x_k}u$, that is,

$$u_{tt} - \triangle u + \sum_{k=1}^{n} a_k(t,x)\partial_{x_k}u = 0.$$

3.2 Classical wave equations

3.2.1 D'Alembert's representation in R^1

We devote to the Cauchy problem

$$u_{tt} - u_{xx} = 0\ ,\ \ u(0,x) = \varphi(x)\ ,\ \ u_t(0,x) = \psi(x).$$

A transformation of co-ordinates $\xi = x - t\,,\ \eta = x + t$ (motivated by the *notion of characteristics*) leads to $-4u_{\xi\eta} = 0$. The last partial differential equation has the general solution $u = u(\xi,\eta) = u_1(\xi)+u_2(\eta)$ with arbitrary functions u_1 and u_2. The backward transformation gives $u = u(t,x) = u_1(x-t) + u_2(x+t)$. The general solution u is a linear superposition of two waves, the wave $u_1(x-t)$ is a perturbation moving with the velocity 1 to the right-hand side. The wave $u_2(x+t)$ is moving with velocity 1 to the left-hand side. Both solutions (let us suppose for the moment that u_1 and u_2 are twice differentiable in the classical sense) are called *travelling wave solutions.* Using both Cauchy conditions we obtain

$$u(0,x) = \varphi(x) = u_1(x) + u_2(x)\ ,\quad u_t(0,x) = \psi(x) = -u_1'(x) + u_2'(x).$$

Integration of the second equation yields $-u_1(x) + u_2(x) = \int_{x_0}^{x} \psi(r)dr$, x_0 is an arbitrary constant. Hence,

$$u_1(x) = \frac{1}{2}\varphi(x) + \frac{1}{2}\int_{x_0}^{x} \psi(r)dr\ ,\ u_2(x) = \frac{1}{2}\varphi(x) - \frac{1}{2}\int_{x_0}^{x} \psi(r)dr\ .$$

Summarizing we derived the so-called *d'Alembert's representation of solution*

$$u(t,x) = \frac{1}{2}\Big(\varphi(x-t) + \varphi(x+t)\Big) + \frac{1}{2}\int\limits_{x-t}^{x+t} \psi(r)dr \ .$$

3.2.2 What kind of properties do we conclude from d'Alembert's representation formula?

3.2.2.1 Regularity of solutions

Let us consider the Cauchy problem

$u_{tt} - u_{xx} = 0$, $u(0,x) = \varphi(x)$, $u_t(0,x) = \psi(x)$ with data $\varphi \in C^k(\mathbb{R}^1)$ and $\psi \in C^{k-1}(\mathbb{R}^1)$.

Theorem 3.1 *The Cauchy problem possesses one and only one solution $u \in C^k([0,\infty) \times \mathbb{R}^1)$. The solution depends continuously on the data, that is, if we change φ and ψ a bit with respect to the topologies of $C^k(\mathbb{R}^1)$ and $C^{k-1}(\mathbb{R}^1)$, then the solution u changes a bit with respect to the topology of $C^k([0,\infty) \times \mathbb{R}^1)$.*

Proof: The existence of a solution is given by d'Alembert's representation formula. The uniqueness follows from the fact that the general solution of $u_{tt} - u_{xx} = 0$ is given by the formula $u(t,x) = u_1(x-t) + u_2(x+t)$. The continuous dependence of the solution from the data is concluded from the representation formula. □

Exercise 16 Explain the statement about the continuous dependence of the solution from the data by formulas!

Exercise 17 Let us consider the Cauchy problem with data $\varphi = \psi = 0$ outside of the interval $[-l,l]$. Show that to each $x_0 \in \mathbb{R}^1$ there exist constants $T(x_0)$ and U with $u(x_0,t) = U$ for $t \geq T(x_0)$. Determine these constants.

3.2.2.2 Qualitative properties of solutions

From d'Alembert's representation formula we conclude remarkable properties for the solutions of wave equations which are characteristic for solutions of *hyperbolic partial differential equations*. The wave equation is one representant of this class.

Finite speed of propagation of perturbations

Let us devote to the Cauchy problem with data $\varphi \in C^2(\mathbb{R}^1)$ and $\psi \in C^1(\mathbb{R}^1)$. We perturb these data by the aid of data $\varphi_s \in C^2(\mathbb{R}^1)$ and $\psi_s \in C^1(\mathbb{R}^1)$ supported on the interval $[a,b]$. We are interested in the propagation of these perturbations. For this reason we study the Cauchy problem

$$u_{tt} - u_{xx} = 0\,, \ \ u(0,x) = \varphi_s(x)\,, \ \ u_t(0,x) = \psi_s(x)$$

with $\varphi_s = \psi_s = 0$ outside of $[a, b]$. As the solution we get

$$u_s(t,x) = \frac{1}{2}\Big(\varphi_s(x-t) + \varphi_s(x+t)\Big) + \frac{1}{2}\int\limits_{x-t}^{x+t} \psi_s(r)dr \ .$$

Question: When do we feel the perturbations in a point $x_0 \in \mathbb{R}^1$ lying outside of $[a, b]$? For small times t we have $u(t, x_0) = 0$ in x_0.

Answer: We feel the perturbations after finite time $T = \text{dist}\ (x_0, [a, b])$. This property is called *finite propagation speed of perturbations or existence of a forward wave front.*

Domain of dependence

Question: Which information about the data has an influence on the solution in a given point (t_0, x_0)?

Answer: To determine the solution $u(t_0, x_0)$ in the point (t_0, x_0) we need the datum φ in the points $x_0 - t_0$ and $x_0 + t_0$ and the datum ψ on the interval $[x_0 - t_0, x_0 + t_0]$. The interval $[x_0 - t_0, x_0 + t_0]$ is called *domain of dependence* for the solution u in the point (t_0, x_0).

Huygens' principle

The *Huygens' principle* describes the existence of a *backward wave front*, that is, the property, that in a point $x_0 \in \mathbb{R}^1$ the solution vanishes after the time $T(x_0)$ if we are interested in the propagation of perturbations located in an interval $[a, b]$. In general we cannot expect the existence of a *backward wave front* having in mind that the *domain of dependence* for the solution u in the point (t_0, x_0) is the interval $[x_0 - t_0, x_0 + t_0]$. If we choose $\psi \equiv 0$, then the solution u in (t_0, x_0) is determined by the values of φ in $(x_0 - t_0)$ and $(x_0 + t_0)$. Consequently, after the time $T = \max(x_0 - a, b - x_0)$ we have $u \equiv 0$ in x_0. Summarizing the *Huygens' principle* holds under the assumption $\psi \equiv 0$.

3.2.2.3 Wave models with sources or sinks

Let us consider the wave model

$$u_{tt} - u_{xx} = F(t,x)\ ,\ \ u(0,x) = \varphi(x)\ ,\ \ u_t(0,x) = \psi(x).$$

We suppose that the source F is integrable, let us say, $F \in L^1_{\text{loc}}([0,\infty) \times \mathbb{R}^1))$. Thus we are interested in non-classical solutions. For the solution u we choose the ansatz $u = v + w$, where v and w are solutions of the Cauchy problems (here we take account of the linearity)

$$\begin{aligned} &v_{tt} - v_{xx} = F(t,x),\ \ v(0,x) = 0,\ \ v_t(0,x) = 0, \\ &w_{tt} - w_{xx} = 0,\ \ w(0,x) = \varphi(x),\ \ w_t(0,x) = \psi(x). \end{aligned}$$

The Cauchy problem for w is studied in Section 3.2.1, thus let us devote to the Cauchy problem for v.

Exercise 18 Derive the representation of solution

$$v(t,x) = \frac{1}{2}\int\limits_0^t \int\limits_{x-(t-t')}^{x+(t-t')} F(x',t')dx'dt'.$$

Which values of F determine the solution v in the point (t_0, x_0)? Find the domain of dependence! Denoting the domain of dependence by $\Omega(t_0, x_0)$ we conclude $\Omega(t_0, x_0) = \{(t,x) \in \mathbb{R}^{n+1} : (t,x) \in [0,t_0) \times \{|x - x_0| \leq t_0 - t\}\}$.

3.2.3 Kirchhoff's representation in R^3

As before we are interested in the Cauchy problem

$$u_{tt} - \triangle u = 0, \ \ u(0,x) = \varphi(x), \ \ u_t(0,x) = \psi(x), \ \ x \in \mathbb{R}^3.$$

To find a solution is more complicate than in the 1-d case. A simple observation tells us the following:

Lemma 3.1 *If $u_p = u_p(t,x)$ solves the Cauchy problem*

$$u_{tt} - \triangle u = 0, \ \ u(0,x) = 0, \ \ u_t(0,x) = p(x),$$

$p = p(x)$ is sufficiently smooth, then $\partial_t u_p =: v$ solves the Cauchy problem

$$v_{tt} - \triangle v = 0, \ \ v(0,x) = p(x), \ \ v_t(0,x) = 0.$$

Exercise 19 Prove the statement of this lemma!

Corollary 3.1 *A solution $u = u(t,x)$ of*

$$u_{tt} - \triangle u = 0, \ \ u(0,x) = \varphi(x), \ \ u_t(0,x) = \psi(x)$$

is representable in the form $u(t,x) = u_\psi(t,x) + \partial_t u_\varphi(t,x)$, where the data φ and ψ are supposed to be smooth.

Thus it is sufficient to derive a formula for $u_p = u_p(t,x)$. First we sketch how to guess such a formula, then we will prove that the formula really gives a solution (see Theorem 3.2).

We consider the auxiliary Cauchy problem

$$u_{tt} - \triangle u = 0, \ \ u(0,x) = 0, \ \ u_t(0,x) = \delta_\varepsilon(x), \ \ x \in \mathbb{R}^3,$$

where $\delta_\varepsilon(x) = (4\pi\varepsilon)^{-\frac{3}{2}} \exp(-\frac{|x|^2}{4\varepsilon}), \ \ \varepsilon > 0$. It holds $\int_{\mathbb{R}^3} \delta_\varepsilon(x)dx = 1$ and $\lim\limits_{\varepsilon \to 0} \delta_\varepsilon(x) = 0$ for all $x \neq 0$.

The data depends on the polar distance, it is radially symmetric. Then we should expect that the solution is radially symmetric, too, thus it depends only on r and t.

Exercise 20 Show that every radially symmetric solution $u = u(t,r)$ of $u_{tt} - \triangle u = 0,\ x \in \mathbb{R}^3$, is representable in the following form:

$$u(t,r) = \frac{u_1(r+t)}{r} + \frac{u_2(r-t)}{r}$$

with arbitrary given twice differentiable functions u_1, u_2 (transform the Laplace operator into polar co-ordinates). Here $u_1 = u_1(r+t)$ is called *contracting wave* and $u_2 = u_2(r-t)$ is called *expanding wave.*

Using the Cauchy conditions then one integration leads to

$$u_2(r) = \int -\frac{r}{2}\,(4\pi\varepsilon)^{-\frac{3}{2}}\exp(-\frac{r^2}{4\varepsilon})dr = \varepsilon\,(4\pi\varepsilon)^{-\frac{3}{2}}\exp(-\frac{r^2}{4\varepsilon}) + C.$$

This gives the representation of solution

$$u(t,x) = I_\varepsilon(r,t) - J_\varepsilon(r,t) :=$$
$$\frac{1}{4\pi r}\,\frac{1}{\sqrt{4\pi\varepsilon}}\,\exp\left(-\frac{(r-t)^2}{4\varepsilon}\right) - \frac{1}{4\pi r}\,\frac{1}{\sqrt{4\pi\varepsilon}}\,\exp\left(-\frac{(r+t)^2}{4\varepsilon}\right).$$

The data $p = p(y)$ is supposed to be continuous. Thus it is nearly constant in a small cube $\triangle y$. Consequently, the solution $u = u(t,x)$ to the data $p(y)\delta_\varepsilon(|x-y|) \triangle y$ ($\triangle y$ means localization near y!) is

$$u(t,x) = p(y)\Big(I_\varepsilon(|x-y|,t) - J_\varepsilon(|x-y|,t)\Big) \triangle y.$$

The superposition of all localized influences leads to

$$u(t,x) = \int_{\mathbb{R}^3} p(y)\Big(I_\varepsilon(|x-y|,t) - J_\varepsilon(|x-y|,t)\Big)dy\ .$$

The desired formula results from

$$u(t,x) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} p(y)\Big(I_\varepsilon(|x-y|,t) - J_\varepsilon(|x-y|,t)\Big)dy\ = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} p(y)I_\varepsilon(|x-y|,t)dy.$$

Introducing spherical harmonics we get with $y = x + \rho\,\omega$, ω is a unit vector in $\mathbb{R}^3$,

$$u(t,x) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} p(y)\,\frac{1}{4\pi|x-y|}\,\frac{1}{\sqrt{4\pi\varepsilon}}\exp\Big(-\frac{(|x-y|-t)^2}{4\varepsilon}\Big)dy$$
$$= \lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_0^\infty \exp\Big(-\frac{(\rho-t)^2}{4\varepsilon}\Big)\frac{1}{\sqrt{4\pi\varepsilon}}\Big(\frac{1}{\rho}\int_{|\omega|=1} p(x+\rho\omega)\rho^2 d\sigma_\omega\Big)d\rho.$$

Finally, setting $\rho - t = 2\sqrt{\varepsilon}z$ and changing the order of integration it follows

$$\begin{aligned} u(t,x) &= \frac{1}{4\pi} \int\limits_{|\omega|=1} \lim_{\varepsilon\to 0} \int\limits_{\frac{-t}{2\sqrt{\varepsilon}}}^{\infty} p(x+(t+2\sqrt{\varepsilon}z)\omega)(t+2\sqrt{\varepsilon}z)\exp(-z^2)dz\, d\sigma_\omega \\ &= \frac{t}{4\pi} \int\limits_{|\omega|=1} p(x+t\omega)d\sigma_\omega. \end{aligned}$$

The element of surface of a ball with radius t is $d\sigma_y = t^2 d\sigma_\omega$. Setting $x + t\omega = y$ we arrive at the equivalent representation

$$u(t,x) = \frac{1}{4\pi t} \int\limits_{S_t(x)} p(y)d\sigma_y,$$

where $S_t(x)$ is the surface of a ball of radius t and center x.

Remark: The above considerations serve to guess a representation of solutions to the Cauchy problem $u_{tt} - \triangle u = 0,\ u(0,x) = 0,\ u_t(0,x) = p(x),\ x \in \mathbb{R}^3$.

Theorem 3.2 *Let $p \in C^k(\mathbb{R}^3)$ with $k \geq 2$. Then the solution of the above Cauchy problem is given by the aid of Kirchhoff's formula*

$$u_p(t,x) = \frac{1}{4\pi t} \int\limits_{S_t(x)} p(y)d\sigma_y.$$

The solution belongs to $C^k([0,\infty) \times \mathbb{R}^3)$.

Proof: We introduce $y = x + t\alpha,\ \alpha = (\alpha_1, \alpha_2, \alpha_3)$, where α is a unit vector in the direction $y - x$. Using $d\sigma_t = t^2 d\sigma_1$ gives

$$u_p(x,t) = \frac{t}{4\pi} \int\limits_{S_1(0)} p(x+t\alpha)d\sigma_1.$$

Thus we get $\lim\limits_{t\to 0} u_p(x,t) = 0$. Differentiating with respect to t implies together with the supposed regularity for p the relation

$$\partial_t u_p(x,t) = \frac{1}{4\pi} \int\limits_{S_1(0)} p(x+t\alpha)d\sigma_1 + \frac{t}{4\pi} \int\limits_{S_1(0)} \nabla p(x+t\alpha)\cdot \alpha\, d\sigma_1.$$

From this equation it follows $\lim_{t\to 0}\partial_t u_p(x,t) = p(x)$. It remains to show, that u_p solves the wave equation, that is, $\Box u_p(x,t) = 0$. We use the representation

$$\partial_t u_p(x,t) = \frac{1}{t}\, u_p(x,t) + \frac{1}{4\pi t} \int\limits_{S_t(x)} \nabla p(y)\cdot\alpha \; d\sigma_t(y).$$

Now, using the fact that α is the exterior unit normal vector to $S_t(x)$ and applying the Divergence Theorem, we obtain

$$\partial_t u_p(x,t) = \frac{1}{t}\, u_p(x,t) + \frac{1}{4\pi t} \int\limits_{B(x,t)} \triangle p(y) dy.$$

Here $B(x,t) \subset \mathbb{R}^3$ denotes the ball around the center x with radius t. Differentiation with respect to t yields

$$\partial_t^2 u_p(x,t) = -\frac{1}{t^2}\, u_p(x,t) + \frac{1}{t}\, \partial_t u_p(x,t) - \frac{1}{4\pi t^2} \int\limits_{B(x,t)} \triangle p(y) dy + \frac{1}{4\pi t}\, \frac{\partial}{\partial t} \int\limits_{B(x,t)} \triangle p(y) dy.$$

Setting into this equation the above relation for $\partial_t u_p$ we obtain

$$\partial_t^2 u_p(x,t) = \frac{1}{4\pi t}\, \partial_t \int\limits_{B(x,t)} \triangle p(y) dy.$$

Taking account of

$$\partial_t \int\limits_{B(x,t)} \triangle p(y) dy = \partial_t \int\limits_0^t \int\limits_{S_r(x)} \triangle p(x + r\alpha) d\sigma_r \; dr = \int\limits_{S_t(x)} \triangle p(x + t\alpha) d\sigma_t$$

we derive

$$\partial_t^2\, u_p(x,t) = \frac{1}{4\pi t} \int\limits_{S_t(x)} \triangle p(y) d\sigma_t = \frac{t}{4\pi} \int\limits_{S_1(0)} \triangle p(x + \alpha t) d\sigma_1.$$

Finally, the relation

$$\triangle u_p(x,t) = \frac{t}{4\pi} \int\limits_{S_1(0)} \triangle p(x + \alpha t) d\sigma_1,$$

guarantees that $u_p = u_p(x,t)$ is a solution of our Cauchy problem. $\Box$

Corollary 3.2 *The Cauchy problem*

$$u_{tt} - \triangle u = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x)$$

with data $\varphi \in C^k(\mathbb{R}^3)$ *and* $\psi \in C^{k-1}(\mathbb{R}^3)$ *has one solution* $u \in C^{k-1}([0,\infty) \times \mathbb{R}^3)$. *This solution is representable in the form*

$$u(t,x) = \frac{1}{4\pi t} \int\limits_{S_t(x)} \psi(y) d\sigma_y + \partial_t \Big(\frac{1}{4\pi t} \int\limits_{S_t(x)} \varphi(y) d\sigma_y \Big).$$

Question: Do we see differences between the statements of Theorems 3.1 and 3.2?

Answer: We have no uniqueness in the formulation of Theorem 3.2. Moreover, the solution from Theorem 3.2 belongs only to the space $C^{k-1}([0,\infty) \times \mathbb{R}^3)$. Thus *we lose one order of regularity.*

Exercise 21 (Duhamel's principle) (compare with Exercise 18)
Show that the solution u of

$$u_{tt} - \triangle u = F(x,t), \;\; u(x,0) = u_t(x,0) = 0, \;\; x \in \mathbb{R}^3,$$

is given by

$$u(x,t) = \int\limits_0^t w(x,t,\tau) d\tau,$$

where $w = w(x,t,\tau)$ solves the following Cauchy problem:

$$w_{tt} - \triangle w = 0, \;\; w(x,\tau,\tau) = 0, \;\; w_t(x,\tau,\tau) = F(x,\tau).$$

3.2.4 General dimension

3.2.4.1 Odd space dimension Let us devote to the Cauchy problem

$$u_{tt} - \triangle u = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x), \;\; x \in \mathbb{R}^{2n+1}, \;\; n \geq 1.$$

Theorem 3.3 *To given data* $\varphi \in C^k(\mathbb{R}^{2n+1})$ *and* $\psi \in C^{k-1}(\mathbb{R}^{2n+1})$ *with* $k \geq n+2, \;\; n \geq 1$ *there exists one solution* $u \in C^{k-n}([0,\infty) \times \mathbb{R}^{2n+1})$. *The solution has the representation*

$$\begin{aligned} u(t,x) &= \sum_{j=0}^{n-1} \Big((j+1) a_j t^j \partial_t^j + a_j t^{j+1} \partial_t^{j+1} \Big) \frac{1}{\omega_{2n+1}} \int\limits_{|y|=1} \varphi(x+ty) d\sigma_y \\ &+ \sum_{j=0}^{n-1} a_j t^{j+1} \partial_t^j \, \frac{1}{\omega_{2n+1}} \int\limits_{|y|=1} \psi(x+ty) d\sigma_y, \end{aligned}$$

where $a_j = a_j(n)$ are constants with $a_{n-1} \neq 0$, and where ω_{2n+1} denotes the measure of the unit sphere in $\mathbb{R}^{2n+1}$.

Question: What do we conclude from this representation of solution?

Answer: We obtain immediately the following properties:

- The loss of regularity is n.
- The properties of finite propagation speed of perturbations, of existence of a domain of dependence and of existence of a forward and of a backward wave front are fulfilled.

Question: How can we prove the uniqueness of solutions?

Example: If $n = 1$, then $a_0 = 1$, and we conclude Kirchhoff's representation formula in 3-d case

$$u(t,x) = (1 + t\,\partial_t)\,\frac{1}{4\pi}\int\limits_{|y|=1}\varphi(x+ty)d\sigma_y + \frac{t}{4\pi}\int\limits_{|y|=1}\psi(x+ty)d\sigma_y.$$

3.2.4.2 Even space dimension Let us devote to the Cauchy problem

$$u_{tt} - \triangle u = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x), \;\; x \in \mathbb{R}^{2n}, \;\; n \geq 1.$$

Theorem 3.4 *To given data $\varphi \in C^k(\mathbb{R}^{2n})$ and $\psi \in C^{k-1}(\mathbb{R}^{2n})$ with $k \geq n+2, \;\; n \geq 1$ there exists a solution $u \in C^{k-n}([0,\infty) \times \mathbb{R}^{2n})$ having the representation*

$$\begin{aligned} u(t,x) \;\; = \;\; & \sum_{j=0}^{n-1}\Big((j+1)b_j t^j \partial_t^j + b_j t^{j+1}\partial_t^{j+1}\Big)\,\frac{2\Gamma(\frac{2n+1}{2})}{\sqrt{\pi}\Gamma(n)t^{2n-1}} \\ & \times \int\limits_0^t \frac{r^{2n-1}}{\omega_{2n}(t^2-r^2)^{1/2}}\int\limits_{|y|=1}\varphi(x+ry)d\sigma_y dr \\ & + \sum_{j=0}^{n-1} b_j t^{j+1}\partial_t^j\,\frac{2\Gamma(\frac{2n+1}{2})}{\sqrt{\pi}\Gamma(n)t^{2n-1}}\int\limits_0^t \frac{r^{2n-1}}{\omega_{2n}(t^2-r^2)^{1/2}}\int\limits_{|y|=1}\psi(x+ry)d\sigma_y dr, \end{aligned}$$

where $b_j = b_j(n)$ are constants with $b_{n-1} \neq 0$, and where ω_{2n} denotes the measure of the unit sphere in $\mathbb{R}^{2n}$.

Question: What do we conclude from this representation of solution?

Answer: We obtain immediately the following properties:

- The loss of regularity is n.

- The properties of finite propagation speed of perturbations, of existence of a domain of dependence and of existence of a forward wave front are fulfilled.

Example: For $n = 1$ we get Kirchhoff's representation formula in 2-d case

$$
\begin{aligned}
u(t,x) &= (b_0 + b_0 t\, \partial_t) \frac{2\Gamma(\frac{3}{2})}{\sqrt{\pi}\Gamma(1)t} \int_0^t \frac{r}{\omega_2 (t^2 - r^2)^{1/2}} \int_{|y|=1} \varphi(x+ry) d\sigma_y dr \\
&\quad + b_0 \frac{2\Gamma(\frac{3}{2})}{\sqrt{\pi}\Gamma(1)} \int_0^t \frac{r}{\omega_2 (t^2 - r^2)^{1/2}} \int_{|y|=1} \psi(x+ry) d\sigma_y dr \\
&= b_0 \frac{2\Gamma(\frac{3}{2})}{\sqrt{\pi}\Gamma(1)} \Big(\partial_t \int_{K_t(x)} \frac{\varphi(y)}{\sqrt{t^2 - |y-x|^2}} dy + \int_{K_t(x)} \frac{\psi(y)}{\sqrt{t^2 - |y-x|^2}} dy \Big)
\end{aligned}
$$

after a suitable choice of the constant $b_0 \neq 0$.

Exercise 22 One can derive Kirchhoff's representation formula in 2-d case from the Kirchhoff's formula in 3-d case. Therefore one has to apply the *method of descent*. Study in the literature the method of descent!

3.2.5 Energy method

The notion of *energy of solution of a wave equation* is a basic tool to derive uniqueness results for the wave models are discussed in Theorems 3.2 to 3.4. Let u be a given function from $C([0,T], H^1(\mathbb{R}^n)) \cap C^1([0,T], L^2(\mathbb{R}^n))$.

Exercise 23 Recall the notation $C([0,T], H^1(\mathbb{R}^n)) \cap C^1([0,T], L^2(\mathbb{R}^n))$.

Then we denote by

$$E(u)(t) := \frac{1}{2} \int_{\mathbb{R}^n} \Big(|u_t(t,x)|^2 + |\nabla_x u(t,x)|^2 \Big) dx = \frac{1}{2} \|u_t(t,\cdot)\|_{L^2}^2 + \frac{1}{2} \|\nabla_x u(t,\cdot)\|_{L^2}^2$$

the *energy or total energy*, which depends only on the time variable t. Here $\frac{1}{2}\|u_t(t,\cdot)\|_{L^2}^2$ denotes the *kinetic energy* and $\frac{1}{2}\|\nabla_x u(t,\cdot)\|_{L^2}^2$ denotes the *elastic energy*.
If we are not so interested in the total energy, then we can define to a given set $K \subset \mathbb{R}^n$ (K is a closure of a domain) the energy

$$E(u,K)(t) := \frac{1}{2} \int_K \Big(|u_t(t,x)|^2 + |\nabla_x u(t,x)|^2 \Big) dx.$$

Let $(t_0, x_0), t_0 > 0$, be a fixed point in $\mathbb{R}^{n+1}$. Then the set $\{(t,x) : |x - x_0| = |t - t_0|\}$ describes the lateral surface of a double cone with apex at (t_0, x_0). The *forward (backward)* *characteristic cone* is for $t \geq t_0$ ($t \leq t_0$) the upper (lower) cone with apex at (t_0, x_0). Let $T \leq t_0$. The part of the plane $t = T$ lying inside the backward characteristic cone will be denoted by $K(x_0, t_0 - T)$. This part is a closed ball around the center $x = x_0$ with radius $t_0 - T$. The following remarkable statement holds:

Theorem 3.5 *(domain of dependence inequality)*
Let $(t_0, x_0) \in \mathbb{R}^{n+1}$ with $t_0 > 0$. We denote by Ω the conical domain bounded by the backward characteristic cone with apex at (t_0, x_0) and by the plane $t = 0$. Let $u \in C^2(\overline{\Omega})$ be a classical solution of the wave equation $u_{tt} - \triangle u = 0$. Then the following inequality holds:

$$E(u, K(x_0, t_0 - t)) \leq E(u, K(x_0, t_0)) \text{ for } t \in [0, t_0].$$

Proof: Let Ω_T be the part of Ω below the plane $t = T$ and let C_T be the lateral surface of Ω_T. The energy method is basing on the identity

$$2u_t \,\Box u = -\nabla_x \cdot (2u_t \nabla_x u) + (|\nabla_x u|^2 + u_t^2)_t = 0.$$

It holds

$$0 = \int_{\Omega_T} (\nabla_x \cdot (2u_t \nabla_x u) - (|\nabla_x u|^2 + u_t^2)_t) d(x,t).$$

The integrand is equal to the divergence of the vector field $(2u_t \nabla_x u, -(|\nabla_x u|^2 + u_t^2))$. Applying the Divergence Theorem we obtain

$$0 = \int_{\partial\Omega_T} (2u_t \nabla_x u, -(|\nabla_x u|^2 + u_t^2)) \cdot \vec{n} \, d\sigma,$$

where $\vec{n}$ is the exterior unit normal vector to $\partial\Omega_T$. The surface $\partial\Omega_T$ consists of three parts. We study how the above integral can be written on each of the three parts.

a) Top ball $K(x_0, t_0 - T) : \vec{n} = (0, \cdots, 0, 1)$. The above integral reduces to $-\int\limits_{K(x_0, t_0 - T)} (|\nabla_x u|^2 + u_t^2) dx$.

b) Bottom ball $K(x_0, t_0) : \vec{n} = (0, 0, \cdots, 0, -1)$. The above integral reduces to $\int\limits_{K(x_0, t_0)} (|\nabla_x u|^2 + u_t^2) dx$.

c) Lateral surface C_T: The above integral reduces to
$\int\limits_{C_T} (2u_t \nabla_x u, -(|\nabla_x u|^2 + u_t^2)) \cdot \vec{\eta} \, d\sigma$
$= \sqrt{2} \int\limits_{C_T} \Big(2u_t u_{x_1} \eta_{n+1} \eta_1 + \cdots + 2u_t u_{x_n} \eta_{n+1} \eta_n - (u_{x_1}^2 + \cdots + u_{x_n}^2 + u_t^2) \eta_{n+1}^2 \Big) d\sigma$
$= -\sqrt{2} \int\limits_{C_T} \Big(u_{x_1} \eta_{n+1} - u_t \eta_1)^2 + \cdots + (u_{x_n} \eta_{n+1} - u_t \eta_n)^2 \Big) d\sigma \leq 0.$
Here we used $\eta_{n+1}^2 = \eta_1^2 + \cdots + \eta_n^2$.

Summarizing we have shown

$$\int\limits_{K(x_0,t_0-T)} \Big(\Big|\nabla_x u(\cdot,t)\Big|^2 + u_t(\cdot,t)^2\Big)\Big|_{t=T} dx \le \int\limits_{K(x_0,t_0)} (|\nabla_x u(\cdot,0)|^2 + u_t(\cdot,0)^2) dx.$$

Hence, the statement is proved. □

Summarizing the statements from Theorems 3.2 to 3.5 we conclude the next result.

Corollary 3.3 *The Cauchy problem*

$$u_{tt} - \triangle u = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x)$$

has a unique classical solution $u \in C^{k-n}([0,\infty)\times\mathbb{R}^{2n+1})$, $u \in C^{k-n}([0,\infty)\times\mathbb{R}^{2n})$, *respectively, for* $k \ge n+2$ *and* $n \ge 1$.

Exercise 24 Use Duhamel's principle and Kirchhoff's representation of solution to derive a solution to the Cauchy problem

$$u_{tt} - \triangle u = F(t,x), \;\; u(0,x) = u_t(0,x) = 0, \;\; x \in \mathbb{R}^3.$$

We assume $F \in C^2([0,T], C^2(\mathbb{R}^3))$. Why?

Exercise 25 Find the solution of the Cauchy problem

$$u_{tt} - \triangle u = 0, \;\; u(0,x) = 1, \;\; u_t(0,x) = \frac{1}{1+|x|^2}, \;\; x \in \mathbb{R}^3.$$

Try to find two different ways to derive the representation of the solution.

Theorem 3.6 *(conservation of energy)*
Let $u \in C([0,T], H^1(\mathbb{R}^n)) \cap C^1([0,T], L^2(\mathbb{R}^n))$ *be a Sobolev solution of*

$$u_{tt} - \triangle u = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x),$$

with data $\varphi \in H^1(\mathbb{R}^n)$ *and* $\psi \in L^2(\mathbb{R}^n)$. *Then it holds*

$$E(u)(t) = E(u)(0) = \frac{1}{2}\Big(\|u_1\|^2_{L^2} + \|\nabla u_0\|^2_{L^2}\Big) \quad \textit{for all } \; t \ge 0.$$

Proof: Using the density of the function space $C_0^\infty(\mathbb{R}^n)$ in $H^1(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ we are able to approximate the given data $\varphi \in H^1(\mathbb{R}^n)$ and $\psi \in L^2(\mathbb{R}^n)$ by sequences of data $\{\varphi_k\}$, $\{\psi_k\}$ with $\varphi_k, \psi_k \in C_0^\infty(\mathbb{R}^n)$. We consider the family of auxiliary problems

$$u_{tt} - \triangle u = 0, \;\; u(0,x) = \varphi_k(x), \;\; u_t(0,x) = \psi_k(x).$$

From the Theorems 3.1 to 3.5 we obtain a unique solution $u_k \in C^\infty([0,T], C_0^\infty(\mathbb{R}^n))$. Differentiating $E(u_k)(t)$ gives

$$E'(u_k)(t) = \int\limits_{\mathbb{R}^n} \Big(\partial_t u_k(t,x)\partial_t^2 u_k(t,x) + \nabla_x u_k(t,x)\cdot\nabla_x\partial_t u_k(t,x)\Big)dx.$$

For each $t \in [0,T]$ the function $u_k(t,\cdot)$ belongs to $C_0^\infty(\mathbb{R}^n)$. After partial integration (all boundary integrals are vanishing) we obtain immediately from the wave equation

$$E'(u_k)(t) = \int\limits_{\mathbb{R}^n} (\partial_t u_k(t,x) \triangle u_k(t,x) - \triangle u_k(t,x)\partial_t u_k(t,x))dx = 0.$$

Hence, $E(u_k)(t) = E(u_k)(0) = \frac{1}{2}\Big(\|\psi_k\|_{L^2}^2 + \|\nabla\varphi_k\|_{L^2}^2\Big)$. Together with the assumption we have $\lim\limits_{k\to\infty} E(u_k)(0) = E(u)(0)$.
From the well-posedness of the Cauchy problem in Sobolev spaces (see the next section) it follows $\lim\limits_{k\to\infty} E(u_k)(t) = E(u)(t)$. This completes the proof. □

Remark: We proved the energy conservation for the whole space $\mathbb{R}^n$. But the *energy conservation* remains true for *bounded domains* $G \subset \mathbb{R}^n$ and classical solutions of the wave equation satisfying a homogeneous boundary condition of Dirichlet- or Neumann type. The *energy conservation* holds also for unbounded domains G, for example for exterior domains, if the initial data have a compact support and if classical solutions to the wave equation satisfy a homogeneous boundary condition of Dirichlet- or Neumann type. In the proof we use that the initial data influence due to the *finite propagation speed* the solution only in the set $\{x \in G : |x| \le R + t\ ,\ t \ge 0\}$. Here R denotes the radius of a ball around the origin containing the support of the data.

3.2.6 Representatation of solutions by using Fourier multipliers - application of partial Fourier transformation

We are interested again in the Cauchy problem

$$u_{tt} - \triangle u = 0,\ \ u(0,x) = \varphi(x),\ \ u_t(0,x) = \psi(x),\ \ x \in \mathbb{R}^n,\ \ n \ge 1.$$

After application of *partial Fourier transformation* ($v(t,\xi) = F_{x\to\xi}(u(t,x))$) we get the auxiliary Cauchy problem

$$v_{tt} + |\xi|^2 v = 0,\ \ v(0,\xi) = F(\varphi)(\xi),\ \ v_t(0,\xi) = F(\psi)(\xi)$$

for an *ordinary differential equation depending on the parameter* $\xi \in \mathbb{R}^n$. For $\xi \neq 0$ we have the general solution

$$v(t,\xi) = c_1(\xi)e^{-i|\xi|t} + c_2(\xi)e^{i|\xi|t}.$$

The Cauchy conditions imply

$$c_1(\xi) + c_2(\xi) = F(\varphi)(\xi),\ -i|\xi|c_1(\xi) + i|\xi|c_2(\xi) = F(\psi)(\xi).$$

It follows

$$c_1(\xi) = \frac{1}{2}\, F(\varphi)(\xi) - \frac{1}{2i|\xi|}\, F(\psi)(\xi), \quad c_2(\xi) = \frac{1}{2}\, F(\varphi)(\xi) + \frac{1}{2i|\xi|}\, F(\psi)(\xi).$$

Setting these coefficients into the general solution gives

$$v(t,\xi) = \cos(|\xi|t)F(\varphi)(\xi) + \frac{\sin(|\xi|t)}{|\xi|}\, F(\psi)(\xi).$$

Supposing for a moment the validity of the *Fourier inversion formula* $u(t,x) = F^{-1}_{\xi\to x}(F_{x\to\xi}(u(t,x))$ (this relation must be checked at the end of our considerations) we arrive at the following representation for u:

$$u(t,x) = F^{-1}_{\xi\to x}\Big(\cos(|\xi|t)F(\varphi)(\xi)\Big) + F^{-1}_{\xi\to x}\left(\frac{\sin(|\xi|t)}{|\xi|}\, F(\psi)(\xi)\right).$$

One can also use the equivalent representation

$$\begin{aligned} u(t,x) &= F^{-1}_{\xi\to x}\left(e^{-i|\xi|t}\frac{1}{2}\, F(\varphi)(\xi)\right) - F^{-1}_{\xi\to x}\left(e^{-i|\xi|t}\frac{1}{2i|\xi|}\, F(\psi)(\xi)\right) \\ &+ F^{-1}_{\xi\to x}\left(e^{i|\xi|t}\frac{1}{2}\, F(\varphi)(\xi)\right) + F^{-1}_{\xi\to x}\left(e^{i|\xi|t}\frac{1}{2i|\xi|}\, F(\psi)(\xi)\right). \end{aligned}$$

This representation consists of so-called *Fourier multipliers*

$$F^{-1}_{\xi\to x}\left(e^{i\phi(t,\xi)}a(t,\xi)\ F(u_0)(\xi)\right).$$

Here $\phi = \phi(t,\xi)$ is the so-called *phase function* and $a = a(t,\xi)$ is the so-called *amplitude function.*

For given data we assume $\varphi \in H^s(\mathbb{R}^n)$ and $\psi \in H^{s-1}(\mathbb{R}^n)$ with $s \geq 1$. Taking into consideration Theorem 3.6 we know that the solution possesses an energy for all $t \geq 0$.

Theorem 3.7 *Let $\varphi \in H^s(\mathbb{R}^n)$ and $\psi \in H^{s-1}(\mathbb{R}^n)$, $s \geq 1$, $n \geq 1$ in the Cauchy problem*

$$u_{tt} - \Delta u = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x).$$

Then there exists a unique solution $u \in C([0,T], H^s(\mathbb{R}^n)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^n))$.

Proof: The uniqueness follows from Theorem 3.6. The existence of a solution is given in the form

$$u(t,x) = F^{-1}_{\xi\to x}\Big(\cos(|\xi|t)F(\varphi)(\xi)\Big) + F^{-1}_{\xi\to x}\left(\frac{\sin(|\xi|t)}{|\xi|}F(\psi)(\xi)\right)$$

if this solution satisfies the desired regularity. Let us transfer the assumptions for the data into the Fourier image. Then we have

$$F(\varphi)(\xi) \in L^{2,s}, \quad \text{that is,} \quad \langle\xi\rangle^s F(\varphi)(\xi) \in L^2, \langle\xi\rangle = (1+|\xi|^2)^{1/2},$$
$$F(\psi)(\xi) \in L^{2,s-1}, \quad \text{that is,} \quad \langle\xi\rangle^{s-1} F(\psi)(\xi) \in L^2.$$

We use the following estimates:

- $|\cos(|\xi|t)| \leq 1$,
- $|\sin(|\xi|t)| \leq |\xi|t \leq |\xi|T$, for $|\xi| \leq \varepsilon$ and $t \in [0,T]$,
- $|\sin(|\xi|t)| \leq 1$ for $|\xi| \geq \varepsilon$ and $t \in [0,T]$.

Thus we can conclude

$$|v(t,\xi)| \leq |F(\varphi)(\xi)| + C(\varepsilon,T)\tfrac{|F(\psi)(\xi)|}{\langle\xi\rangle},$$
$$\langle\xi\rangle^s|v(t,\xi)| \leq \langle\xi\rangle^s|F(\varphi)(\xi)| + C(\varepsilon,T)\langle\xi\rangle^{s-1}|F(\psi)(\xi)|.$$

This leads to $v \in L^\infty([0,T],L^{2,s})$. In the same way we derive $\partial_t v \in L^\infty([0,T],L^{2,s-1})$. It remains to prove $v \in C([0,T],L^{2,s}) \cap C^1([0,T],L^{2,s-1})$. The property $v \in C([0,T],L^{2,s})$ follows from $\lim\limits_{t_1\to t_2} ||v(t_1,\cdot) - v(t_2,\cdot)||_{L^{2,s}} = 0$.

Using the explicit representation of solution we conclude as follows:

$$\lim_{t_1\to t_2}\int_{\mathbb{R}^n} |v(t_1,\xi)-v(t_2,\xi)|^2\langle\xi\rangle^{2s}d\xi$$

$$\leq \lim_{t_1\to t_2}\int_{\mathbb{R}^n}\Big|\sin(\tfrac{|\xi|(t_1+t_2)}{2})\sin(\tfrac{|\xi|(t_1-t_2)}{2})\Big|^2|F(\varphi)(\xi)|^2\langle\xi\rangle^{2s}d\xi$$

$$+\lim_{t_1\to t_2}\int_{\mathbb{R}^n}\Big|\cos(\tfrac{|\xi|(t_1+t_2)}{2})\sin(\tfrac{|\xi|(t_1-t_2)}{2})\Big|^2\tfrac{1}{|\xi|^2}|F(\psi)(\xi)|^2\langle\xi\rangle^{2s}d\xi.$$

Let $K_R(0) \subset \mathbb{R}^n$ be a sufficiently large ball around the origin with radius R. We divide the integral $\int\limits_{\mathbb{R}^n}$ in two integrals $\int\limits_{K_R(0)} + \int\limits_{\mathbb{R}^n\setminus K_R(0)}$. Using the above estimates it holds

$$\int_{\mathbb{R}^n}\Big|\sin(\tfrac{|\xi|(t_1+t_2)}{2})\sin(\tfrac{|\xi|(t_1-t_2)}{2})\Big|^2|F(\varphi)(\xi)|^2\langle\xi\rangle^{2s}d\xi$$

$$=\int_{K_R(0)}\Big|\sin(\tfrac{|\xi|(t_1+t_2)}{2})\sin(\tfrac{|\xi|(t_1-t_2)}{2})\Big|^2|F(\varphi)(\xi)|^2\langle\xi\rangle^{2s}d\xi$$

$$+\int_{\mathbb{R}^n\setminus K_R(0)}\Big|\sin(\tfrac{|\xi|(t_1+t_2)}{2})\sin(\tfrac{|\xi|(t_1-t_2)}{2})\Big|^2|F(\varphi)(\xi)|^2\langle\xi\rangle^{2s}d\xi$$

$$\leq \int_{K_R(0)}\tfrac{|\xi|^2(t_1-t_2)}{4}^2|F(\varphi)(\xi)|^2\langle\xi\rangle^{2s}d\xi + \int_{\mathbb{R}^n\setminus K_R(0)}|F(\varphi)(\xi)|^2\langle\xi\rangle^{2s}d\xi$$

for $|t_1 - t_2| < \varepsilon(R)$. The first integral at the right-hand side is estimated by $C_R(t_1 - t_2)^2||F(\varphi)||^2_{L^{2,s}}$. Using the continuity of the Lebesgue measure the second integral is estimated by $\widetilde{\varepsilon}(R) \to 0$. Summarizing we obtain

$$\lim_{t_1 \to t_2} \int_{\mathbb{R}^n} \left| \sin\left(\frac{|\xi|(t_1+t_2)}{2}\right) \sin\left(\frac{|\xi|(t_1-t_2)}{2}\right)\right|^2 |F(\varphi)(\xi)|^2 \langle \xi \rangle^{2s} d\xi$$
$$\leq \lim_{t_1 \to t_2} C_R(t_1 - t_2)^2 ||F(\varphi)||^2_{L^{2,s}} + \widetilde{\varepsilon}(R) = \widetilde{\varepsilon}(R).$$

Taking account of $\widetilde{\varepsilon}(R) \to 0$ for $R \to \infty$ we conclude

$$\lim_{t_1 \to t_2} \int_{\mathbb{R}^n} \left| \sin\left(\frac{|\xi|(t_1+t_2)}{2}\right) \sin\left(\frac{|\xi|(t_1-t_2)}{2}\right)\right|^2 |F(\varphi)(\xi)|^2 \langle \xi \rangle^{2s} d\xi = 0.$$

Repeating this approach yields

$$\lim_{t_1 \to t_2} \int_{\mathbb{R}^n} \left| \cos\left(\frac{|\xi|(t_1+t_2)}{2}\right) \sin\left(\frac{|\xi|(t_1-t_2)}{2}\right)\right|^2 \frac{|F(\psi)(\xi)|^2}{|\xi|^2} \langle \xi \rangle^{2s} d\xi = 0,$$

where we now divide $\int\limits_{\mathbb{R}^n}$ into $\int\limits_{K_\varepsilon(0)} + \int\limits_{K_R(0)\setminus K_\varepsilon(0)} + \int\limits_{\mathbb{R}^n \setminus K_R(0)}$.

Summarizing we have shown $v \in C([0,T], L^{2,s})$. The validity of the Fourier inversion formula $u = F^{-1}_{\xi \to x}(F_{x \to \xi}(u(t,x)))$ implies $u \in C([0,T], H^s)$. An analogous reasoning brings $v \in C^1([0,T], L^{2,s-1})$, $u \in C^1([0,T], H^{s-1})$, respectively. Here we have again to use the inversion formula $\partial_t u = F^{-1}_{\xi \to x}(F_{x \to \xi}(\partial_t u(t,x)))$. Thus the proof is complete. □

Exercise 26 Carry out the step of the proof

$$\lim_{t_1 \to t_2} \int_{\mathbb{R}^n} \left| \cos\left(\frac{|\xi|(t_1+t_2)}{2}\right) \sin\left(\frac{|\xi|(t_1-t_2)}{2}\right)\right|^2 \frac{|F(\psi)(\xi)|^2}{|\xi|^2} \langle \xi \rangle^{2s} d\xi = 0.$$

The considerations of this section show that the Cauchy problem is H^s well-posed.

Corollary 3.4 *The Cauchy problem*

$$u_{tt} - \Delta u = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x),\ x \in \mathbb{R}^n,\ n \geq 1$$

is H^s well-posed, that is, to given data $\varphi \in H^s(\mathbb{R}^n)$, $\psi \in H^{s-1}(\mathbb{R}^n)$ there exists a uniquely determined solution $u \in C([0,T], H^s(\mathbb{R}^n)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^n))$. The solution depends continuously on the data, that is, to each $\varepsilon > 0$ there exists a $\delta(\varepsilon)$ such that $||\varphi_1 - \varphi_2||_{H^s} + ||\psi_1 - \psi_2||_{H^{s-1}} < \delta$ implies $||u_1 - u_2||_{C([0,T],H^s) \cap C^1([0,T],H^{s-1})} < \varepsilon$.

In the moment we are familiar with two ways to represent solutions of wave equations. On the one hand we know the representations from Theorems 3.1 to 3.4. On the other hand we are acquainted with representations consisting of Fourier multipliers. Is it possible to transfer one representation into another one?

Exercise 27 In the $1-d$ case we have the representation

$$u(t,x) = F^{-1}_{\xi\to x}\left(\left(e^{i\xi t}+e^{-i\xi t}\right)\frac{1}{2}F(\varphi)(\xi)\right) + F^{-1}_{\xi\to x}\left(\left(e^{i\xi t}-e^{-i\xi t}\right)\frac{1}{2i\xi}F(\psi)(\xi)\right).$$

How can we get from this representation the d'Alembert's representation formula from Section 3.2.1.

From the representation

$$v(t,\xi) = \cos(|\xi|t)F(\varphi)(\xi) + \frac{\sin(|\xi|t)}{|\xi|}F(\psi)(\xi) = \partial_t\left(\frac{\sin(|\xi|t)}{|\xi|}F(\varphi)(\xi)\right) + \frac{\sin(|\xi|t)}{|\xi|}F(\psi)(\xi)$$

it follows

$$u(t,x) = \partial_t F^{-1}_{\xi\to x}\left(\frac{\sin(|\xi|t)}{|\xi|}F(\varphi)(\xi)\right) + F^{-1}_{\xi\to x}\left(\frac{\sin(|\xi|t)}{|\xi|}F(\psi)(\xi)\right).$$

Therefore we only have to understand

$$F^{-1}_{\xi\to x}\left(\frac{\sin(|\xi|t)}{|\xi|}F(\psi)(\xi)\right).$$

Exercise 28 What is the main difficulty in the discussion of the last Fourier multiplier? Which methods do we find in the literature to overcome these difficulties?

Question: What are the advantages or disadvantages of the application of the method of Fourier transformation to study wave equations?

Answer:

advantages: Choosing data from Sobolev spaces we have *no loss of regularity* (see Theorem 3.7). The approach is independent of the spatial dimension n.

disadvantages: Special qualitative properties of solutions of the wave equation as existence of forward or backward wave front, or finite propagation speed of perturbations or domain of dependence are difficult to understand by using Fourier multipliers in the representation of solution.

3.2.7 Propagation of singularities

Let us recall d'Alembert's representation of solution to the 1-d wave equation

$$u(t,x) = \frac{1}{2}\Big(\varphi(x-t) + \varphi(x+t)\Big) + \frac{1}{2}\int\limits_{x-t}^{x+t} \psi(r)dr.$$

Let us assume that the data φ or ψ have a *jump* in $x = x_0$. Then there will be still a jump in the solution and the jump propagates along the characteristics $x - x_0 = t$ and $x - x_0 = -t$. If there is only a jump in ψ, then we feel it in the first derivatives of the solution. This observation can be generalized to higher-dimensional cases. Thus singularities in the data propagate along the forward characteristic cone. If we have an obstacle, then singularities from the data for solutions of the wave equation will be reflected. There exist special situations where the study of propagation of singularities for solutions of mixed problems, and thus the property of *reflection of singularities*, can be understood from the study of the propagation of singularities for solutions to Cauchy problems.

In the following we introduce three different models.

Model 1: Vibrations of an infinite string with fixed end

The mathematical model is $u_{tt} - u_{xx} = 0,\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x)$ for $x \geq 0$ and $u(0,t) = 0$ for $t \geq 0$. The compatibility conditions imply $\varphi(0) = \psi(0) = 0$. Let $\tilde{\varphi}$ and $\tilde{\psi}$ be the *odd continuations* of φ and ψ onto the negative part of the real axis. We consider the Cauchy problem $v_{tt} - v_{xx} = 0,\ v(x,0) = \tilde{\varphi}(x), v_t(x,0) = \tilde{\psi}(x)$. From d'Alembert's formula we get

$$v(x,t) = \frac{1}{2}\left(\tilde{\varphi}(x+t) + \tilde{\varphi}(x-t)\right) + \frac{1}{2}\int\limits_{x-t}^{x+t} \tilde{\psi}(s)\ ds\ .$$

Thus $v(0,t) = 0$. The solution $u = u(x,t)$ is

$$u(x,t) = \frac{1}{2}\left(\tilde{\varphi}(x+t) + \tilde{\varphi}(x-t)\right) + \frac{1}{2}\int\limits_{x-t}^{x+t} \tilde{\psi}(s)\ ds$$

for $(x,t) \in [0,\infty) \times [0,\infty)$.

Model 2: Vibrations of an infinite string with a free end

The mathematical model is $u_{tt} - u_{xx} = 0,\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x)$, for $x \geq 0$, and $u_x(0,t) = 0$ for $t \geq 0$. The compatibility conditions imply $\varphi'(+0) = \psi'(+0) = 0$. Let $\tilde{\varphi}$ and $\tilde{\psi}$ be the *even continuations* of φ and ψ onto the negative part of the real axis. We consider the Cauchy problem $v_{tt} - v_{xx} = 0,\ v(x,0) = \tilde{\varphi}(x), v_t(x,0) = \tilde{\psi}(x)$. Then we get the solution

$$u(x,t) = \frac{1}{2}\Big(\tilde{\varphi}(x+t) + \tilde{\varphi}(x-t)\Big) + \frac{1}{2}\int\limits_{x-t}^{x+t} \tilde{\psi}(s)ds.$$

It is easy to see that the boundary condition $u_x(0,t) = 0$ for $t \geq 0$ is satisfied.

Model 3: Reflection of sound waves on an obstacle

The mathematical model is $\Box u = 0$ in $\{x \in \mathbb{R}^3 : x_3 > 0 \text{ and } t > 0\}$, $u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x),\ \partial_{x_3} u(x_1,x_2,0,t) = 0$ for $(x_1,x_2) \in \mathbb{R}^2$ and $t \geq 0$.

Thus we consider a mixed problem in the half space. The normal derivative of u vanishes on $x_3 = 0$. We choose again the *even continuations*

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x) \quad , & x_3 > 0, \\ \varphi(x_1,x_2,-x_3)\ , & x_3 \leq 0, \end{cases}$$

and $\tilde{\psi}$ in the same way. Then we determine the solution $\tilde{u} = \tilde{u}(x,t)$ of the Cauchy problem $\Box\tilde{u} = 0,\ \tilde{u}(x,0) = \tilde{\psi}(x),\ \tilde{u}_t(x,0) = \tilde{\psi}(x)$. The restriction of $\tilde{u}$ on $x_3 \geq 0$ is the desired solution $u = u(x,t)$ of our starting problem. Let us suppose that the data vanishes outside of a small ball around the point $(0,0,a)$. Then the solution consists of two expanding spherical waves with the origins in $(0,0,a)$ and $(0,0,-a)$. The wave with the origin in $(0,0,a)$ is the direct wave, the wave with the origin in $(0,0,-a)$ is the reflection of the direct wave at the plane $x_3 = 0$.

Something more about the propagation of singularities

The above introduced propagation of singularities along characteristics is not so precise. Let us introduce another concept basing on the notion of *wave front set*. Let x_0 be a point in $\mathbb{R}^n$ and let $u = u(x)$ which is C^∞ in a neighborhood $U_0(x_0)$ of x_0. Further let us choose a *cut-off* function $\chi \in C_0^\infty(U_0(x_0))$ which is identically to 1 in a smaller neighborhood $U_1(x_0)$, $\overline{U_1(x_0)} \subset U_0(x_0)$, of x_0. If we form the Fourier transform of χu (this is a Schwartz function), then we have the following estimate:

to each N there exists a constant C_N such that $|F(\chi u)(\xi)| \leq C_N \langle\xi\rangle^{-N}$.

This estimate opens the way to the following definitions for the notion of *wave front set*.

Definitions 3.1

1. *Let us consider a distribution $u \in D'(G)$. A point $(x_0, \xi_0) \in T^*G \setminus \{0\}$ does not belong to the* **wave front set** *WFu of a given distribution $u \in D'(G)$ if there exist two functions ψ and χ with the following properties:*
   • $\chi \in C_0^\infty(U_0(x_0))$, $\chi \equiv 1$ *on a neighborhood* $U_1(x_0)$, $\overline{U_1(x_0)} \subset U_0(x_0)$ *of* x_0,
   • $\psi \in C^\infty(\mathbb{R}^n)$, $\psi \equiv 1$ *on a conical neighborhood* $V_1(\xi_0)$ *of* ξ_0, $\psi \equiv 0$ *outside a larger conical neighborhood* $V_0(\xi_0)$, $\overline{V_1(\xi_0)} \subset V_0(\xi_0)$, *of* ξ_0,
   • $\psi(D)(\chi u) \in C^\infty(\mathbb{R}^n)$.

2. *Let us consider a distribution $u \in D'(G)$. A point $(x_0, \xi_0) \in T^*G \setminus \{0\}$ does not belong to the* **wave front set** *WFu of a given distribution $u \in D'(G)$ if there exists a function χ with the following properties:*
   • $\chi \in C_0^\infty(U_0(x_0))$, $\chi \equiv 1$ *on a neighborhood* $U_1(x_0)$, $\overline{U_1(x_0)} \subset U_0(x_0)$ *of* x_0,
   • *to each N there exists a constant C_N such that $|F(\chi u)(\xi)| \leq C_N \langle \xi \rangle^{-N}$ in a conical neighborhood $V_0(\xi_0)$ of ξ_0.*

Example: Let δ_x be the Dirac δ-distribution in a point $x \in \mathbb{R}^n$. Then $\mathrm{WF}\delta_x = \{x\} \times \mathbb{R}^n \setminus \{0\}$.

Exercise 29 Determine the wave front set of $g(y)\delta_x$, where $g \in C_0^\infty(\mathbb{R}^m)$. Determine the wave front set of δ_x as a distribution on $\mathbb{R}^n_x \times \mathbb{R}^m_y$.

It is clear that if a function u is only H^s in a neighborhood of x_0, then the function χu (χ from Definition 3.1) does not satisfy the conditions from Definition 3.1. This is clear because the Fourier transform $F(\chi u)$ satisfies $\langle \xi \rangle^s F(\chi u) \in L^2(\mathbb{R}^n)$.

To apply the concept of wave front set to wave equations we have to introduce an important result. This result explains the structure of the wave front set for *Fourier integral operators*.

Theorem 3.8 *Let A be the Fourier integral operator*

$$Au(x) := \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi) F(u)(\xi) d\xi, \tag{3.1}$$

where the amplitude $a = a(x,\xi)$ is from $S^m(G)$ and $u \in E'(G)$ is a distribution with compact support in G. Let us suppose that the set

$$\{(y,\xi) \in WFu : \text{ there exists } x \in G : y = \nabla_\xi \phi(x,\xi), \nabla_x \phi(x,\xi) = 0\}$$

is empty. Then it holds

$$WFAu \subset \{(x,\xi) : \text{ there exists } \eta \in \mathbb{R}^n \setminus \{0\} \text{ such that } \xi = \nabla_x \phi(x,\eta), (\nabla_\eta \phi(x,\eta), \eta) \in WFu\}.$$

This tool at hand we are able to consider the *propagation of microlocal singularities* for the solutions to the Cauchy problem

$$u_{tt} - \triangle u = 0, \; u(0,x) = \varphi(x), \; u_t(0,x) = \psi(x).$$

In the previous section we have seen how the solution to this Cauchy problem can be represented by Fourier multipliers.

Theorem 3.9 *The wave front set of $u(t,\cdot)$ can be described as follows:*

$$WFu(t,\cdot) \subset \{(x \pm t\xi|\xi|^{-1}, \xi) : (x,\xi) \in WF\varphi \cup WF\psi\}.$$

This result explains that the microlocal singularities of the solution are contained on the lateral surface of the characteristic cone with apex in the singularities of the data, that is, in those points having no small neighborhood where the data are C^∞. More precisely, if $(x_0,\xi_0) \in \mathrm{WF}\varphi \cup \mathrm{WF}\psi$, that is, in the direction ξ_0 the estimate from Definition 3.1 (second part) does not hold, then the wave front $\mathrm{WF}u(t,\cdot)$ is contained on the set (with respect to x) of points on the lateral surface of the characteristic cone with apex in x_0 in the direction $\omega_0 := \xi_0|\xi_0|^{-1}$. With respect to ξ the ξ_0 direction is bad.

Proof: From the representation of the solution to the wave equation by Fourier multipliers we know that we have to care for the phase functions $\phi_\pm(t,x,\xi) := x \cdot \xi \pm t|\xi|$. These phase functions satisfy the assumption from Theorem 3.8 because $\nabla_x \phi_\pm(t,x,\eta) = \eta = 0$ is forbidden in the Definition 3.1 for $\xi = \eta$. Hence, with $\xi = \eta$ we get from $(x_0,\xi_0) \in \mathrm{WF}\varphi \cup \mathrm{WF}\psi$ immediately

$$\nabla_\xi \phi_\pm(t,x,\xi) = x_0 \pm t\xi_0|\xi_0|^{-1},\ \xi = \xi_0.$$

This proves the theorem. □

3.3 Klein-Gordon equation

The Cauchy problem for the Klein-Gordon equation is

$$u_{tt} - \triangle u + m^2 u = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x)$$

with a constant $m^2 > 0$.

How can we define the total energy of a solution?

The mass term or potential forces to include into the total energy besides the *elastic and the kinetic energy* a third component, it is the *potential energy*. Thus we define the total energy

$$E(u)(t) := \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla_x u(t,\cdot)|^2 + |u_t(t,\cdot)|^2 + m^2 |u(t,x)|^2 \right) dx.$$

3.3.1 Energy estimates

Repeating the proof to Theorem 3.6 one can show the following result:

Theorem 3.10 *(conservation of energy)*
Let $u \in C([0,T], H^1(\mathbb{R}^n)) \cap C^1([0,T], L^2(\mathbb{R}^n))$ be a Sobolev solution of

$$u_{tt} - \triangle u + m^2 u = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x),$$

with data $\varphi \in H^1(\mathbb{R}^n)$ and $\psi \in L^2(\mathbb{R}^n)$. Then the conservation of energy holds,

$$E(u)(t) = E(u)(0) = \frac{1}{2}\Big(\|\psi\|^2_{L^2} + \|\nabla\varphi\|^2_{L^2} + m^2\|\varphi\|^2_{L^2}\Big) \quad \textit{for all } t \geq 0.$$

Exercise 30 Prove the statement of Theorem 3.10.

Another way to show the *conservation of energy* is to use the *partial Fourier transformation*. The Fourier transform $v(t,\xi) = F_{x\to\xi}(u(t,x))(t,\xi)$ satisfies the ordinary differential equation with parameter ξ, $v_{tt} + |\xi|^2 v + m^2 v = v_{tt} + \langle\xi\rangle^2_m v = 0$ with $\langle\xi\rangle^2_m = |\xi|^2 + m^2$. Taking into consideration the explicit representations of solutions for $v(t,\cdot)$ and $v_t(t,\cdot)$, the assumption $(\varphi, \nabla_x\varphi, \psi) \in L^2 \times L^2 \times L^2$ and *Parseval's formula* from the theory of Fourier transformation, then we conclude as follows:

$$\begin{aligned}
E(u)(t) &= \frac{1}{2}\big(\|\nabla_x u(t,\cdot)\|^2_{L^2} + \|u_t(t,\cdot)\|^2_{L^2} + m^2\|u(t,\cdot)\|^2_{L^2}\big)\\
&= \frac{1}{2}\big(\|\,|\xi|\,v(t,\cdot)\|^2_{L^2} + \|v_t(t,\cdot)\|^2_{L^2} + m^2\|v(t,\cdot)\|^2_{L^2}\big)\\
&= \frac{1}{2}\big(\|\langle\xi\rangle_m v(t,\cdot)\|^2_{L^2} + \|v_t(t,\cdot)\|^2_{L^2}\big)\\
&= \frac{1}{2}\big(\|\langle\xi\rangle_m v_0(\xi)\|^2_{L^2} + \|v_1(\xi)\|^2_{L^2}\big)\\
&= \frac{1}{2}\int_{\mathbb{R}^n}\big(|\nabla_x\varphi(x)|^2 + |\psi(x)|^2 + m^2|\varphi(x)|^2\big)\,dx = E(u)(0).
\end{aligned}$$

<u>*Question:*</u> Do we have a similar statement to Theorem 3.5 if we are not interested in the *total energy*?

<u>*Answer:*</u> Let $K \subset \mathbb{R}^n$ be the closure of a domain. We define the energy

$$E(u,K)(t) := \frac{1}{2}\int_K \Big(|u_t(t,x)|^2 + |\nabla_x u(t,x)|^2 + m^2|u(t,x)|^2\Big)dx.$$

The following remarkable result holds by using the same notations as in Section 3.2.5.

Theorem 3.11 *(domain of dependence inequality)*
Let $(t_0, x_0) \in \mathbb{R}^{n+1}$ with $t_0 > 0$. We denote by Ω the conical domain bounded by the backward characteristic cone with apex at (t_0, x_0) and by the plane $t = 0$. Let $u \in C^2(\overline{\Omega})$ be a classical

solution of the Klein-Gordon equation $u_{tt} - \triangle u + m^2 u = 0$. Then the following inequality holds:

$$E(u, K(x_0, t_0 - t)) \leq E(u, K(x_0, t_0)) \text{ for } t \in [0, t_0].$$

The domain of dependence property helps to get a uniqueness result.

Corollary 3.5 *The Cauchy problem*

$$u_{tt} - \triangle u + m^2 u = 0 \ , \quad u(0,x) = \varphi(x) \ , \quad u_t(0,x) = \psi(x)$$

possesses at most one classical solution $u \in C^2([0,\infty) \times \mathbb{R}^n)$ if the data are supposed to be sufficiently smooth.

Exercise 31 Prove the statement of Theorem 3.11!

3.3.2 Representation of solutions by using Fourier multipliers

We will study the Cauchy problem

$$u_{tt} - \triangle u + m^2 u = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x), \;\; x \in \mathbb{R}^n, \;\; n \geq 1.$$

Applying the *partial Fourier transformation* ($v(t,\xi) = F_{x \to \xi}(u(t,x))$) we obtain the auxiliary Cauchy problem

$$v_{tt} + \langle \xi \rangle_m^2 v = 0, \;\; v(0,\xi) = F(\varphi)(\xi), \;\; v_t(0,\xi) = F(\psi)(\xi).$$

Analogous to the approach from Section 3.2.6 we have

$$v(t,\xi) = \cos(\langle \xi \rangle_m t) F(\varphi)(\xi) + \frac{\sin(\langle \xi \rangle_m t)}{\langle \xi \rangle_m} F(\psi)(\xi).$$

Supposing for the moment the validity of *Fourier's inversion formula* $u(t,x) = F^{-1}_{\xi \to x}(F_{x \to \xi}(u(t,x))$ (this we have to check at the end) brings

$$u(t,x) = F^{-1}_{\xi \to x}\Big(\cos(\langle \xi \rangle_m t) F(\varphi)(\xi) \Big) + F^{-1}_{\xi \to x} \left(\frac{\sin(\langle \xi \rangle_m t)}{\langle \xi \rangle_m} F(\psi)(\xi) \right).$$

This is the desired *representation of solutions*. Let us given data $\varphi \in H^s(\mathbb{R}^n)$ and $\psi \in H^{s-1}(\mathbb{R}^n)$ with $s \geq 1$. From Theorem 3.11 it follows that the solution has a total energy for all $t \geq 0$. Analogous to the proof of Theorem 3.7 one can show the following statement:

Theorem 3.12 *Under the assumptions $\varphi \in H^s(\mathbb{R}^n)$ and $\psi \in H^{s-1}(\mathbb{R}^n)$, $s \geq 1$, $n \geq 1$ the Cauchy problem*

$$u_{tt} - \Delta u + m^2 u = 0, \; u(0,x) = \varphi(x), \; u_t(0,x) = \psi(x)$$

has a unique solution $u \in C([0,T], H^s(\mathbb{R}^n)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^n))$. The solution depends continuously on the data.

Remark: The statements of Theorem 3.7 and Theorem 3.12 coincide. The mass term or potential has no important influence on the regularity of solutions. But mass terms have an influence on energy estimates as one can see in Theorems 3.10 and 3.11.

3.3.3 von Wahl's transformation

The von Wahl's transformation allows to *transfer Cauchy problems for the Klein-Gordon equation to Cauchy problems for the wave equation.* With $x = (x_1, \cdots, x_n)$ let us consider the Cauchy problem

$$u_{tt} - \triangle u + m^2 u = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x).$$

We introduce

$$v(t, x_1, \cdots, x_n, x_{n+1}) := e^{-i\,m\,x_{n+1}} u(t, x_1, \cdots, x_n).$$

Evidently,

$$v_{tt} - \triangle v = 0, \quad v(0,x) = e^{-i\,m\,x_{n+1}} u_0(x_1, \cdots, x_n), \quad v_t(0,x) = e^{-i\,m\,x_{n+1}} u_1(x_1, \cdots, x_n).$$

In order to apply results from Section 3.2.6 we have to guarantee the correct function spaces for the data. We need for example $(\nabla_x v_0, v_1) \in L^2 \times L^2$. For v_1 it holds:

$$\int\limits_{\mathbb{R}^{n+1}} |v_1(x_1, \cdots, x_{n+1})|^2 dx = \int\limits_{\mathbb{R}^{n+1}} |u_1(x_1, \cdots, x_n)|^2 dx$$
$$= \int\limits_{-\infty}^{\infty} \int\limits_{\mathbb{R}^n} |u_1(x_1, \cdots, x_n)|^2 dx\, dx_{n+1} = \infty.$$

Consequently, the desired L^2 property of the data is not satisfied. Thus we cannot apply directly the results from Section 3.2.6.

Question: Why could one apply three decades before the von Wahl's transformation?
Answer: If we are interested in well-posedness results for the Cauchy problem

$$u_{tt} - \triangle u + m^2 u = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x)$$

with $(\varphi, \psi) \in C^k(\mathbb{R}^n) \cap C^{k-1}(\mathbb{R}^n)$, then one can apply von Wahl's transformation. One obtains a Cauchy problem for the wave equation with data from $C^k(\mathbb{R}^{n+1}) \cap C^{k-1}(\mathbb{R}^{n+1})$. Hence, we are able to apply Theorems 3.3 or 3.4. After backward transformation one has to prove the optimality of the obtained results.

3.4 Damped wave equation

Exercise 32 Refresh your knowledge about the *damped harmonic oscillator* from the course "Ordinary differential equations".

Let us devote to the Cauchy problem

$$u_{tt} - \triangle u + u_t = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x).$$

As for the classical wave equation we introduce the total energy

$$E(u)(t) = \frac{1}{2} \int\limits_{\mathbb{R}^n} \Big(|\nabla_x u(t,x)|^2 + |u_t(t,x)|^2 \Big) dx.$$

3.4.1 Energy estimates

First of all we are interested in energy estimates following from differentiation of the energy $E(u)(t)$ with respect to t and partial integration. It holds

$$\begin{aligned} E'(u)(t) &= \frac{1}{2} \int\limits_{\mathbb{R}^n} (2\nabla_x u \cdot \nabla_x u_t + 2 u_t u_{tt}) dx \\ &= \int\limits_{\mathbb{R}^n} (\nabla_x u \cdot \nabla_x u_t + u_t(\triangle u - u_t)) dx = \int\limits_{\mathbb{R}^n} -u_t(t,x)^2 dx \leq 0. \end{aligned}$$

Thus the energy is decreasing for increasing t. This seems to be no surprise because of the damping term. It arises the question for the behavior of the energy for $t \to \infty$. Of special interest is the question if the energy $E(u)(t)$ tends to 0 for $t \to \infty$. Such a behavior is called *decay.*

3.4.2 Representation of solutions by using Fourier multipliers

We will deal with the Cauchy problem

$$u_{tt} - \triangle u + u_t = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x).$$

Step 1 Transformation of the dissipation into a mass or a potential

We introduce the function $w = w(t,x)$ as $w(t,x) := e^{\frac{1}{2}t} u(t,x)$. Then w satisfies the partial differential equation

$$w_{tt} - \triangle w - \frac{1}{4} w = 0\,, \;\; w(0,x) = \varphi(x), \quad w_t(0,x) = \frac{1}{2}\varphi(x) + \psi(x).$$

In opposite to the Klein-Gordon equation it appears a *negative mass term.* This negative mass needs some special considerations.

Step 2 Application of partial Fourier transformation

The application of partial Fourier transformation gives an ordinary differential equation for $v = v(t,\xi) = F_{x\to\xi}(w)(t,\xi)$:

$$v_{tt} + \Big(|\xi|^2 - \frac{1}{4}\Big)v = 0, \;\; v(0,\xi) = v_0(\xi) = F(\varphi)(\xi), \;\; v_t(0,\xi) = v_1(\xi) = \frac{1}{2}\,F(\varphi)(\xi) + F(\psi)(\xi).$$

We carry out a distinction of cases for $\{\xi \in \mathbb{R}^n : |\xi| < \frac{1}{2}\}$, the mass term $|\xi|^2 - \frac{1}{4}$ is negative, and for $\{\xi \in \mathbb{R}^n : |\xi| > \frac{1}{2}\}$, the mass term $|\xi|^2 - \frac{1}{4}$ is positive.

Case 1 $\{\xi : |\xi| > \frac{1}{2}\}$

Using $|\xi|^2 > \frac{1}{4}$ we can define a new positive variable $|\eta|$ satisfying $|\eta|^2 := |\xi|^2 - \frac{1}{4} > 0$. So we get the ordinary differential equation $v_{tt} + |\eta|^2 v = 0$. Taking account of the results from Section 3.2.6 we obtain immediately the following representation of solution $v(t,\xi)$:

$$v(t,\xi) \;=\; \cos\Big(\sqrt{|\xi|^2 - \frac{1}{4}}\;t\Big)\,v_0(\xi) + \frac{\sin\Big(\sqrt{|\xi|^2 - \frac{1}{4}}\;t\Big)}{\sqrt{|\xi|^2 - \frac{1}{4}}}\,v_1(\xi).$$

Case 2 $\{\xi : |\xi| < \frac{1}{2}\}$

The solution for the transformed differential equation is

$$\begin{aligned} v(t,\xi) &= \left(\frac{v_0(\xi)}{2} - \frac{v_1(\xi)}{\sqrt{1-4|\xi|^2}}\right)e^{-\frac{1}{2}\sqrt{1-4|\xi|^2}t} + \left(\frac{v_0(\xi)}{2} + \frac{v_1(\xi)}{\sqrt{1-4|\xi|^2}}\right)e^{\frac{1}{2}\sqrt{1-4|\xi|^2}t} \\ &= v_0(\xi)\cosh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big) + \frac{2v_1(\xi)}{\sqrt{1-4|\xi|^2}}\sinh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big). \end{aligned}$$

If we consider the Cauchy problem

$$u_{tt} - \triangle u + u_t = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x)$$

with data $\varphi \in H^s$ and $\psi \in H^{s-1}$, then we conclude from the above representations of solutions the next result (pay attention that only the *behavior for large frequencies is important for the regularity of solutions*, the continuity with respect to t is proved as in Theorem 3.7):

Theorem 3.13 *Let the data $\varphi \in H^s(\mathbb{R}^n)$ and $\psi \in H^{s-1}(\mathbb{R}^n)$, $s \geq 1$, $n \geq 1$ be given for the Cauchy problem*

$$u_{tt} - \Delta u + u_t = 0,\; u(0,x) = \varphi(x),\; u_t(0,x) = \psi(x).$$

Then there exists a uniquely determined solution $u \in C([0,T], H^s(\mathbb{R}^n)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^n))$. The solution depends continuously on the data.

Remark: The statements of Theorem 3.7 and Theorem 3.13 coincide. The dissipation term has no important influence on the regularity of solutions. Dissipation terms have an essential influence on energy estimates, they produce a *decay of the energy.* This will be explained in the next section.

Exercise 33 Let us consider the Cauchy problem for a very large damped membrane

$$u_{tt} - c^2 \triangle u + k u_t = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),\ x \in \mathbb{R}^2.$$

Solve this problem by the aid of the following transformations:

$$u(t,x) = \exp(-kt/2) w(t,x), \quad v(t,x_1,x_2,x_3) = w(t,x_1,x_2)\exp(kx_3/(2c)).$$

Exercise 34 We are interested in the Cauchy problem

$$u_{tt} - u_{xx} + \varepsilon u_t = 0,\ u(0,x,\varepsilon) = \varphi(x),\ u_t(0,x,\varepsilon) = \psi(x),\ x \in \mathbb{R}^1,$$

with sufficiently smooth data φ and ψ. Let $u = u(t,x,\varepsilon)$ be the unique solution of this Cauchy problem. Show that we have for every fixed (t,x) the relation $\lim_{\varepsilon \to 0} u(t,x,\varepsilon) = w(t,x)$, where $w = w(t,x)$ solves the Cauchy problem

$$w_{tt} - w_{xx} = 0,\ \ w(0,x) = \varphi(x),\ \ w_t(0,x) = \psi(x),\ x \in \mathbb{R}^1.$$

3.4.3 Decay behavior and decay rate

The *application of the partial Fourier transformation* and a very precise *WKB analysis, the abbreviation is due to the physicists Wentzel, Kramer and Broullion* (this is a precise analysis to study the Fourier multipliers appearing in the representation of solutions) allows us to estimate in a better way than it is done in Section 3.4.1 the energy of solutions to the damped wave equation. We are able to derive an *optimal decay behavior* with an optimal *decay rate.*

Theorem 3.14 *The solution to the Cauchy problem*

$$u_{tt} - \triangle u + u_t = 0,\ \ u(0,x) = \varphi(x),\ \ u_t(0,x) = \psi(x)$$

with data $\varphi \in H^1$ and $\psi \in L^2$ satisfies the following estimates:

$$\begin{aligned} \|\nabla_x u(t,\cdot)\|_{L^2} &\le C(1+t)^{-\frac{1}{2}}\Big(\|\psi\|_{L^2} + \|\varphi\|_{H^1}\Big), \\ \|u_t(t,\cdot)\|_{L^2} &\le C(1+t)^{-1}\Big(\|\psi\|_{L^2} + \|\varphi\|_{H^1}\Big), \end{aligned}$$

and consequently, the energy satisfies

$$E(u)(t) \le C(1+t)^{-1}\Big(\|\psi\|_{L^2}^2 + \|\varphi\|_{H^1}^2\Big).$$

Proof: *Step 1 Transformation of energy into the phase space*
Let $\hat{u}$ the Fourier transform of u, that is, $\hat{u}(t,\xi) = F_{x\to\xi}(u)(t,\xi)$. As in Section 3.3.1 we can transfer the energy into the phase space as follows:

$$E(u)(t) = \frac{1}{2}\left(\|\nabla_x u(t,\cdot)\|^2_{L^2} + \|u_t(t,\cdot)\|^2_{L^2}\right) = \frac{1}{2}\left(\||\xi|\hat{u}(t,\cdot)\|^2_{L^2} + \|\hat{u}_t(t,\cdot)\|^2_{L^2}\right).$$

By $u(t,x) = e^{-\frac{1}{2}t}w(t,x)$ and $v(t,\xi) = F_{x\to\xi}(w)(t,\xi)$ it follows $\hat{u}(t,\xi) = e^{-\frac{1}{2}t}v(t,\xi)$. For the *elastic energy* we will use

$$|\xi|\hat{u}(t,\xi) = e^{-\frac{1}{2}t}\ |\xi|v(t,\xi),$$

for the kinetic energy we will use

$$\hat{u}_t(t,\xi) = e^{-\frac{1}{2}t}\ \Big(v_t(t,\xi)\ - \frac{1}{2}\ v(t,\xi)\Big).$$

Step 2 Estimate of the elastic energy
We will distinguish several cases.
Case 1 $\{\xi:\ |\xi| > \frac{1}{2}\}$
First we notice $|\xi|\hat{u}(t,\xi) = e^{-\frac{1}{2}t}\Big(\cos(\sqrt{|\xi|^2-\frac{1}{4}}\ t)\ |\xi|v_0(\xi) + t\ \frac{\sin(\sqrt{|\xi|^2-\frac{1}{4}}t)}{\sqrt{|\xi|^2-\frac{1}{4}}t}\ |\xi|v_1(\xi)\Big)$. This helps us to estimate the elastic energy $\|\nabla_x u(t,\cdot)\|^2_{L^2}$. We have

$$\||\xi|\hat{u}(t,\xi)\|^2_{L^2\{|\xi|>\frac{1}{2}\}} = \int\limits_{|\xi|>\frac{1}{2}} |\xi|^2|\hat{u}(t,\xi)|^2 d\xi \le 2\Big(\int\limits_{|\xi|>\frac{1}{2}} e^{-t}|\xi|^2|v_0(\xi)|^2 d\xi$$

$$+ \int\limits_{\frac{1}{2}<|\xi|\le 1} \underbrace{\frac{\sin^2(\sqrt{|\xi|^2-\frac{1}{4}}t)}{\sqrt{|\xi|^2-\frac{1}{4}}^2\ t^2}}_{\frac{\sin^2\alpha}{\alpha^2}\le C} t^2 e^{-t}|\xi|^2|v_1(\xi)|^2 d\xi\ + \int\limits_{|\xi|\ge 1} \underbrace{\frac{1}{|\xi|^2-\frac{1}{4}}|\xi|^2}_{\le C} e^{-t}|v_1(\xi)|^2 d\xi\Big)$$

$$\le 2e^{-t}\int\limits_{\mathbb{R}^n} |\xi|^2|v_0(\xi)|^2 d\xi\ +\ Ct^2e^{-t}\int\limits_{\mathbb{R}^n}|v_1(\xi)|^2 d\xi\ +\ Ce^{-t}\int\limits_{\mathbb{R}^n}|v_1(\xi)|^2 d\xi.$$

Summarizing *these terms decay exponentially.* It holds

$$\int\limits_{|\xi|>\frac{1}{2}} |\xi|^2|\hat{u}(t,\xi)|^2 d\xi \le Ct^2e^{-t}\int\limits_{\mathbb{R}^n}(|\xi|^2|v_0(\xi)|^2 + |v_1(\xi)|^2)d\xi.$$

Case 2 $\{\xi : |\xi| < \frac{1}{2}\}$

To estimate the elastic energy we use

$$
\begin{aligned}
|\xi|\hat{u}(t,\xi) &= |\xi|e^{-\frac{1}{2}t}\Big(\Big(\frac{v_0(\xi)}{2} - \frac{v_1(\xi)}{\sqrt{1-4|\xi|^2}}\Big)e^{-\frac{1}{2}\sqrt{1-4|\xi|^2}t} \\
&\quad + \Big(\frac{v_0(\xi)}{2} + \frac{v_1(\xi)}{\sqrt{1-4|\xi|^2}}\Big)e^{\frac{1}{2}\sqrt{1-4|\xi|^2}t}\Big) \\
&= v_0(\xi)|\xi|\cosh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big)e^{-\frac{1}{2}t} + \frac{2v_1(\xi)|\xi|}{\sqrt{1-4|\xi|^2}}\sinh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big)e^{-\frac{1}{2}t}.
\end{aligned}
$$

We divide the interval $[0,\frac{1}{2})$ in two subintervals.

a) $\{\xi : |\xi| \in [\frac{1}{4},\ \frac{1}{2})\}$:

Here we estimate the elastic energy as follows:

$$
\begin{aligned}
|\xi||\hat{u}(t,\xi)| &= \Big|\ v_0(\xi)|\xi|\underbrace{\cosh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big)}_{\leq \cosh(\frac{\sqrt{3}}{4}t)}e^{-\frac{1}{2}t} + \underbrace{\frac{\sinh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big)}{\frac{1}{2}\sqrt{1-4|\xi|^2}t}t}_{\leq C\cosh(\frac{\sqrt{3}}{4}t)}\ v_1(\xi)|\xi|e^{-\frac{1}{2}t}\ \Big| \\
&\leq \Big|\ v_0(\xi)|\xi|\underbrace{\cosh(\frac{\sqrt{3}}{4}t)e^{-\frac{1}{2}t}}_{\leq e^{-\delta t},\ \delta>0} + C\underbrace{v_1(\xi)|\xi|}_{\leq |v_1(\xi)|}\underbrace{\cosh(\frac{\sqrt{3}}{4}t)\ e^{-\frac{1}{2}t}}_{\leq e^{-\delta t},\ \delta>0}\ \Big|,
\end{aligned}
$$

and obtain

$$
\int\limits_{\frac{1}{4}\leq|\xi|<\frac{1}{2}} |\xi|^2|\hat{u}(t,\xi)|^2 d\xi \leq Ce^{-\delta t}\int\limits_{\mathbb{R}^n}(|\xi|^2|v_0(\xi)|^2 + |v_1(\xi)|^2)d\xi.
$$

b) $\{\xi : |\xi| \in [0,\ \frac{1}{4})\}$:

Now we use for $|\xi| < \frac{1}{2}$ the inequality $-4|\xi|^2 \leq -1 + \sqrt{1-4|\xi|^2} \leq -2|\xi|^2$. With this inequality we proceed as follows:

$$
\begin{aligned}
\int\limits_{|\xi|<\frac{1}{4}} |\xi|^2|\hat{u}(t,\xi)|^2 d\xi &\leq \int\limits_{|\xi|<\frac{1}{4}} (|v_1(\xi)|^2|\xi|^2 + |v_0(\xi)|^2|\xi|^2)(\underbrace{e^{-t-\sqrt{1-4|\xi|^2}t}}_{\leq e^{-t}} + \underbrace{e^{-t+\sqrt{1-4|\xi|^2}t}}_{\leq e^{-2|\xi|^2t}})d\xi \\
&\leq Ce^{-t}\int\limits_{|\xi|<\frac{1}{4}} (|v_1(\xi)|^2|\xi|^2 + |v_0(\xi)|^2|\xi|^2)d\xi \\
&\quad + C\int\limits_{|\xi|<\frac{1}{4}} (|v_1(\xi)|^2 + |v_0(\xi)|^2)|\xi|^2 e^{-2|\xi|^2t}d\xi.
\end{aligned}
$$

By using the norm inequality $\|\cdot\|_{L^2} \leq \|\cdot\|_{L^\infty}\|\cdot\|_{L^2}$ we get for the second term of the right-hand side of the last inequality

$$
\begin{aligned}
&C \int\limits_{|\xi|<\frac{1}{4}} (|v_1(\xi)|^2 + |v_0(\xi)|^2)|\xi|^2 e^{-2|\xi|^2 t} d\xi \\
&\quad \leq C \sup_{|\xi|<\frac{1}{4},\, t\geq 1} \frac{t|\xi|^2}{t} e^{-2|\xi|^2 t} \int\limits_{\mathbb{R}^n} (|v_1(\xi)|^2 + |v_0(\xi)|^2) d\xi \\
&\quad \leq C \frac{1}{t} \underbrace{\sup_{|\xi|<\frac{1}{4},\, t\geq 1} t|\xi|^2 e^{-2|\xi|^2 t}}_{\leq C} \int\limits_{\mathbb{R}^n} (|v_1(\xi)|^2 + |v_0(\xi)|^2) d\xi.
\end{aligned}
$$

Summarizing we have shown for small frequencies

$$
\int\limits_{|\xi|<\frac{1}{4}} |\xi|^2 |\hat{u}(t,\xi)|^2 d\xi \;\;\leq\;\; C(1+t)^{-1} \int\limits_{\mathbb{R}^n} (|v_0(\xi)|^2 + |\xi|^2 |v_0(\xi)|^2 + |v_1(\xi)|^2) d\xi.
$$

Step 3   Estimate of the kinetic energy

Finally, we deal with the kinetic energy. We will use the identity $\|u_t(t,\xi)\|^2_{L^2} = \|\hat{u}_t(t,\xi)\|^2_{L^2}$ with $\hat{u}_t(t,\xi) = e^{-\frac{1}{2}t}\Big(v_t(t,\xi) \; - \frac{1}{2}\, v(t,\xi)\Big)$.

Case 1 $\{\xi : |\xi| > \frac{1}{2}\}$

We need

$$
v_t(t,\xi) = -\sqrt{|\xi|^2 - \frac{1}{4}} \sin\Big(\sqrt{|\xi|^2 - \frac{1}{4}}\; t\Big) v_0(\xi) + \cos\Big(\sqrt{|\xi|^2 - \frac{1}{4}}\; t\Big) v_1(\xi).
$$

By using the last equation we obtain

$$
\begin{aligned}
\hat{u}_t(t,\xi) \;\;=\;\; & e^{-\frac{1}{2}t}\Big(v_1(\xi)\Big(\cos(\sqrt{|\xi|^2 - \frac{1}{4}}\; t) \;-\; \frac{1}{2}\frac{\sin(\sqrt{|\xi|^2 - \frac{1}{4}}\; t)}{\sqrt{|\xi|^2 \;-\; \frac{1}{4}}}\Big) \\
& - v_0(\xi)\Big(\frac{1}{2}\cos(\sqrt{|\xi|^2 - \frac{1}{4}}\; t) \;+\; \sqrt{|\xi|^2 - \frac{1}{4}} \sin(\sqrt{|\xi|^2 - \frac{1}{4}}\; t)\Big)\Big).
\end{aligned}
$$

Repeating the reasoning to estimate the elastic energy gives

$$
\begin{aligned}
\|u_t(t,\cdot)\|^2_{L^2\{|\xi|>\frac{1}{2}\}} \leq C & \int\limits_{|\xi|>\frac{1}{2}} e^{-t}|v_1(\xi)|^2 \underbrace{\Big(\cos(\sqrt{|\xi|^2 - \frac{1}{4}}\; t) - \frac{1}{2}\frac{\sin(\sqrt{|\xi|^2 - \frac{1}{4}}\; t)}{\sqrt{|\xi|^2 \;-\; \frac{1}{4}}}\Big)^2}_{\leq C\, t^2} d\xi \\
& + C \int\limits_{|\xi|>\frac{1}{2}} e^{-t}|v_0(\xi)|^2 \Big(\sqrt{|\xi|^2 - \frac{1}{4}} \sin(\sqrt{|\xi|^2 - \frac{1}{4}}\; t) \;+\; \underbrace{\frac{1}{2}\cos(\sqrt{|\xi|^2 - \frac{1}{4}}\; t)}_{\leq C}\Big)^2 d\xi.
\end{aligned}
$$

The inequality $(|\xi|^2 - \frac{1}{4})\sin^2(\sqrt{|\xi|^2 - \frac{1}{4}}\, t) \leq |\xi|^2$ brings for $\{\xi : |\xi| > \frac{1}{2}\}$

$$\int\limits_{|\xi|>\frac{1}{2}} |\xi|^2 |\hat{u}(t,\xi)|^2 d\xi \leq C t^2 e^{-t} \int\limits_{\mathbb{R}^n} (|\xi|^2 |v_0(\xi)|^2 + |v_1(\xi)|^2) d\xi.$$

Case 2: $\{\xi : |\xi| < \frac{1}{2}\}$
After the determination of $v_t(t,\xi)$ we get immediately

$$\begin{aligned}\hat{u}_t(t,\xi) &= \frac{1}{2} e^{-\frac{1}{2}t}\Big(\sqrt{1-4|\xi|^2}\sinh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big) - \cosh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big)\Big)v_0(\xi)\\ &+ e^{-\frac{1}{2}t}\Big(\cosh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big) - \frac{1}{\sqrt{1-4|\xi|^2}}\sinh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big)\Big)v_1(\xi).\end{aligned}$$

We divide again the interval $[0,\ \frac{1}{2})$.
a) $\{\xi : |\xi| \in [\frac{1}{4},\ \frac{1}{2})\}$:
Here we can show the exponential decay of the energy. On the one hand we use

$$\cosh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big) + \sinh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big) \leq 2\cosh(\frac{\sqrt{3}}{4}t),$$

on the other hand we use

$$\Big|\frac{1}{\sqrt{1-4|\xi|^2}}\sinh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big)\Big| \leq C_\varepsilon t \text{ für } \frac{1}{2}\sqrt{1-4|\xi|^2}t \leq \varepsilon.$$

Both together gives

$$\|\hat{u}_t(t,\xi)\|^2_{L^2\{|\xi|\in[\frac{1}{4},\ \frac{1}{2})\}} \;\leq\; C e^{-\delta t}\int\limits_{\mathbb{R}^n}(|\xi|^2|v_0(\xi)|^2 + |v_1(\xi)|^2)d\xi$$

with a suitable positive δ.
b) $\{\xi : |\xi| < \frac{1}{4}\}$:
In this case we obtain

$$\hat{u}_t(t,\xi) = \Big(\frac{v_0(\xi)}{4} + \frac{v_1(\xi)}{2\sqrt{1-4|\xi|^2}}\Big)(\sqrt{1-4|\xi|^2}-1)e^{-\frac{1}{2}t-\frac{1}{2}\sqrt{1-4|\xi|^2}t}.$$

Hence, we can estimate as follows:

$$|\hat{u}_t(t,\xi)| \leq \Big|\Big(\frac{v_1(\xi)}{2\sqrt{1-4|\xi|^2}} + \frac{v_0(\xi)}{4}\Big)\Big(\underbrace{\sqrt{1-4|\xi|^2}-1}_{\leq -2|\xi|^2}\Big)\underbrace{e^{-\frac{1}{2}t+\frac{1}{2}\sqrt{1-4|\xi|^2}t}}_{\leq e^{-|\xi|^2 t},\ |\xi|<\frac{1}{2}}\Big|.$$

Recalling the estimates for the elastic energy a similar approach leads to

$$\begin{aligned}
\|\hat{u}_t(t,\xi)\|^2_{L^2\{|\xi|<\frac{1}{4}\}} &\le C \int\limits_{|\xi|<\frac{1}{4}} \left(|v_1(\xi)|^2+|v_0(\xi)|^2\right)|\xi|^4\left(e^{-t}+e^{-2|\xi|^2t}\right)d\xi \\
&\le Ce^{-t}\int\limits_{|\xi|<\frac{1}{4}}\left(|v_1(\xi)|^2+|v_0(\xi)|^2\right)d\xi \;+\; C\int\limits_{|\xi|<\frac{1}{4}}\left(|v_1(\xi)|^2+|v_0(\xi)|^2\right)|\xi|^4e^{-2|\xi|^2t}d\xi \\
&\le Ce^{-t}\int\limits_{|\xi|<\frac{1}{4}}\left(|v_1(\xi)|^2+|v_0(\xi)|^2\right)d\xi \\
&+ C\frac{1}{t^2}\underbrace{\sup_{|\xi|<\frac{1}{4},\,t\ge1} t^2|\xi|^4e^{-2|\xi|^2t}}_{\le c}\int\limits_{|\xi|<\frac{1}{4}}\left(|v_1(\xi)|^2+|v_0(\xi)|^2\right)d\xi \\
&\le Ce^{-t}\int\limits_{|\xi|<\frac{1}{4}}\left(|v_1(\xi)|^2+|v_0(\xi)|^2\right)d\xi \;+\; \frac{C}{(1+t)^2}\int\limits_{|\xi|<\frac{1}{4}}\left(|v_1(\xi)|^2+|v_0(\xi)|^2\right)d\xi \\
&\le \frac{C}{(1+t)^2}\int\limits_{\mathbb{R}^n}\left(|v_1(\xi)|^2+|v_0(\xi)|^2\right)d\xi.
\end{aligned}$$

Thus all statements from the theorem are proved. □

Question: Which part of the phase space does the decay behavior of the energy determine?
Answer: The decay behavior is determined by the small frequencies.
Question: Which property do the large frequencies influence?

3.5 Damped plate equation

The calculations of this section were carried out by Ms. Nguyen Thi Thu Huong during her stay at Bergakademie Freiberg from May to July 2007.

In this section we consider the following Cauchy problem for the damped plate equation

$$u_{tt}+(-\Delta)^2u+u_t=0,\;\; u(0,x)=\varphi(x),\;\; u_t(0,x)=\psi(x).$$

Let $u = u(t,x)$ be a solution of this Cauchy problem. Then we introduce its energy as follows:

$$E(u)(t)=\frac{1}{2}\int_{\mathbb{R}^n}(|u_t(t,x)|^2+|\triangle u(t,x)|^2)dx.$$

Some simple calculations imply

$$E'(u)(t)=-\int_{\mathbb{R}^n}u_t^2(t,x)dx\le 0.$$

So, the energy decreases and we will investigate it when t tends to infinity.

3.5.1 Estimates for the energy

Using partial Fourier transformation with respect to x, and denote $F_{x\to\xi}(u(t,x)) = v(t,\xi)$ we get

$$v_{tt} + v_t + |\xi|^4 v = 0,\ v(0,\xi) = F(\varphi)(\xi),\ v_t(0,\xi) = F(\psi)(\xi).$$

The characteristic polynomial of the above differential equation is

$$\lambda^2 + \lambda + |\xi|^4 = 0.$$

To obtain eigenvalues we must divide into two cases:
1.case: $\left\{\xi : |\xi| < \frac{1}{\sqrt{2}}\right\}$

Then $\lambda = \dfrac{-1 \pm \sqrt{1-4|\xi|^4}}{2}$ and

$$v(t,\xi) = e^{-\frac{t}{2}}\Big[\cosh\Big(\sqrt{\frac{1}{4}-|\xi|^4}t\Big)v_0(\xi) + \frac{\sinh\Big(\sqrt{\frac{1}{4}-|\xi|^4}t\Big)}{\sqrt{\frac{1}{4}-|\xi|^4}}v_1(\xi)\Big].$$

2.case: $\left\{\xi : |\xi| > \dfrac{1}{\sqrt{2}}\right\}$

Then $\lambda = \dfrac{-1 \pm i\sqrt{4|\xi|^4-1}}{2}$ and

$$v(t,\xi) = e^{-\frac{t}{2}}\Big[\cos\Big(\sqrt{|\xi|^4-\frac{1}{4}}t\Big)v_0(\xi) + \frac{\sin\Big(\sqrt{|\xi|^4-\frac{1}{4}}t\Big)}{\sqrt{|\xi|^4-\frac{1}{4}}}v_1(\xi)\Big],$$

where

$$v_0(\xi) = F(\varphi)(\xi),\ v_1(\xi) = \frac{1}{2}F(\varphi)(\xi) + F(\psi)(\xi).$$

The line $|\xi| = \dfrac{1}{\sqrt{2}}$ is called the *separating line*. In fact this line divides between elliptic WKB analysis (1.case) and hyperbolic WKB analysis (2.case).
Applying the inverse Fourier transformation we obtain the desired representation of solutions. In the *large frequencies domain* the solution has *hyperbolic structure* and the *exponential decay* comes in. The regularity of the solution does not been effected. Indeed, we have

$$\langle\xi\rangle^s v(t,\xi) = e^{-\frac{t}{2}}\Big[\langle\xi\rangle^s\cos\Big(\sqrt{|\xi|^4-\frac{1}{4}}t\Big)v_0(\xi) + \langle\xi\rangle^s\frac{\sin\Big(\sqrt{|\xi|^4-\frac{1}{4}}t\Big)}{\sqrt{|\xi|^4-\frac{1}{4}}}v_1(\xi)\Big],$$

$$
\begin{aligned}
\langle\xi\rangle^s|v(t,\xi)| &\leq e^{-\frac{t}{2}}\Big[\langle\xi\rangle^s|v_0(\xi)| + \langle\xi\rangle^{s-2}\Big|\frac{\sin\Big(\sqrt{|\xi|^4-\frac{1}{4}}t\Big)(1+|\xi|^2)}{\sqrt{|\xi|^4-\frac{1}{4}}}\Big||v_1(\xi)|\Big],\\
\langle\xi\rangle^s|v(t,\xi)| &\leq e^{-\frac{t}{2}}\Big[\langle\xi\rangle^s|v_0(\xi)| + \langle\xi\rangle^{s-2}C(\varepsilon,T)|v_1(\xi)|\Big].
\end{aligned}
$$

So if $\varphi \in H^s$ and $\psi \in H^{s-2}$, then $u(t,\cdot) \in H^s$.

But in the *small frequencies domain*, like in the case of the damped wave equation, the solution has *elliptic structure* and the decay rate changes.

Using Parseval's relation we have

$$
E(u)(t) = \frac{1}{2}(\|u_t(t,\cdot)\|_{L_2}^2 + \|\triangle u(t,\cdot)\|_{L_2}^2) = \frac{1}{2}\int_{\mathbb{R}^n}(|v_t(t,\xi)|^2 + ||\xi|^2 v(t,\xi)|^2)d\xi.
$$

We will prove that for data $\varphi \in H^2$ and $\psi \in L^2$ the decay rates are as follows:

$$
\begin{aligned}
\|u_t(t,\cdot)\|_{L_2} &\leq C(1+t)^{-1}(\|\varphi\|_{H^2} + \|\psi\|_{L_2}),\\
\|\triangle u(t,\cdot)\|_{L_2} &\leq C(1+t)^{-\frac{1}{2}}(\|\varphi\|_{H^2} + \|\psi\|_{L_2}),
\end{aligned}
$$

thus,

$$
E(u)(t) \leq C(1+t)^{-1}(\|\varphi\|_{H^2} + \|\psi\|_{L_2}).
$$

In the following considerations we will use the next inequalities: $\cos\alpha \leq 1$, $\dfrac{\sin\alpha}{\alpha} \leq 1$ when $|\alpha|$ is small, $\sin\alpha \leq 1$, $\dfrac{\sinh t}{t} \leq \cosh t, t \geq 0$.

Estimate of $\|\triangle u(t,\cdot)\|_{L_2} = \||\xi|^2 v(t,\cdot)\|_{L_2}$

a) Case $|\xi| > \dfrac{1}{\sqrt{2}}$. The representation of the solution shows

$$
\begin{aligned}
\int_{|\xi|>\frac{1}{\sqrt{2}}}|\xi|^4|v(t,\xi)|^2 d\xi &= \int_{|\xi|>\frac{1}{\sqrt{2}}}|\xi|^4 e^{-t}\Big[\cos\Big(\sqrt{|\xi|^4-\frac{1}{4}}t\Big)v_0(\xi) + \frac{\sin\Big(\sqrt{|\xi|^4-\frac{1}{4}}t\Big)}{\sqrt{|\xi|^4-\frac{1}{4}}}v_1(\xi)\Big]^2 d\xi\\
&\leq 2\int_{|\xi|>\frac{1}{\sqrt{2}}} e^{-t}\Big[|\xi|^4|v_0(\xi)|^2 + |\xi|^4\frac{\sin^2\Big(\sqrt{|\xi|^4-\frac{1}{4}}t\Big)}{|\xi|^4-\frac{1}{4}}|v_1(\xi)|^2\Big]d\xi.
\end{aligned}
$$

When $1 \geq |\xi| > \dfrac{1}{\sqrt{2}}$, then $\dfrac{\sin^2\left(\sqrt{|\xi|^4 - \frac{1}{4}}t\right)}{|\xi|^4 - \frac{1}{4}} \leq t^2$, and when $|\xi| > 1$, then $\dfrac{|\xi|^4}{|\xi|^4 - \frac{1}{4}} \leq C$.
Summarizing we have

$$\begin{aligned}\int_{|\xi|>\frac{1}{\sqrt{2}}} |\xi|^4|v(t,\xi)|^2 d\xi &\leq Ce^{-t}\int_{\mathbb{R}^n} |\xi|^4|v_0(\xi)|^2 d\xi + Ct^2e^{-t}\int_{1\geq|\xi|>\frac{1}{\sqrt{2}}} |v_1(\xi)|^2 d\xi + Ce^{-t}\int_{|\xi|>1} |v_1(\xi)|^2 d\xi \\ &\leq Ct^2e^{-t}\int_{\mathbb{R}^n}(|\xi|^4|v_0(\xi)|^2 + |v_1(\xi)|^2)d\xi.\end{aligned}$$

b) Case $|\xi| < \dfrac{1}{\sqrt{2}}$. The representation of the solution shows

$$|\xi|^2|v(t,\xi)| = e^{-\frac{t}{2}}\left||\xi|^2 v_0(\xi)\cosh\left(\sqrt{-|\xi|^4 + \frac{1}{4}}t\right) + |\xi|^2\frac{\sinh\left(\sqrt{-|\xi|^4 + \frac{1}{4}}t\right)}{\sqrt{-|\xi|^4 + \frac{1}{4}}}v_1(\xi)\right|.$$

When $\dfrac{1}{2} \leq |\xi| \leq \dfrac{1}{\sqrt{2}}$, using the fact that the cosine-hyperbolic function is increasing for non-negative arguments, we have

$$e^{-\frac{t}{2}}\cosh\left(\sqrt{-|\xi|^4 + \frac{1}{4}}t\right) \leq \cosh\left(\frac{\sqrt{3}}{4}t\right)e^{-\frac{t}{2}} \leq Ce^{-\delta t},$$

and

$$e^{-\frac{t}{2}}|\xi|^2\left|t\frac{\sinh\left(\sqrt{-|\xi|^4 + \frac{1}{4}}t\right)}{t\sqrt{-|\xi|^4 + \frac{1}{4}}}\right| \leq Ce^{-\frac{t}{2}}t\cosh\left(\frac{\sqrt{3}}{4}t\right) \leq Ce^{-\delta t}.$$

Hence

$$\int\limits_{|\xi|<\frac{1}{\sqrt{2}}} |\xi|^4|v(t,\xi)|^2 d\xi \leq Ce^{-\delta t}\int\limits_{|\xi|<\frac{1}{\sqrt{2}}}(|\xi|^4|v_0(\xi)|^2 + |v_1(\xi)|^2)d\xi \leq Ce^{-\delta t}\int\limits_{\mathbb{R}^n}(|\xi|^4|v_0(\xi)|^2 + |v_1(\xi)|^2)d\xi.$$

When $\dfrac{1}{2} \geq |\xi| \geq 0$, using that

$$v(t,\xi) = e^{-\frac{t}{2}}\left[\left(\frac{v_0(\xi)}{2} - \frac{v_1(\xi)}{\sqrt{1-4|\xi|^4}}\right)e^{-\frac{1}{2}\sqrt{1-4|\xi|^4}t} + \left(\frac{v_0(\xi)}{2} + \frac{v_1(\xi)}{\sqrt{1-4|\xi|^4}}\right)e^{\frac{1}{2}\sqrt{1-4|\xi|^4}t}\right],$$

then

$$|\xi|^2|v(t,\xi)|^2 \leq e^{-t}|\xi|^4[|v_0(\xi)|^2+|v_1(\xi)|^2]\Big(1+\frac{1}{1-4|\xi|^4}\Big)\Big(e^{-\sqrt{1-4|\xi|^4}t}+e^{\sqrt{1-4|\xi|^4}t}\Big),$$

$$\int\limits_{\frac{1}{2}\geq|\xi|}|\xi|^2|v(t,\xi)|^2d\xi \leq C\int\limits_{\frac{1}{2}\geq|\xi|}(|v_0(\xi)|^2|\xi|^4+|v_1(\xi)|^2|\xi|^4)(e^{-t-\sqrt{1-4|\xi|^4}t}+e^{-t+\sqrt{1-4|\xi|^4}t})d\xi.$$

Because in this domain we have $-4|\xi|^4 \leq -1+\sqrt{1-4|\xi|^4} \leq -2|\xi|^4$ we may conclude

$$\int\limits_{\frac{1}{2}\geq|\xi|}(|v_0(\xi)|^2|\xi|^4+|v_1(\xi)|^2|\xi|^4)e^{-t-\sqrt{1-4|\xi|^4}t}d\xi \leq Ce^{-t}\int\limits_{\mathbb{R}^n}(|v_0(\xi)|^2+|v_1(\xi)|^2)d\xi,$$

$$\int\limits_{\frac{1}{2}\geq|\xi|}(|v_0(\xi)|^2|\xi|^4+|v_1(\xi)|^2|\xi|^4)e^{-t+\sqrt{1-4|\xi|^4}t}d\xi \leq C\frac{1}{t}\sup_{|\xi|\leq\frac{1}{2},t\geq1}t|\xi|^4e^{-2t|\xi|^4}\int\limits_{\mathbb{R}^n}(|v_0(\xi)|^2+|v_1(\xi)|^2)d\xi,$$

$$\int\limits_{\frac{1}{2}\geq|\xi|}|v(t,\xi)|\xi|^2|^2d\xi \leq C(1+t)^{-1}\int\limits_{\mathbb{R}^n}(|v_0(\xi)|^2+|\xi|^4|v_0(\xi)|^2+|v_1(\xi)|^2)d\xi.$$

Summarizing we have shown that the decay rate for $\| \triangle u(t,\cdot)\|_{L_2}$ is $(1+t)^{-1}$.

Estimate of $\|u_t(t,\cdot)\|_{L_2} = \|v_t(t,\cdot)\|_{L_2}$

a) Case $|\xi| > \dfrac{1}{\sqrt{2}}$. The representation of the solution shows

$$\begin{aligned} v_t(t,\xi) = \quad & e^{-\frac{t}{2}}\Big[v_1(\xi)\Big(\cos\Big(\sqrt{|\xi|^4-\frac{1}{4}}t\Big)-\frac{1}{2}\frac{\sin\Big(\sqrt{|\xi|^4-\frac{1}{4}}t\Big)}{\sqrt{|\xi|^4-\frac{1}{4}}}\Big)- \\ & -v_0(\xi)\Big(\frac{1}{2}\cos\Big(\sqrt{|\xi|^4-\frac{1}{4}}t\Big)+\sqrt{|\xi|^4-\frac{1}{4}}\sin\Big(\sqrt{|\xi|^4-\frac{1}{4}}t\Big)\Big)\Big]. \end{aligned}$$

Using similar inequalities as in the corresponding proof in the case of damped wave equation we have

$$\Big(\cos\Big(\sqrt{|\xi|^4-\frac{1}{4}}t\Big)-\frac{1}{2}\frac{\sin\Big(\sqrt{|\xi|^4-\frac{1}{4}}t\Big)}{\sqrt{|\xi|^4-\frac{1}{4}}}\Big)^2 \leq Ct^2,$$

$$\Big|\frac{1}{2}\cos\Big(\sqrt{|\xi|^4-\frac{1}{4}}t\Big)+\sqrt{|\xi|^4-\frac{1}{4}}\sin\Big(\sqrt{|\xi|^4-\frac{1}{4}}t\Big)\Big| \leq C|\xi|^2,$$

$$\int\limits_{|\xi|>\frac{1}{\sqrt{2}}}|v_t(t,\xi)|^2d\xi \leq Ct^2e^{-t}\int\limits_{\mathbb{R}^n}\Big[|v_1(\xi)|^2+|v_0(\xi)|^2|\xi|^4d\xi\Big].$$

b) Case $|\xi| < \dfrac{1}{\sqrt{2}}$. The representation of the solution shows

$$v_t(t,\xi) = e^{-\frac{t}{2}}\Big[v_1(\xi)\Big(\cosh\Big(\sqrt{-|\xi|^4+\frac{1}{4}}t\Big) - \frac{\sinh\Big(\sqrt{-|\xi|^4+\frac{1}{4}}t\Big)}{\sqrt{-|\xi|^4+\frac{1}{4}}}\Big)$$

$$+\frac{1}{2}v_0(\xi)\Big(-\cosh\Big(\sqrt{-|\xi|^4+\frac{1}{4}}t\Big) + \sqrt{-|\xi|^4+\frac{1}{4}}\sinh\Big(\sqrt{-|\xi|^4+\frac{1}{4}}t\Big)\Big)\Big].$$

When $\dfrac{1}{2} \le |\xi| \le \dfrac{1}{\sqrt{2}}$ we have

$$|A| = \Big|\cosh\Big(\sqrt{-|\xi|^4+\frac{1}{4}}t\Big) - \frac{\sinh\Big(\sqrt{-|\xi|^4+\frac{1}{4}}t\Big)}{\sqrt{-|\xi|^4+\frac{1}{4}}}\Big|$$

$$\le \cosh\Big(\sqrt{-|\xi|^4+\frac{1}{4}}t\Big) + \frac{\sinh\Big(\sqrt{-|\xi|^4+\frac{1}{4}}t\Big)}{\sqrt{-|\xi|^4+\frac{1}{4}}t}t \le (Ct+1)\cosh\Big(\frac{\sqrt{3}}{4}\Big)t,$$

consequently, $\Big|e^{-\frac{t}{2}}A\Big| \le Cte^{-\delta_1 t} \le Ce^{-\delta t}$ with $\delta > 0$,

$$|B| = \Big|-\cosh\Big(\sqrt{-|\xi|^4+\frac{1}{4}}t\Big) + \sqrt{-|\xi|^4+\frac{1}{4}}\sinh\Big(\sqrt{-|\xi|^4+\frac{1}{4}}t\Big)\Big|$$

$$\le \cosh\Big(\sqrt{-|\xi|^4+\frac{1}{4}}t\Big) + \frac{1}{2}\sinh\Big(\sqrt{-|\xi|^4+\frac{1}{4}}t\Big) = \exp\Big(\frac{\sqrt{3}}{4}t\Big)$$

consequently, $\Big|\dfrac{1}{2}e^{-\frac{t}{2}}B\Big| \le Ce^{-\delta t}$ with $\delta > 0$.

Hence,

$$\int\limits_{|\xi|<\frac{1}{\sqrt{2}}} |v_t(t,\xi)|^2 d\xi \le Ce^{-\delta t} \int\limits_{|\xi|<\frac{1}{\sqrt{2}}} |v_0(\xi)|^2 + |v_1(\xi)|^2 d\xi \le Ce^{-\delta t}\int\limits_{\mathbb{R}^n} |v_0(\xi)|^2 + |v_1(\xi)|^2 d\xi.$$

If $\dfrac{1}{2} \ge |\xi| > 0$ by using

$$v(t,\xi) = e^{-\frac{t}{2}}\Big[\Big(\frac{v_0(\xi)}{2} - \frac{v_1(\xi)}{\sqrt{1-4|\xi|^4}}\Big)e^{-\frac{1}{2}\sqrt{1-4|\xi|^4}t} + \Big(\frac{v_0(\xi)}{2} + \frac{v_1(\xi)}{\sqrt{1-4|\xi|^4}}\Big)e^{\frac{1}{2}\sqrt{1-4|\xi|^4}t}\Big]$$

we have

$$v_t(t,\xi) = e^{-\frac{t}{2}+\sqrt{\frac14-|\xi|^4}t}\Big[\Big(v_0(\xi)+\frac{v_1(\xi)}{\sqrt{\frac14-|\xi|^4}}\Big)\Big(-\frac12+\sqrt{\frac14-|\xi|^4}\Big)$$

$$+e^{-\frac{t}{2}-\sqrt{\frac14-|\xi|^4}t}\Big[\Big(v_0(\xi)-\frac{v_1(\xi)}{\sqrt{\frac14-|\xi|^4}}\Big)\Big(-\frac12-\sqrt{\frac14-|\xi|^4}\Big)\Big].$$

Because for those $|\xi|$ we have $-4|\xi|^4 \le -1+\sqrt{1-4|\xi|^4} \le -2|\xi|^4$ and $\Big(v_0(\xi)\pm\frac{v_1(\xi)}{\sqrt{\frac14-|\xi|^4}}\Big)^2 \le C\Big(|v_0(\xi)|^2+|v_1(\xi)|^2\Big)$ we conclude

$$|v_t(t,\xi)|^2 \le Ce^{-|\xi|^4 t}\Big(|v_0(\xi)|^2+|v_1(\xi)|^2\Big)|\xi|^8 + Ce^{-t}\Big(|v_0(\xi)|^2+|v_1(\xi)|^2\Big),$$

and, finally,

$$\|v_t(t,\cdot)\|^2_{L_2} \le Ce^{-t}\int\limits_{\mathbb{R}^n}(|v_0(\xi)|^2+|v_1(\xi)|^2)d\xi + C\int\limits_{\mathbb{R}^n}|\xi|^8 e^{-2|\xi|^4 t}(|v_0(\xi)|^2+|v_1(\xi)|^2)d\xi$$

$$\le \int\limits_{\mathbb{R}^n} Ce^{-t}(|v_0(\xi)|^2+|v_1(\xi)|^2)d\xi + \frac{C}{t^2}\sup_{|\xi|<\frac12, t\ge1} t^2|\xi|^8 e^{-2|\xi|^4 t}\int\limits_{|\xi|<\frac12}(|v_0(\xi)|^2+|v_1(\xi)|^2)d\xi$$

$$\le \int\limits_{\mathbb{R}^n} Ce^{-t}(|v_0(\xi)|^2+|v_1(\xi)|^2)d\xi + \frac{C}{(1+t)^2}\int\limits_{|\xi|<\frac12}(|v_0(\xi)|^2+|v_1(\xi)|^2)d\xi.$$

Summarizing we have the decay rate for $\|u_t(t,\cdot)\|^2_{L_2}$ is $(1+t)^{-2}$. So the total energy will decay with the rate $(1+t)^{-1}$. Thus we have proved the foolwing result:

Theorem 3.15 *The energy solution to the Cauchy problem*

$$u_{tt}+(-\triangle)^2 u + u_t = 0,\ \ u(0,x)=\varphi(x),\ \ u_t(0,x)=\psi(x)$$

with data $\varphi \in H^2$ and $\psi \in L^2$ satisfies the following decay estimates:

$$\begin{aligned}\|\nabla_x u(t,\cdot)\|_{L^2} &\le C(1+t)^{-\frac14}\Big(\|\psi\|_{L^2}+\|\varphi\|_{H^2}\Big),\\ \|u_t(t,\cdot)\|_{L^2} &\le C(1+t)^{-1}\Big(\|\psi\|_{L^2}+\|\varphi\|_{H^2}\Big),\end{aligned}$$

and consequently, the energy satisfies

$$E(u)(t) \le C(1+t)^{-1}\Big(\|\psi\|^2_{L^2}+\|\varphi\|^2_{H^2}\Big).$$

3.5.2 Estimates for the solution itself

We want to have the estimate of weak solutions with respect to the norm of L_2. Let us devote to the Cauchy problem for the damped plate equation

$$u_{tt} + (-\triangle)^2 u + u_t = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x)$$

with data $\varphi(x) \in L_2,\ \psi(x) \in H^{-2}$.
Using Parseval's relation and the explicit representation of solution we have

$$\|u(t,\cdot)\|^2_{L_2} = \int\limits_{\mathbb{R}^n} |v(t,\xi)|^2 d\xi$$

$$= e^{-t} \int\limits_{|\xi|<\frac{1}{\sqrt{2}}} \Big[F(\varphi)(\xi)\Big(\cosh\Big(\sqrt{\frac{1}{4} - |\xi|^4}t\Big) + \frac{\sinh\Big(\sqrt{\frac{1}{4} - |\xi|^4}t\Big)}{\sqrt{1-4|\xi|^4}}\Big)$$

$$+ \frac{F(\psi)(\xi)}{\langle\xi\rangle^2}\langle\xi\rangle^2 \frac{\sinh\Big(\sqrt{\frac{1}{4} - |\xi|^4}t\Big)}{\sqrt{1-4|\xi|^4}}\Big]^2 d\xi$$

$$+ e^{-t} \int\limits_{|\xi|>\frac{1}{\sqrt{2}}} \Big[F(\varphi)(\xi)\Big(\cos\Big(\sqrt{\frac{1}{4} - |\xi|^4}t\Big) + \frac{\sin\Big(\sqrt{\frac{1}{4} - |\xi|^4}t\Big)}{\sqrt{1-4|\xi|^4}}\Big)$$

$$+ \frac{F(\psi)(\xi)}{\langle\xi\rangle^2}\langle\xi\rangle^2 \frac{\sin\Big(\sqrt{\frac{1}{4} - |\xi|^4}t\Big)}{\sqrt{1-4|\xi|^4}}\Big]^2 d\xi.$$

a) Case $\frac{1}{2} \le |\xi| < \frac{1}{\sqrt{2}}$. We use

$$\cosh\Big(\sqrt{\frac{1}{4} - |\xi|^4}t\Big) \le \cosh\Big(\frac{\sqrt{3}}{4}t\Big),\quad \frac{\sinh\Big(\sqrt{\frac{1}{4} - |\xi|^4}t\Big)}{\sqrt{1-4|\xi|^4}} \le t\cosh\Big(\sqrt{\frac{1}{4} - |\xi|^4}t\Big) \le t\cosh\Big(\frac{\sqrt{3}}{4}t\Big),$$

$$\langle\xi\rangle^2 \frac{\sinh\Big(\sqrt{\frac{1}{4} - |\xi|^4}t\Big)}{\sqrt{1-4|\xi|^4}} \le Ct\cosh\Big(\frac{\sqrt{3}}{4}t\Big).$$

Thus

$$\int\limits_{\frac{1}{2}\le|\xi|<\frac{1}{\sqrt{2}}} |v(t,\xi)|^2 d\xi \le e^{-t} \int\limits_{\frac{1}{2}\le|\xi|<\frac{1}{\sqrt{2}}} \Big[|F(\varphi)(\xi)|^2 + \Big|\frac{F(\psi)(\xi)}{\langle\xi\rangle^2}\Big|^2\Big] d\xi C t^2 \cosh^2(\frac{\sqrt{3}}{4}t)$$

$$\le Ce^{-\delta t}(||\varphi||^2_{L_2} + ||\psi||^2_{H^{-2}}) \text{ with } \delta > 0.$$

b) Case $0 < |\xi| \le \dfrac{1}{2}$. Here we use

$$\cosh\left(\sqrt{\frac{1}{4} - |\xi|^4 t}\right) \le \cosh\left(\frac{t}{2}\right) \le Ce^{\frac{t}{2}}, \quad \frac{\sinh\left(\sqrt{\frac{1}{4} - |\xi|^4 t}\right)}{\sqrt{1-4|\xi|^4}} \le C \sinh\left(\sqrt{\frac{1}{4} - |\xi|^4 t}\right) \le Ce^{\frac{t}{2}}.$$

Hence,

$$\int_{0<|\xi|\le\frac{1}{2}} |v(t,\xi)|^2 d\xi \le C(\|\varphi\|_{L_2}^2 + \|\psi\|_{H^{-2}}^2).$$

The second integral in the above representation of solution will decay exponentially, indeed it can be estimated by

$$Ce^{-t}t^2(\|\varphi\|_{L_2}^2 + \|\psi\|_{H^{-2}}^2).$$

Summarizing we have the following conclusion.

Theorem 3.16 *The distributional solution to the Cauchy problem*

$$u_{tt} + (-\triangle)^2 u + u_t = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x)$$

with data $\varphi \in L^2$ and $\psi \in H^{-2}$ satisfies the following estimate:

$$\|u(t,\cdot)\|_{L_2} \le C(\|\varphi\|_{L_2} + \|\psi\|_{H^{-2}}).$$

3.5.3 Estimates for the derivatives

Derivatives with respect to x

We want to estimate

$$\|\partial_x^\alpha u(t,\cdot)\|_{L_2}^2 = \|(i\xi)^\alpha v(t,\cdot)\|_{L_2}^2 \le \int_{\mathbb{R}^n} |\xi|^{2|\alpha|} |v(t,\xi)|^2 d\xi.$$

Similarly to the approach in the previous sections the second integral which is over $|\xi| > \frac{1}{\sqrt{2}}$ decays exponentially. We also use the obtained estimates for the integral over $\frac{1}{2} \le |\xi| < \frac{1}{\sqrt{2}}$ which yields a decay rate $Ce^{-\delta t}$. When $0 < |\xi| \le \frac{1}{2}$ we have

$$\int_{0<|\xi|\le\frac{1}{2}} |\xi|^{2|\alpha|} |v(t,\xi)|^2 d\xi = \int_{0<|\xi|\le\frac{1}{2}} |\xi|^{2|\alpha|} (|v_1(\xi)|^2 + |v_0(\xi)|^2)(e^{-t-\sqrt{1-4|\xi|^4}t} + e^{-t+\sqrt{1-4|\xi|^4}t}) d\xi$$

$$\le C \int_{0<|\xi|\le\frac{1}{2}} |\xi|^{|\alpha|} (\langle\xi\rangle^{|\alpha|-2} |v_1(\xi)|^2 + \langle\xi\rangle^{|\alpha|} |v_0(\xi)|^2)(e^{-t-\sqrt{1-4|\xi|^4}t} + e^{-t+\sqrt{1-4|\xi|^4}t}) d\xi$$

$$\le Ce^{-t}(\|\varphi\|_{H^{|\alpha|}}^2 + \|\psi\|_{H^{|\alpha|-2}}^2) + C \int_{0<|\xi|\le\frac{1}{2}} (\langle\xi\rangle^{|\alpha|-2} |v_1(\xi)|^2 + \langle\xi\rangle^{|\alpha|} |v_0(\xi)|^2) e^{-2|\xi|^4 t} |\xi|^{|\alpha|} d\xi.$$

Using the norm inequality $\|\cdot\|_{L_2} \leq \|\cdot\|_{L_\infty}\|\cdot\|_{L_2}$ we get for the second integral

$$\int\limits_{0<|\xi|\leq\frac{1}{2}} (\langle\xi\rangle^{|\alpha|-2}|v_1(\xi)|^2 + \langle\xi\rangle^{|\alpha|}|v_0(\xi)|^2)e^{-2|\xi|^4t}|\xi|^{2|\alpha|}d\xi$$
$$\leq C(\|\varphi\|^2_{H^{|\alpha|}} + \|\psi\|^2_{H^{|\alpha|-2}}) \sup_{|\xi|\ ,t\geq 1} e^{-|\xi|^4t}|\xi|^{|\alpha|} \leq C(1+t)^{-\frac{|\alpha|}{4}}(\|\varphi\|^2_{H^{|\alpha|}} + \|\psi\|^2_{H^{|\alpha|-2}}).$$

Summarizing we conclude

$$\|\partial_x^\alpha u(t,\cdot)\|_{L_2} \leq C(1+t)^{-\frac{|\alpha|}{4}}(\|\varphi\|_{H^{|\alpha|}} + \|\psi\|_{H^{|\alpha|-2}}).$$

Derivatives with respect to t
When $0 < |\xi| \leq \frac{1}{2}$ we have

$$v(t,\xi) = \frac{1}{2}\Big(v_0(\xi) + \frac{v_1(\xi)}{\sqrt{\frac{1}{4}-|\xi|^4}}\Big)\exp\Big(-\frac{t}{2} + \sqrt{\frac{1}{4}-|\xi|^4t}\Big)$$
$$+\frac{1}{2}\Big(v_0(\xi) - \frac{v_1(\xi)}{\sqrt{\frac{1}{4}-|\xi|^4}}\Big)\exp\Big(-\frac{t}{2} - \sqrt{\frac{1}{4}-|\xi|^4t}\Big).$$

Thus we get for the derivatives

$$\partial_t^k v(t,\xi) = \frac{1}{2}\Big(v_0(\xi) + \frac{v_1(\xi)}{\sqrt{\frac{1}{4}-|\xi|^4}}\Big)\exp\Big(-\frac{t}{2} + \sqrt{\frac{1}{4}-|\xi|^4t}\Big)\Big(-\frac{1}{2} + \sqrt{\frac{1}{4}-|\xi|^4}\Big)^k$$
$$+\frac{1}{2}\Big(v_0(\xi) - \frac{v_1(\xi)}{\sqrt{\frac{1}{4}-|\xi|^4}}\Big)\exp\Big(-\frac{t}{2} - \sqrt{\frac{1}{4}-|\xi|^4t}\Big)(-\frac{1}{2} - \sqrt{\frac{1}{4}-|\xi|^4}\Big)^k.$$

In the following we use

$$\Big(\frac{1}{2} + \sqrt{\frac{1}{4}-|\xi|^4}\Big)^k \leq C_k,\ \Big(-\frac{1}{2} + \sqrt{\frac{1}{4}-|\xi|^4}\Big) \sim -C|\xi|^4,\ \Big(1 + \frac{1}{\sqrt{\frac{1}{4}-|\xi|^4}}\Big) \leq C.$$

Then we obtain

$$\int\limits_{\frac{1}{2}\leq|\xi|<\frac{1}{\sqrt{2}}} |\partial_t^k v(t,\xi)|^2 d\xi \leq (Ce^{-t} + C \sup_{\frac{1}{2}\leq|\xi|<\frac{1}{\sqrt{2}},t\geq 1} e^{-|\xi|^4t}|\xi|^{4k})(\|\varphi\|^2_{L_2} + \|\psi\|^2_{L_2})$$
$$\leq C(1+t)^{-k}(\|\varphi\|^2_{L_2} + \|\psi\|^2_{L_2}) \leq C(1+t)^{-k}(\|\varphi\|^2_{H^{2k}} + \|\psi\|^2_{H^{2k-2}}).$$

When $\frac{1}{2} \le |\xi| < \frac{1}{\sqrt{2}}$ we have

$$v(t,\xi) = e^{-\frac{t}{2}}\Big[v_0(\xi)\cosh\Big(\sqrt{\frac{1}{4}-|\xi|^4}t\Big) + v_1(\xi)\frac{\sinh\Big(\sqrt{\frac{1}{4}-|\xi|^4}t\Big)}{\sqrt{\frac{1}{4}-|\xi|^4}}\Big],$$

$$\partial_t^k v(t,\xi) = \sum_{l=0}^{k}(e^{-\frac{t}{2}})^{(l)}\Big[v_0(\xi)\Big(\cosh\Big(\sqrt{\frac{1}{4}-|\xi|^4}t\Big)\Big)^{(k-l)} + v_1(\xi)\Big(\frac{\sinh\Big(\sqrt{\frac{1}{4}-|\xi|^4}t\Big)}{\sqrt{\frac{1}{4}-|\xi|^4}}\Big)^{(k-l)}\Big].$$

To estimate each term in the above equality we use

$$\sqrt{\frac{1}{4}-|\xi|^4} \le C,\ e^{-\frac{t}{2}}\cosh\Big(\sqrt{\frac{1}{4}-|\xi|^4}t\Big) \le Ce^{-\delta t},\ e^{-\frac{t}{2}}\frac{\sinh\Big(\sqrt{\frac{1}{4}-|\xi|^4}t\Big)}{\sqrt{\frac{1}{4}-|\xi|^4}} \le Ce^{-\delta t} \text{ with } \delta > 0.$$

Then we obtain

$$\int\limits_{0<|\xi|\le\frac{1}{2}} |\partial_t^k v(t,\xi)|^2 d\xi \le Ce^{-2\delta t}(\|\varphi\|_{L_2}^2 + \|\psi\|_{L_2}^2) \le Ce^{-2\delta t}(\|\varphi\|_{H^{2k}}^2 + \|\psi\|_{H^{2k-2}}^2).$$

When $\frac{1}{\sqrt{2}} < |\xi| \le 1$ we have

$$v(t,\xi) = e^{-\frac{t}{2}}\Big[v_0(\xi)\cos\Big(\sqrt{-\frac{1}{4}+|\xi|^4}t\Big) + v_1(\xi)\frac{\sin\Big(\sqrt{-\frac{1}{4}+|\xi|^4}t\Big)}{\sqrt{-\frac{1}{4}+|\xi|^4}}\Big],$$

$$\partial_t^k v(t,\xi) = \sum_{l=0}^{k}(e^{-\frac{t}{2}})^{(l)}\Big[v_0(\xi)\Big(\cos\Big(\sqrt{-\frac{1}{4}+|\xi|^4}t\Big)\Big)^{(k-l)} + v_1(\xi)\Big(\frac{\sin\Big(\sqrt{-\frac{1}{4}+|\xi|^4}t\Big)}{\sqrt{-\frac{1}{4}+|\xi|^4}}\Big)^{(k-l)}\Big].$$

To estimate each term in the above equality we use

$$-\frac{1}{4}+|\xi|^4 \le C,\ e^{-\frac{t}{2}}\cos\Big(\sqrt{-\frac{1}{4}+|\xi|^4}t\Big) \le Ce^{-\frac{t}{2}},\ e^{-\frac{t}{2}}\frac{\sin\Big(\sqrt{-\frac{1}{4}+|\xi|^4}t\Big)}{\sqrt{-\frac{1}{4}+|\xi|^4}} \le Ce^{-\frac{t}{2}}t.$$

Then we obtain

$$\int\limits_{\frac{1}{\sqrt{2}}<|\xi|\le 1} |\partial_t^k v(t,\xi)|^2 d\xi \le Ct^2e^{-t}(\|\varphi\|_{H^{2k}}^2 + \|\psi\|_{H^{2k-2}}^2).$$

When $1 \leq |\xi|$ we have

$$v(t,\xi) = \frac{1}{2}\Big(v_0(\xi) - \frac{iv_1(\xi)}{\sqrt{-\frac{1}{4}+|\xi|^4}}\Big)\exp\Big(-\frac{t}{2}+i\sqrt{-\frac{1}{4}+|\xi|^4}t\Big)$$
$$+\frac{1}{2}\Big(v_0(\xi) + \frac{iv_1(\xi)}{\sqrt{-\frac{1}{4}+|\xi|^4}}\Big)\exp\Big(-\frac{t}{2}-i\sqrt{-\frac{1}{4}+|\xi|^4}t\Big)$$
$$\partial_t^k v(t,\xi) = \frac{1}{2}\Big(v_0(\xi) - \frac{iv_1(\xi)}{\sqrt{-\frac{1}{4}+|\xi|^4}}\Big)\exp\Big(-\frac{t}{2}+i\sqrt{-\frac{1}{4}+|\xi|^4}t\Big)\Big(-\frac{1}{2}+i\sqrt{-\frac{1}{4}+|\xi|^4}\Big)^k$$
$$+\frac{1}{2}\Big(v_0(\xi) + \frac{iv_1(\xi)}{\sqrt{-\frac{1}{4}+|\xi|^4}}\Big)\exp\Big(-\frac{t}{2}-i\sqrt{-\frac{1}{4}+|\xi|^4}t\Big)\Big(-\frac{1}{2}-i\sqrt{-\frac{1}{4}+|\xi|^4}\Big)^k.$$

Here we use that

$$1+\frac{1}{|\xi|^4-\frac{1}{4}} \leq C, \ \Big| -\frac{1}{2} \pm i\sqrt{|\xi|^4-\frac{1}{4}}\Big| = |\xi|^2, \text{ hence } \int\limits_{|\xi|\geq 1} |\partial_t^k v(t,\xi)|^2 d\xi \leq e^{-\frac{t}{2}}(\|\varphi\|^2_{H^{2k}} + \|\psi\|^2_{H^{2k-2}}).$$

Summarizing we have shown the following result:

Theorem 3.17 *The energy solution to the Cauchy problem*

$$u_{tt} + (-\triangle)^2 u + u_t = 0, \ \ u(0,x) = \varphi(x), \ \ u_t(0,x) = \psi(x)$$

with data $\varphi \in H^{2k+|\alpha|}$ *and* $\psi \in H^{2k+|\alpha|-2}$ *satisfies the decay estimate*

$$\|\partial_t^k \partial_x^\alpha u(t,\cdot)\|_{L_2} \leq C(1+t)^{-k-\frac{|\alpha|}{4}}(\|\varphi\|_{H^{2k+|\alpha|}} + \|\psi\|_{H^{2k+|\alpha|-2}}).$$

3.6 A general result for linear Cauchy problems

Let us consider the strict hyperbolic Cauchy problem

$$D_t^m u + \sum_{k+|\alpha|\leq m,\, k\neq m} a_{k,\alpha}(t,x) D_x^\alpha D_t^k u = f(t,x),$$
$$u(0,x) = u_0(x), \ D_t u(0,x) = u_1(x), \cdots, D_t^{(m-1)} u(0,x) = u_{m-1}(x).$$

Then the following existence and uniqueness result holds (see [38]):

Theorem 3.18 *Let us assume*

- *for the coefficients:* $a_{k,\alpha} \in C([0,T],\mathcal{B}^l(\mathbb{R}^n)) \cap C^1([0,T],\mathcal{B}^0)$ *if* $k+|\alpha| = m$, *where* $l \geq 1$;

$a_{k,\alpha} \in C([0,T], \mathcal{B}^l(\mathbb{R}^n))$ *if* $k + |\alpha| < m$*;*
- *for the right-hand side:* $f \in C([0,T], H^l)$*;*
- *for the data:* $u_j \in H^{l+m-1-j}$.

Then there exists a unique solution $u \in \bigcap_{0\le j\le m-1} C^j([0,T], H^{l+m-1-j})$ *satisfying the energy inequality*

$$\sum_{j=0}^{m-1} \|\partial_t^j u(t,\cdot)\|_{H^{l+m-j-1}} \le C(T,j)\Big(\sum_{j=0}^{m-1} \|u_j(\cdot)\|_{H^{l+m-j-1}} + \int_0^t \|f(s,\cdot)\|_{H^l}\Big).$$

Here $\mathcal{B}^l$ denotes the space of l time continuously differentiable function on $\mathbb{R}^n$ with bounded derivatives up to order l.

Remark: We transform the equation to a pseudo-differential system of first order and apply energy method. This needs C^1 with respect to t only for the coefficients of the principal part.

4 Nonlinear Cauchy problems

4.1 Global existence of small data solutions

If one is interested in the Cauchy problem for non-linear wave equations like

$$u_{tt} - \triangle u = f(u_t, \nabla u, \nabla u_t, \nabla^2 u), \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x), \tag{4.1}$$

then one should be prepared to meet several phenomena. Under reasonable regularity assumptions one can expect *a local (in time) well-posedness result for solutions valued in Sobolev spaces*. On the other hand the nonlinearity may cause a *blow-up behavior of these solutions*. Thus it seems to be reasonable to ask if under the assumption that the *zero solution is a steady state solution* there exist *global (in time) small data solutions*. This question applied to the above model (4.1) means to prove that for all $\varepsilon \in (0, \varepsilon_0(\varphi,\psi)]$ the Cauchy problem

$$u_{tt} - \triangle u = f(u_t, \nabla u, \nabla u_t, \nabla^2 u), \;\; u(0,x) = \varepsilon\varphi(x), \;\; u_t(0,x) = \varepsilon\psi(x) \tag{4.2}$$

has a global in time solution. A positive answer to this question yields a *stability result* for the solution (here the trivial one) *generating an equilibrium*.
One of the key tools to prove such a global existence result is the so-called *Strichartz' decay estimate* (see [61]) for the energy $E(u)(t) := (\nabla u(t,\cdot), u_t(t,\cdot))|_{L^q}$ basing on the $L^q(\mathbb{R}^n)$-norm

$$E(u)(t)|_{L^q} \le C(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})} E(u)(0)|_{W_p^{N_p}} \tag{4.3}$$

on the conjugate line $2 \le q \le \infty,\;\; 1/p + 1/q = 1$ for solutions of the Cauchy problem to the classical wave equation

$$u_{tt} - \triangle u = 0,\; u(0,x) = \varphi(x),\; u_t(0,x) = \psi(x),$$

where $N_p > n\Big(\frac{1}{p} - \frac{1}{q}\Big)$.
One can derive such type of *Strichartz' decay estimates* for the wave energy $E(u)(t) := (\nabla u(t,\cdot), u_t(t,\cdot))|_{L^q}$ for solutions of the Cauchy problem to the classical Klein-Gordon equation

$$u_{tt} - \triangle u + m^2 u = 0,\; u(0,x) = \varphi(x),\; u_t(0,x) = \psi(x).$$

We obtain

$$E(u)(t)|_{L^q} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} E(u)(0)|_{W_p^{K_p}}$$

on the conjugate line $2 \le q \le \infty,\;\; 1/p + 1/q = 1$ with a suitable K_p depending on p and q similar as N_p for the wave case.
Finally, one can derive such type of *Strichartz' decay estimates* for the wave energy $E(u)(t) := (\nabla u(t,\cdot), u_t(t,\cdot))|_{L^q}$ for solutions of the Cauchy problem to the classical damped wave equation

$$u_{tt} - \triangle u + u_t = 0,\; u(0,x) = \varphi(x),\; u_t(0,x) = \psi(x).$$

We obtain

$$E(u)(t)|_{L^q} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} E(u)(0)|_{W_p^{N_p}}$$

on the conjugate line $2 \le q \le \infty,\;\; 1/p + 1/q = 1$.
These *Strichartz' decay estimates* are an important tool to prove the global (in time) existence of small data solutions for

$$u_{tt} - \triangle u + m^2 u = f(u_t, \nabla u, \nabla u_t, \nabla^2 u),\;\; u(0,x) = \varepsilon\varphi(x),\; u_t(0,x) = \varepsilon\psi(x),\; m > 0;$$
$$u_{tt} - \triangle u + u_t = f(u_t, \nabla u, \nabla u_t, \nabla^2 u),\;\; u(0,x) = \varepsilon\varphi(x),\; u_t(0,x) = \varepsilon\psi(x).$$

The parameter ε describes the smallness of the data. In all these applications the right-hand side $f(u_t, \nabla u, \nabla u_t, \nabla^2 u)$ is supposed to possess a structure related to the energy $E(u)(t) := (\nabla u(t,\cdot), u_t(t,\cdot))|_{L^q}$ and to satisfy a *special asymptotic behavior* around 0, thus the terms $\triangle u$ in the nonlinear Klein-Gordon model or the dissipation term u_t or $\triangle u$ in the nonlinear damped wave model are not allowed to be included into the right-hand side $f(u_t, \nabla u, \nabla u_t, \nabla^2 u)$.
Those *Strichartz' decay estimates* are proved in the literature for special partial differential equations or systems of partial differential equations with constant coefficients in connection

with the study of models from e.g. elasticity, thermo-elasticity, thermo-viscoelasticity or electrodynamics (see [43] and references therein) (see also Section 8.2).

Now let us devote to the Cauchy problem (4.2). We introduce the notation $Du = (u_t, \nabla u)$. Then one can show the following result.

Theorem 4.1 *Let us assume for the nonlinear right-hand side* $(w := Du)$

$$f \in C^\infty(\mathbb{R}^{(n+1)^2}, \mathbb{R}) : f(w, \nabla w) = O((|w| + |\nabla w|)^{\alpha+1}) \text{ as } |w| + |\nabla w| \to 0. \tag{4.4}$$

Moreover, $\frac{1}{\alpha}(1 + \frac{1}{\alpha}) < \frac{n-1}{2}$. *Then there exist positive constants* $s_0 > \frac{n}{2} + 1$ *and* ε_0 *such that the following holds:*
If $(\psi, \nabla\varphi) \in H^s \cap H^{s,p}$, $s \in \mathbb{N}$, *with* $s \geq s_0$ *and* $p = \frac{2\alpha+2}{2\alpha+1}$, *then there is a unique solution* u *to (4.2) for all* $\varepsilon \leq \varepsilon_0$. *The first derivatives have the regularity*

$$(u_t, \nabla u) \in C([0, \infty), H^s) \cap C^1([0, \infty), H^{s-1}).$$

Finally, the decay properties can be described by

$$\|(u_t, \nabla u)\|_{L^\infty} + \|(u_t, \nabla u)\|_{L^{2\alpha+2}} = O(t^{-\frac{n-1}{2}\frac{\alpha}{\alpha+1}}), \quad \|(u_t, \nabla u)\|_{H^s} = O(1).$$

The statement of this theorem was given in [27]. It contains the *global existence of small data solutions*. The condition $\frac{1}{\alpha}(1+\frac{1}{\alpha}) < \frac{n-1}{2}$ connects the space dimension n and the degree $\alpha + 1$ of vanishing order of the nonlinearity near 0. The larger α or n are, the better the situation is. This condition is not sharp in all cases. Thus quadratic nonlinearities ($\alpha = 1$) require n to be at least 6. But the optimal condition is $n \geq 4$ for quadratic nonlinearities. To show this one has to use special properties of the wave operator.

Sketch of the proof to Theorem 4.1:
Let us devote to

$$u_{tt} - \triangle u = f(u_t, \nabla u, \nabla u_t, \nabla^2 u), \;\; u(0, x) = \varphi(x), \;\; u_t(0, x) = \psi(x),$$

where we will only discuss the quasi-linear case

$$f(u_t, \nabla u, \nabla u_t, \nabla^2 u) = \sum_{k,l=1}^n a_{kl}(u_t, \nabla u)\partial^2_{x_k x_l} u.$$

Introducing $U := (u_t, \nabla u)^T$ the quasi-linear Cauchy problem is equivalent to the symmetric hyperbolic system

$$A^0(U)\partial_t U + \sum_{j=1}^n A^j(U)\partial_{x_j} U = 0, \;\; U(0, x) = U_0(x),$$

where $A^0(U)$ is positive definite for small U (here the condition $a_{kl}(0,0) = 0$ helps) and $A^j(U)$ are symmetric for $j = 1, \cdots, n$.

Hence we can apply the local existence result from Section 5.7. If we assume $U_0 \in H^s, s \in \mathbb{N}$ and $s > \frac{n}{2} + 1$ to guarantee the existence of a classical solution, then there exists a small T and a unique solution $U \in C([0,T], H^s)$. In this way we get a *local (in time) solution* to the hyperbolic system

$$A^0(U)\partial_t U + \sum_{j=1}^{n} A^j(U)\partial_{x_j} U = 0, \ U(0,x) = U_0(x).$$

This system has an *autonomous structure*, that is, the coefficients depend only on the solution. If we are interested in the Cauchy problem with Cauchy condition on the hyperplane $t = s$

$$A^0(U)\partial_t U + \sum_{j=1}^{n} A^j(U)\partial_{x_j} U = 0, \ U(s,x) = U_0(x),$$

then we have the existence of the solution in the time interval $[s, s+T]$.

After having a solution which is valued in H^s it is necessary to estimate the *energy of higher order* s. The goal is to derive an estimate like

$$\|U(t,\cdot)\|_{H^s} \leq C\|U_0(\cdot)\|_{H^s} \exp\Big(C\int_0^t \|U(r,\cdot)\|^{\alpha}_{H^{\infty,b}} dr\Big),$$

where $C = C(s)$, and where b is independent of s. This is an important fact. Thus we can be sure, that the exponential term does not involve higher derivatives in the L^∞ norm. Such an inequality is shown after using general inequalities for composite functions as Gagliardo-Nirenberg inequality, Moser type inequalities and inequalities for mollifiers.

In the last step *weighted a-priori estimates* for $U(t,\cdot)$ are proved. They read as follows:

$$\sup_{t\in[0,T]} (1+t)^d \|U(t,\cdot)\|_{H^{q,s_1}} \leq M_0 < \infty,$$

where M_0 is independent of T. To get such an estimate we need $L^p - L^q$ decay estimates for the solution to the Cauchy problem for the classical wave equation (see (8.23) from Section 8.2). With such a tool at hand we consider

$$u_{tt} - \triangle u = \sum_{k,l=1}^{n} a_{kl}(u_t, \nabla u)\partial^2_{x_k x_l} u, \ u(0,x) = \varphi(x), \ u_t(0,x) = \psi(x).$$

Applying Duhamel's principle (interpreting the right-hand side as a function $f = f(t,x)$), using estimates for composite functions, and the $L^p - L^q$ decay estimate for the classical

wave equation with homogeneous right-hand side we arrive at the desired *weighted a-priori estimate* for $U(t,\cdot)$ assuming that the datum U_0 has a *sufficiently small H^s-norm.*
Using the weighted a-priori estimate

$$\sup_{t\in[0,T]}(1+t)^d\|U(t,\cdot)\|_{H^{q,s_1}}\leq M_0,$$

we can show that under smallness assumption to the data

$$\int_0^t\|U(r,\cdot)\|^{\alpha}_{H^{\infty,b}}dr\leq K$$

for all $t\geq 0$, where K is independent of t. This opens to apply a standard *continuation argument* for the solution. Here the *autonomous character of the right-hand side* (no explicit dependence on the time and on the spatial variables) is important (see the explanations from the previous page). Summarizing all these observations completes the proof of Theorem 4.1. □

Now let us devote again to the Cauchy problem (4.2). We assume that the right-hand side f satisfies condition (4.4) with $\alpha=1$. By $T(\varepsilon)$ we denote the *life span* of a solution to (4.2).

Theorem 4.2 *Let $n>3$. If $\varphi,\ \psi\in C_0^\infty(\mathbb{R}^n)$, then there exists a unique global small data solution. The first derivatives have the regularity*

$$(u_t,\nabla u)\in C([0,\infty),H^{2n+3})\cap C^1([0,\infty),H^{2n+2}).$$

Finally, the decay properties can be described by

$$\|(u_t,\nabla u)\|_{L^\infty}=O(t^{-\frac{n-1}{2}}),\quad \|(u_t,\nabla u)\|_{H^{2n+3}}=O(1).$$

Let $n=3$. There exist positive constants ε_0 and A such that for all $0<\varepsilon\leq\varepsilon_0$ we have $T(\varepsilon)\geq\exp(\frac{A}{\varepsilon})$. Such a life span behavior is called **almost global existence***.*

Theorem 4.2 is taken from [25]. It is optimal in the sense, that in the case $n=3$ quadratic nonlinearities can develop singularities in finite time.

Remark: In the case $n=3$ quadratic nonlinearities can imply the existence of global small data solutions. As an example let us consider the Cauchy problem

$$u_{tt}-\triangle u=a|\nabla u|^2+bu_t^2,\quad u(0,x)=\varphi(x),\ u_t(0,x)=\psi(x).$$

Then global small data solutions exist if and only if $a+b=0$. If $a+b=0$, then the right-hand side satisfies the so-called *null condition.* For the above example this condition means, that the symbol of the right-hand side vanishes on the characteristic set of the wave operator. Another quadratic nonlinearity which develops singularities in finite time was studied in [23], namely the Cauchy problem,

$$u_{tt} - \triangle u = 2u_t u_{tt}, \quad u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x).$$

We studied up to now only Cauchy problems of type (4.1). But there exists also a long history about the existence of *global small data solutions for families of semi-linear Cauchy problems.* As a prototype let us consider the Cauchy problem

$$u_{tt} - \triangle u = |u|^\alpha, \quad u(0,x) = \varepsilon\varphi(x),\ u_t(0,x) = \varepsilon\psi(x). \tag{4.5}$$

Question: What kind of expectations do we have?

Answer: If α is large and the data are small, then the right-hand side will remain small. Thus we have a good chance to prove *global existence of small data solutions*. If α is small and the data are small, then the influence of the right-hand side may become dominant, thus blow-up may appear.

That this is really so shows the next theorem which explains the situation in the 3-d case.

Theorem 4.3 *Let us consider the Cauchy problem (4.5) with $\varphi \in C_0^3(\mathbb{R}^3)$, $\psi \in C_0^2(\mathbb{R}^3)$. If $\alpha > 1+\sqrt{2}$, then we have a unique global small data solution $u \in C^2(\mathbb{R}^3 \times [0,\infty))$. There is a unique solution $u \in C^2(\mathbb{R}^3 \times [0,T(\varepsilon)))$ if $2 \le \alpha < 1+\sqrt{2}$. Finally, if $1 < \alpha < 2$ and if the data $\varphi \in C_0^2(\mathbb{R}^3)$, $\psi \in C_0^1(\mathbb{R}^3)$, then there is a unique weak solution $u \in C^1(\mathbb{R}^3 \times [0,T(\varepsilon)))$. The life-span $T(\varepsilon)$ is equal to*

$$T(\varepsilon) = \begin{cases} \exp(A\varepsilon^{-\alpha(\alpha-1)}) \text{ if } \alpha = 1+\sqrt{2}, \\ A\varepsilon^{\frac{\alpha(\alpha-1)}{\alpha^2-2\alpha-1}} \text{ if } 1 < \alpha < 1+\sqrt{2}. \end{cases}$$

The results of this theorem are given in [22]. It is shown that the estimates for the life span time are sharp.

We formulated the above result for the 3-d case. But what happens in the n-dimensional case? We expect of course a similar situation. The general situation can be described by the aid of the positive root α_{crit} of $(n-1)\alpha^2 - (n+1)\alpha - 2 = 0$. If $n = 3$, then $\alpha_{crit} = 1+\sqrt{2}$.

- If $\alpha > \alpha_{crit}$, then we expect the existence of *global small data solutions.*
- If $\alpha < \alpha_{crit}$, then we expect *a blow-up behavior.*
- What happens in the critical case $\alpha = \alpha_{crit}$?
- What estimates of the life-span time can we get?

The interested reader should follow the mathematical literature to understand the state-of-the-art of already obtained results.

4.2 Global existence of large data solutions

Up to now we only discussed the global existence of small data solutions.

Question: Can we have for nonlinear Cauchy problems the *global existence of large data solutions*?

This can be shown for models with a special right-hand side. Let us discuss for this reason in the 3-d case the Cauchy problem

$$u_{tt} - \triangle u = -|u|^{\alpha-1}u, \quad u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x).$$

Question: Why do we have the chance to prove the *global existence of large data solutions*?

Answer: The special structure of the right-hand side allows to include the nonlinearity as a *potential part* into the energy. Thus we have the conserved energy

$$E(u)(t) = E(u)(0) \text{ with } E(u)(t) := \int\limits_{\mathbb{R}^n} \Big(\frac{1}{2}(|\nabla u(t,x)|^2 + u_t(t,x)^2) + \frac{1}{\alpha+1}|u|^{\alpha+1}(t,x)\Big)dx.$$

The question is now if the *classical part* of the energy dominates the *potential part* or not. Thus we come to the case $\alpha < 5$ (classical part dominates the potential part, *subcritical case*), $\alpha = 5$ (equilibrium, *critical case*) or $\alpha > 5$ (potential part dominates the classical part, *supercritical case*).

Exercise 35 Can you explain this division of cases?

Only very few is known about the supercritical case. The existence of *global large data solutions* was proved for the subcritical case $\alpha < 5$ in [21].

In the following we will follow the exposition from [60] and study a bit more general model

$$u_{tt} - \triangle u = -f_\alpha(u), \quad u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x).$$

Here the function $f_\alpha = f_\alpha(u)$ satisfies the following properties:

- $f_\alpha \in C^2(\mathbb{R})$,
- $|f_\alpha(u)| + |uf'_\alpha(u)| \leq (1+|u|)^\alpha$ with some $\alpha > 1$,
- $F_\alpha(u) \geq 0$, where $F_\alpha(u) := \int_0^u f_\alpha(s)ds$,
- $|u|^{\alpha+1} \leq C(1 + F_\alpha(u))$,
- in the critical case $\alpha = 5$: $uf_\alpha(u) - 4F_\alpha(u) \geq 0$ if $|u|$ is large.

Under these assumptions the following result holds.

Theorem 4.4 *Let us consider the Cauchy problem*

$$u_{tt} - \triangle u = -f_\alpha(u), \quad u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x).$$

The data are supposed to belong to $C_0^3(\mathbb{R}^3)$, $C_0^2(\mathbb{R}^3)$, respectively. Then there exists a unique global solution $u \in C^2(\mathbb{R}^3 \times [0,\infty))$.

The subcritical part of this theorem was proved in [21]. The critical part was proved in [13].

5 Cauchy problem for hyperbolic systems

5.1 Some preparations

We investigate in this chapter among other things systems of linear first-order partial differential equations having the form

$$\partial_t U + \sum_{k=1}^{n} A_k(t,x)\partial_{x_k} U = F(t,x) \quad \text{in} \quad [0,\infty)\times\mathbb{R}^n$$

subject to the initial condition

$$U(0,x) \;=\; U_0(x).$$

For the following we introduce the notation

$$A(t,x;\xi) := \sum_{k=1}^{n} \xi_k\, A_k(t,x) \quad \text{for} \quad t\geq 0,\ (x,\xi)\in\mathbb{R}^{2n}.$$

Definition 5.1 *The above system of first order is* **called hyperbolic** *if the $m\times m$ matrix $A(t,x,\xi)$ has m* **real eigenvalues** *$\lambda_1(t,x;\xi)\leq\lambda_2(t,x;\xi)\leq\cdots\leq\lambda_m(t,x;\xi)$ and corresponding eigenvectors $\{\vec{r}_j(t,x;\xi)\}_{j=1}^m$ that form a basis of $\mathbb{R}^m$.*

There are two important special cases:

Definition 5.2 *We say that the above system is a* **symmetric hyperbolic system** *if $A_k(t,x)$ is a symmetric $m\times m$ matrix, $k=1,\cdots,m$.*
We say that the above system is **strictly hyperbolic** *if for each $x,\xi\in\mathbb{R}^n,\xi\neq 0$, and each $t\geq 0$, the matrix $A(t,x;\xi)$ has m distinct real eigenvalues:*

$$\lambda_1(t,x;\xi)<\lambda_2(t,x;\xi)<\cdots<\lambda_m(t,x;\xi).$$

Remark: We justify the hyperbolicity condition as follows. Assume $F\equiv 0$ and $A(t,x;\xi)=A(\xi)$, thus all matrices are constant. Let us look for plane wave solutions, that is, we seek a solution U having the form

$$U(t,x)=V(x\cdot\xi-\sigma t),\quad \sigma\in\mathbb{R},$$

for some direction $\xi\in\mathbb{R}^n$, velocity $\frac{\sigma}{|\xi|}$, and profile V. Plugging this ansatz into the system we compute

$$\left(-\sigma I+\sum_{k=1}^{n}\xi_k A_k\right)V'=0.$$

This equality asserts that V' is an eigenvector of the matrix $A(\xi)$ corresponding to the eigenvalue σ. The hyperbolicity condition requires that there are m plane wave solutions to the starting system for each direction ξ. These are

$$(x \cdot \xi - \lambda_j(\xi)t)\vec{r}_j(\xi), \quad j = 1, \cdots, m.$$

The eigenvalues for $|\xi| = 1$ are the wave speeds.

5.2 Some examples

Let us introduce some examples from applications. In electrical engineering the study of transmission lines leads to the system

$$\begin{cases} L(x)\partial_t I + \partial_x E + R(x)I \;=\; 0, \\ C(x)\partial_t E + \partial_x I + G(x)E \;=\; 0. \end{cases}$$

The study of fluid dynamics leads to many hyperbolic systems. The form of the system that governs a particular fluid flow depends on the assumptions made for that flow. The 2-d motion of a perfect inviscid fluid is governed by *Euler's equations of motion*

$$\begin{cases} \partial_t u + u\,\partial_x u + v\,\partial_y u + \frac{1}{\rho}\,\partial_x p \;=\; 0, \\ \partial_t v + u\,\partial_x v + v\,\partial_y v + \frac{1}{\rho}\,\partial_y p \;=\; 0, \\ \partial_t \rho + \partial_x(\rho u) + \partial_y(\rho v) \;=\; 0, \\ \partial_t \left(\frac{p}{\rho^\gamma}\right) + u\,\partial_x \left(\frac{p}{\rho^\gamma}\right) + v\,\partial_y \left(\frac{p}{\rho^\gamma}\right) \;=\; 0. \end{cases}$$

Here u and v are the components of the velocity of the fluid, p is the pressure and ρ is the density of the fluid. The constant $\gamma > 1$ is known as the ratio of specific heats.
The 1-d isentropic (or homentropic) flow of an inviscid gas is modelled by

$$\begin{cases} \partial_t u + u\,\partial_x u + \frac{c^2(\rho)}{\rho}\,\partial_x \rho \;=\; 0, \\ \partial_t \rho + u\,\partial_x \rho + \rho\,\partial_x u \;=\; 0. \end{cases}$$

Here u and ρ are the velocity and density of the gas while $c = c(\rho)$ is the local speed of sound which is assumed to be a known function of ρ. Both models are quasi-linear hyperbolic systems of first order. For a derivation of both models see [12].

If the assumption of constant entropy is dropped, the flow is said to be nonisentropic (or nonhomentropic) and is governed by the system of three equations

$$\begin{cases} \partial_t u + u\,\partial_x u + \frac{1}{\rho}\,\partial_x p \;=\; 0, \\ \partial_t v + u\,\partial_x \rho + \rho\,\partial_x u \;=\; 0, \\ \partial_t p + u\,\partial_x p + c^2(\rho)\rho\,\partial_x u \;=\; 0. \end{cases}$$

For this model see [1].

Exercise 36 The study of long gravity waves on the surface of a fluid in a channel leads to the linear system

$$\begin{cases} \partial_t v + g\,\partial_x \rho \;=\; 0, \\ \partial_t \rho + \frac{S_0(x)}{b}\,\partial_x v + \frac{S_0'(x)}{b}\,v \;=\; 0, \end{cases}$$

where b and g are positive constants, $S_0 = S_0(x)$ is a given positive function representing the equilibrium cross-sectional area of the fluid in the channel (see [30]). Show that the system is hyperbolic in the (x,t)-plane.

5.3 Hyperbolic systems with constant coefficients in two variables

Let us start with the system

$$\partial_t U \;+\; A\,\partial_x U \;=\; 0,$$

where A is a constant $m \times m$ matrix. We suppose that A is *strictly hyperbolic*. Let $\vec{r}_j$ and $\vec{l}_j$, $j = 1, \cdots, m$, be linearly independent right and left eigenvectors. We may normalize these so that $|\vec{r}_j| = 1$, $\vec{l}_j \cdot \vec{r}_k = \delta_{jk}$. By left multiplying the above systems by $\vec{l}_j$ we find

$$\vec{l}_j \cdot \partial_t U + \vec{l}_j \cdot A\,\partial_x U = (\vec{l}_j \cdot U)_t + \lambda_j (\vec{l}_j \cdot U)_x = 0.$$

Setting $z_j := \vec{l}_j \cdot U$ gives

$$\partial_t z_j + \lambda_j \partial_x z_j = \;0, \quad j = 1, \cdots, m.$$

This system has the general solution $z_j(t,x) = Z_j(x - \lambda_j t)$, where Z_j are arbitrary C^1 functions. Consequently, we can write

$$U(t,x) = \sum_{j=1}^{m} Z_j(x - \lambda_j t)\vec{r}_j$$

as the general solution to our starting system.

Theorem 5.1 *Let us consider the Cauchy problem*

$$\partial_t U + A\,\partial_x U \;=\; 0, \;\; U(0,x) = U_0(x).$$

If $U_0 \in C^1(\mathbb{R})$*, there is a unique solution* $u \in C^1(\mathbb{R}^2)$*.*

Proof: We have $U_0(x) = \sum_{j=1}^{m} Z_j(x)\vec{r}_j$ because the system of eigenvectors spans the whole space $\mathbb{R}^n$. Hence, $Z_j(x) = \vec{l}_j \cdot U_0(x)$. Then

$$U(t,x) = \sum_{j=1}^{m} \vec{l}_j \cdot U_0(x - \lambda_j t)\vec{r}_j$$

is a solution. The uniqueness is clear. □

The solution is a linear combination of travelling waves each of which is constant along *characteristic lines* $x - \lambda_j t = \text{const}$.

Exercise 37 Solve the system from Exercise 36 with S_0 constant.

Let us devote to the mixed problem

$$\partial_t U + A\,\partial_x U = 0,\ \ (t,x) \in S = \{(t,x) : t > 0\ ,\ 0 < x < L\},\ U(0,x) = U_0(x),\ x \in (0,L),$$
$$\vec{l}_j \cdot U(t,0) = f_j(t),\ t > 0,\ j \in J_+,\ \ \vec{l}_j \cdot U(t,L) = g_j(t),\ t > 0,\ j \in J_-,$$

where for $j = 1,\cdots,m$ we define $J_+ := \{j : \lambda_j > 0\}$, $J_- := \{j : \lambda_j < 0\}$.

Theorem 5.2 *If $U_0 \in C^1[0,L]$, $f_j, g_j \in C^1[0,\infty)$, and if the compatibility conditions*

$$f_j(0) = \vec{l}_j \cdot U_0(0),\ \ f_j'(0) = -\lambda_j \vec{l}_j \cdot U_0'(0),\ \ j \in J_+,$$
$$g_j(0) = \vec{l}_j \cdot U_0(L),\ \ g_j'(0) = -\lambda_j \vec{l}_j \cdot U_0'(L),\ \ j \in J_-,$$

hold, then there is a unique solution $U \in C^1(\overline{S})$.

To get the uniqueness we may define the energy for a solution

$$E(U)(t) := \frac{1}{2}\int_0^L \sum_{j=1}^{m} (\vec{l}_j \cdot U)^2 dx \equiv \frac{1}{2}\int_0^L \sum_{j=1}^{m} z_j^2(t,x)dx.$$

A simple calculation brings

$$E'(U)(t) = -\frac{1}{2}\sum_{j=1}^{m} \lambda_j (z_j^2(t,L) - z_j^2(t,0)).$$

If $f_j(t) = 0\,(z_j^2(t,0) = 0)$ for $j \in J_+$ and $g_j(t) = 0\,(z_j^2(t,L) = 0)$ for $j \in J_-$, then $E'(U)(t) \le 0$ for $t \ge 0$. We see that the energy is *decreasing* for homogeneous boundary conditions. Thus, if $U_0(x) \equiv 0$, then we get $E(U)(t) \equiv 0$, $U \equiv 0$, respectively. Theorem 5.2 implies the rule that the number of boundary conditions to be assigned at a boundary point must be equal to the number of characteristic lines entering the domain at the point.

5.4 General hyperbolic systems with constant coefficients

In this section we apply the Fourier transformation to solve the constant coefficient system

$$\partial_t U + \sum_{k=1}^{n} A_k \partial_{x_k} U = 0, \;\; U(0,x) = U_0(x).$$

We assume that the matrices $A_k,\;\; k = 1, \cdots, n$, are constant $m \times m$ matrices and that the $m \times m$ matrix $A(\xi) := \sum_{k=1}^{n} \xi_k A_k$ has for each $\xi \in \mathbb{R}^n$ m real eigenvalues $\lambda_1(\xi) \le \lambda_2(\xi) \le \cdots \le \lambda_m(\xi)$. There is no hypothesis concerning the eigenvectors, and so we are supposing instead of Definitions 5.1 and 5.2 a *very weak sort of hyperbolicity* here.

Theorem 5.3 *Assume that* $U_0 \in H^s(\mathbb{R}^n;\mathbb{R}^m)$ *with* $s > \frac{n}{2} + m$. *Then there is a unique solution* $U \in C^1([0,\infty) \times \mathbb{R}^n)$.

Proof: Applying the partial Fourier transformation with respect to x, solving the auxiliary problem and assuming the validity of Fourier inversion formula we obtain

$$U(t,x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-it\,A(\xi)} F(U_0)(\xi) d\xi.$$

First we have to check that the integral converges. Since $U_0 \in H^s(\mathbb{R}^n;\mathbb{R}^m)$ it holds $\langle \xi \rangle^s F(U_0) \in L^2$. So in order to investigate the convergence we must estimate $\|e^{-itA(\xi)}\|$. For a fixed ξ, let Γ denote the path $\partial B_r(0)$ in the complex plane, traversed counter-clockwise, the radius r selected so large that the eigenvalues $\lambda_1(\xi), \cdots, \lambda_m(\xi)$ lie within Γ. Then we have the formula

$$e^{-itA(\xi)} = \frac{1}{2\pi i} \int\limits_{\Gamma} e^{-itz} (zI - A(\xi))^{-1} dz.$$

This can be proved by function-theoretical methods. Define a new path $\tilde{\Gamma}$ in the complex plane as follows: For fixed ξ draw circles $B_k = B_1(\lambda_k(\xi))$ of radius 1 centered at $\lambda_k(\xi),\;\; k = 1, \cdots, m$. Then take $\tilde{\Gamma}$ to be the boundary of $\bigcup_{k=1}^{m} B_k$, traversed counterclockwise. Deforming the path Γ into $\tilde{\Gamma}$ we deduce that

$$e^{-itA(\xi)} = \frac{1}{2\pi i} \int\limits_{\tilde{\Gamma}} e^{-itz} (zI - A(\xi))^{-1} dz.$$

Using

$$|e^{-itz}| \le e^t, \;\; \|(zI - A(\xi))^{-1}\| \le C\, \langle \xi \rangle^{m-1}, \;\; z \in \tilde{\Gamma},$$

we derive the estimate

$$\|e^{-itA(\xi)}\| \le C e^t \langle \xi \rangle^{m-1} \quad \text{for} \;\; \xi \in \mathbb{R}^n.$$

Finally, we proceed as follows

$$\int\limits_{\mathbb{R}^n} |e^{ix\cdot\xi}e^{-itA(\xi)}F(U_0)(\xi)|d\xi \leq C \int\limits_{\mathbb{R}^n} \|e^{-itA(\xi)}\|\ |F(U_0)(\xi)|d\xi$$

$$\leq C\ e^t \int\limits_{\mathbb{R}^n} \langle\xi\rangle^{-s+m-1}\langle\xi\rangle^{s}|F(U_0)(\xi)|d\xi$$

$$\leq C\ e^t\|U_0\|_{H^s}\Big(\int\limits_{\mathbb{R}^n}\langle\xi\rangle^{-2s+2m-2}d\xi\Big)^{1/2}.$$

Taking account of $s > \frac{n}{2} + m - 1$ the integral converges and the function

$$U(t,x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi}e^{-itA(\xi)}F(U_0)(\xi)d\xi$$

is continuous on $[0,\infty)\times\mathbb{R}^n$.
To show U is C^1 observe for $0 < |h| \leq 1$ that

$$\frac{U(t+h,x)-U(t,x)}{h} = \frac{1}{(2\pi)^{\frac{n}{2}}h} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi}\left(e^{-i(t+h)A(\xi)} - e^{-itA(\xi)}\right)F(U_0)(\xi)d\xi.$$

Since

$$e^{-i(t+h)A(\xi)} - e^{-itA(\xi)} = -i\int\limits_t^{t+h} A(\xi)e^{-isA(\xi)}ds,$$

we can estimate as above that

$$\Big|\frac{1}{h}\Big(e^{-i(t+h)A(\xi)} - e^{-itA(\xi)}\Big)\Big| \leq C\ e^{t+1}\langle\xi\rangle^m.$$

Therefore,

$$\Big|\frac{U(t+h,x)-U(t,x)}{h}\Big| \leq C\ e^{t+1}\|U_0\|_{H^s}\Big(\int\limits_{\mathbb{R}^n}\langle\xi\rangle^{-2s+2m}d\xi\Big)^{1/2}.$$

From $s > \frac{n}{2} + m$ it follows that U_t exists and is continuous on $[0,\infty)\times\mathbb{R}^n$. The same statement can be concluded for U_{x_k}. According to Lebesgue Convergence Theorem we can furthermore differentiate under the integral sign in the representation for $U = U(t,x)$ to confirm that U solves our Cauchy problem. □

5.5 Method of characteristics

In Section 5.3 we have seen how the eigenvalues of the coefficient matrix A appear in the process of study of hyperbolic systems with constant coefficients in two variables. In this section we will show how ideas from Section 5.3 are generalized to classes of *semi-linear strictly hyperbolic systems in two variables*. The method is called *Method of Characteristics*. This method is already known from the study of partial differential equations of first order.

We will be interested to study the semi-linear system

$$\partial_t U + A(t,x)\partial_x U + B(t,x,U) = 0.$$

We assume that the matrix $A(t,x)$ is strictly hyperbolic in a domain $G \subset \mathbb{R}^2$. We will show that in a neighborhood of a point $(t_0, x_0) \in G$ the system can be transformed into a very simple *canonical form* by introducing new unknowns. We make use of the eigenvalues and the eigenvectors of the matrix A in a neighborhood of $(t_0, x_0) \in \mathbb{R}^2$. Let $\mathcal{D}$ be the $m \times m$ diagonal matrix with diagonal entries the eigenvalues of A. Let N be the $m \times m$ matrix with columns the corresponding eigenvectors of A. Using an implicit function theorem one can show that the assumption $A \in C^1$ implies $\mathcal{D},\ N \in C^1$ in a small neighborhood of $(t_0, x_0) \in G$. In fact, we use $NAN^{-1} = \mathcal{D}$ in G, thus this transformation exists globally in G due to our assumption of strict hyperbolicity. But we need some regularity of N and $\mathcal{D}$. The regularity C^1 we have at least in a neighborhood of the point $(t_0, x_0) \in G$. The matrix N is nonsingular in a neighborhood, so N^{-1} exists. Substituting $U := NV$, then the above system can be transformed into the strictly hyperbolic semi-linear system

$$\partial_t V + \mathcal{D}(t,x)\partial_x V + B_0(t,x,V) = 0, \quad B_0(t,x,V) = N^{-1}(\partial_t N + A\partial_x N)V + N^{-1}B(t,x,NV),$$

in a neighborhood of $(t_0, x_0) \in G$. The simplicity of the canonical form $\partial_t V + \mathcal{D}(t,x)\partial_x V + B_0(t,x,V) = 0$ becomes apparent if we write it out in component form

$$\partial_t v_k + \lambda_k(t,x)\partial_x v_k + B_{0,k}(t,x,v_1,\cdots,v_m) = 0,\ k = 1,\cdots,m,\ V = (v_1,\cdots,v_m)^T.$$

The principal part of the k'th equation involves only the k'th unknown v_k.

Definition 5.3 *Let us consider the semi-linear hyperbolic system*

$$\partial_t U + A(t,x)\partial_x U + B(t,x,U) = 0.$$

A **characteristic curve** *of this system is a curve in the $t - x$ plane which is defined by the semi-linear ordinary differential equation*

$$\frac{dx}{dt} = \lambda_k(t,x)$$

with $\lambda_k(t,x)$ being an eigenvalue of A.

Remark: One should understand that even for *linear hyperbolic systems* the determination of characteristic curves leads to *semi-linear ordinary differential equations.* There exist m characteristic curves passing through a given point $(t_0, x_0) \in G$. Since the eigenvalues are distinct, the characteristic curves are never tangent.

Now let us come back to the canonical transformation from the beginning of this section. It is clear, that the characteristic curves of a given strictly hyperbolic system remain invariant under transformation to its canonical form. Thus we may restrict ourselves to

$$\partial_t v_k + \lambda_k(t,x)\partial_x v_k + B_{0,k}(t,x,v_1,\cdots,v_m) = 0.$$

The following observation is of importance:

Let Γ_k be a characteristic curve corresponding to λ_k. This curve is given by $x_k = x_k(t)$, where $x_k(t)$ is a solution to $d_t x(t) = \lambda_k(t,x)$. Moreover, let $W_k(t)$ be the function $v_k(t,x)$ restricted to Γ_k. Then

$$\frac{dW_k(t)}{dt} = \partial_t v_k(t,x_k(t)) + \lambda_k(t,x_k(t))\partial_x v_k(t,x_k(t)).$$

Thus, the k'th equation of the canonical form when restricted to a characteristic curve corresponding to λ_k is the ordinary differential equation

$$\begin{aligned} &d_t W_k + B_{0,k}(t,x_k(t),V(t,x_k(t))) = 0, \text{ where} \\ &V(t,x_k(t)) := (v_1(t,x_k(t)),\cdots,v_m(t,x_k(t))). \end{aligned}$$

In the following we are interested in the Cauchy problem for solutions to our starting strictly hyperbolic system. The above observation leads to two strategies concerning the Cauchy problem.

- The characteristic curves are exceptional for the Cauchy problem.
- The characteristic curves can be used to solve Cauchy problems.

Let Γ be a given curve in the $t-x$ plane having the parameter representation $x = x_0(s)$, $t = t_0(s)$, $s \in I$, from $C^1(I)$.

Definition 5.4 *A* **curve** $\Gamma = \Gamma(s) \in C^1(I)$ *in the $t-x$ plane is said to be* **characteristic at a point** $(t_0,x_0) \in G$ *with respect to our starting system*

$$\partial_t U + A(t,x)\partial_x U + B(t,x,U) = 0$$

if there is a characteristic curve to this system which is tangent to Γ *at* (t_0,x_0).

Now let us formulate the Cauchy problem. Therefore let $V_0(s) = (v_{0,1}(s),\cdots,v_{0,m}(s))^T$ be a given vector-function in $C^1(I)$.

Definition 5.5 *The Cauchy problem asks for a vector-function $V = V(t,x)$ which is defined in a domain $G \subset \mathbb{R}^2$ (subdomain of domain of definition of "coefficients" with respect to*

(t,x)) containing the curve Γ such that
• *V is a solution of the above system in G, that is,*

$$\partial_t V + A(t,x)\partial_x V + B(t,x,V) = 0;$$

• *the solution V is equal to V_0 on Γ, that is, $V(t_0(s), x_0(s)) = V_0(s)$ for $s \in I$.*

Exercise 38 Show that, if the initial curve is characteristic, then the Cauchy problem has in general no solution. One can use some knowledge from the theory of ordinary differential equations to guess how to solve this exercise.

For this reason let us consider initial curves *being nowhere characteristic*. Without loss of generality we formulate the initial condition $V(0,x) = V_0(x) = (v_{1,0}(x), \cdots, v_{m,0}(x))^T$ for $x \in (a,b)$.

Theorem 5.5 *Let us devote to the strictly hyperbolic Cauchy problem*

$$\partial_t V + \mathcal{D}(t,x)\partial_x V + B_0(t,x,V) = 0, \;\; V(0,x) = V_0(x),$$

in a domain $G \subset \mathbb{R}^2$ containing the interval (a,b). We assume $\mathcal{D},\ V_0 \in C^1$ while $B_0 \in C^0$. Then the Cauchy problem has a unique classical solution in a subdomain $G_0 \subset G$. The solution depends continuously on the datum.

Proof: The proof of this result can be found in the book [41]. It uses the method of characteristics to transform the Cauchy problem to a *system of integral equations* which is solved by the *method of successive approximations.* □

How does the method of characteristics proceed?

Let (t,x) be a fixed point with $x \in (a,b)$ and $t > 0$. Through (t,x) pass m distinct characteristics $\Gamma_1, \cdots, \Gamma_m$ which are never parallel to the x-axis. If t is sufficiently small, all of these characteristics will intersect the x-axis at points in the interval (a,b). Let $(0, x_k(0))$ be the point of intersection of the k'th characteristic Γ_k with the x-axis. Remember that Γ_k is given by the solution $x = x(t)$ of $d_t x = \lambda_k(t,x), x(0) = x_k(0)$. Along Γ_k the k'th equation of the system

$$\partial_t v_k + \lambda_k(t,x)\partial_x v_k + B_{0,k}(t,x,v_1,\cdots,v_m) = 0$$

is the ordinary differential equation

$$d_t W_k + B_{0,k}(t, x_k(t), V(t, x_k(t))) = 0.$$

Integrating this equation along Γ_k over the integral $[0,t]$ we obtain

$$W_k(t) - W_k(0) + \int_0^t B_{0,k}(s, x_k(s), V(s, x_k(s)))ds = 0.$$

On the one hand $W_k(t) = v_k(t, x_k(t))$, on the other hand from the initial condition we know $W_k(0) = v_k(0, x_k(0)) = v_{0,k}(x_k(0))$. Substituting these relations into the above one and remembering that this procedure holds for every $k = 1, 2, \cdots, m$, we obtain the system of integral equations

$$v_k(t,x) = v_{0,k}(x_k(0)(t,x)) - \int_0^t B_{0,k}(s, x_k(s)(t,x), v_1(s, x_k(s)(t,x)), \cdots, v_m(s, x_k(s)(t,x)))ds,$$

$$k = 1, 2, \cdots, m.$$

Here we have to note again, that the characteristic curves and the points of intersection of characteristic curves with the x-axis depend on the chosen (t, x). This is expressed in the above formula by (t, x). Briefly, the *method of successive approximations* for solving this system of integral equations proceeds in the following manner. The first approximation $V^{(0)}(t,x) := (v_1^{(0)}(t,x), \cdots, v_m^{(0)}(t,x))$ is assumed to have components

$$v_k^{(0)}(t,x) = v_k(0, x_k(0)) = v_{0,k}(x_k(0)), \quad k = 1, 2, \cdots, m.$$

To obtain the next approximation $V^{(1)}(t,x) := (v_1^{(1)}(t,x), \cdots, v_m^{(1)}(t,x))$ we have to consider the system of integral equations

$$v_k^{(1)}(t,x) = v_k(0, x_k(0)) - \int_0^t B_{0,k}(s, x_k(s), v_1^{(0)}(s, x_k(s)), \cdots, v_m^{(0)}(s, x_k(s)))ds, \quad k = 1, 2, \cdots, m,$$

and the indicated integration is carried out. The general iteration scheme is

$$v_k^{(l+1)}(t,x) = v_k(0, x_k(0)) - \int_0^t B_{0,k}(s, x_k(s), v_1^{(l)}(s, x_k(s)), \cdots, v_m^{(l)}(s, x_k(s)))ds, \quad k = 1, 2, \cdots, m.$$

It can be shown that the sequence of approximations $V^{(l)}(t,x)$ converges to the desired solution.

Example: Finally, let us explain how the method of characteristics is used to solve the system from electrical engineering with an infinite transmission line (see Section 5.2)

$$\begin{cases} L\partial_t I + \partial_x E + RI = 0, \\ C\partial_t E + \partial_x I + GE = 0, \\ I(0,x) = I_0(x), \quad E(0,x) = E_0(x), \quad x \in \mathbb{R}. \end{cases}$$

This system is hyperbolic in the whole (t, x)-plane with constant eigenvalues $\lambda_1 = 1/\sqrt{LC}$ and $\lambda_2 = -1/\sqrt{LC}$. Our first step is to transform this system to its canonical form. In terms of the new variables v_1 and v_2 related to I and E by

$$\begin{pmatrix} I \\ E \end{pmatrix} = \begin{pmatrix} \sqrt{C} & \sqrt{C} \\ \sqrt{L} & -\sqrt{L} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{2\sqrt{LC}} \begin{pmatrix} \sqrt{L} & \sqrt{C} \\ \sqrt{L} & -\sqrt{C} \end{pmatrix} \begin{pmatrix} I \\ E \end{pmatrix},$$

the canonical form is

$$\frac{\partial}{\partial t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{LC}} & 0 \\ 0 & -\frac{1}{\sqrt{LC}} \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} +$$

$$\frac{1}{2LC}\begin{pmatrix} RC+LG & RC-LG \\ \\ RC-LG & RC+LG \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The new unknowns must satisfy the initial condition

$$v_1(0,x) = \varphi_1(x) \equiv \frac{1}{2\sqrt{LC}}\Big(\sqrt{L}\, I_0(x) + \sqrt{C}\, E_0(x)\Big),$$
$$v_2(0,x) = \varphi_2(x) \equiv \frac{1}{2\sqrt{LC}}\Big(\sqrt{L}\, I_0(x) + \sqrt{C}\, E_0(x)\Big), \quad -\infty < x < \infty.$$

Following the method of characteristics for solving this initial value problem let (t,x) be an arbitrary but fixed point in the upper-half (s,x)-plane. The characteristic curves Γ_1 and Γ_2 corresponding to $\lambda_1 = 1/\sqrt{LC}$ and $\lambda_2 = -1/\sqrt{LC}$ and passing through (t,x) are the lines,

$$\Gamma_1 : x_1 = x_1(s) \equiv x + \frac{1}{\sqrt{LC}}\,(s-t), \quad \Gamma_2 : x_2 = x_2(s) \equiv x - \frac{1}{\sqrt{LC}}\,(s-t),$$

and these lines intersect the x-axis at the points (set $s=0$)

$$x_1(0) = x - \frac{1}{\sqrt{LC}}\,t, \quad x_2(0) = x + \frac{1}{\sqrt{LC}}\,t.$$

The system of integral equations in this case is

$$\begin{cases} v_1(t,x) = \varphi_1\Big(x - \frac{1}{\sqrt{LC}}\,t\Big) - \frac{1}{2LC}\int_0^t \Big((RC+LG)v_1(s,x_1(s)) + (RC-LG)v_2(s,x_1(s))\Big)ds, \\ v_2(t,x) = \varphi_2\Big(x + \frac{1}{\sqrt{LC}}\,t\Big) - \frac{1}{2LC}\int_0^t \Big((RC-LG)v_1(s,x_2(s)) + (RC+LG)v_2(s,x_2(s))\Big)ds, \end{cases}$$

where $x_1(s)$ and $x_2(s)$ are given.

In general, in order to proceed any further with the computation of the solution, it is necessary to know the specific functional form of the initial data and employ a numerical approximation scheme. However, in the *special case in which the electrical parameters of the transmission line satisfy the relation*

$$RC - LG = 0,$$

it is possible to obtain a general formula for the solution. This is due to the fact that when this condition holds, the canonical form is separated in the sense that each equation involves only one of the unknowns v_1, v_2. These unknowns restricted on the characteristics are

$$W_1(s) = v_1\Big(s, x + \frac{1}{\sqrt{LC}}\,(s-t)\Big), \quad W_2(s) = v_2\Big(s, x - \frac{1}{\sqrt{LC}}\,(s-t)\Big).$$

They satisfy the corresponding ordinary differential equations which in this case are

$$\frac{dW_1}{ds} + \frac{R}{L}W_1 = 0, \quad \frac{dW_2}{ds} + \frac{R}{L}W_2 = 0,$$

and the initial conditions

$$W_1(0) = \varphi_1(x_1(0)), \quad W_2(0) = \varphi_2(x_2(0)).$$

Solving these two initial value problems we obtain

$$W_1(s) = \varphi_1(x_1(0))\, e^{-\frac{R}{L}s}, \quad W_2(s) = \varphi_2(x_2(0))\, e^{-\frac{R}{L}s}.$$

We have

$$v_1(t,x) = W_1(t), \quad v_2(t,x) = W_2(t).$$

Consequently, we get under the condition $RC - LG = 0$ the solution

$$v_1(t,x) = \varphi_1\left(x - \frac{1}{\sqrt{LC}}t\right)e^{-\frac{R}{L}t}, \quad v_2(t,x) = \varphi_2\left(x + \frac{1}{\sqrt{LC}}t\right)e^{-\frac{R}{L}t}.$$

The solution of the original problem is obtained in the form

$$\begin{aligned} I(t,x) = \Big\{&\frac{1}{2}\Big(I_0(x - \frac{1}{\sqrt{LC}}t) + I_0(x + \frac{1}{\sqrt{LC}}t)\Big) \\ &+\frac{1}{2}\sqrt{\frac{C}{L}}\Big(E_0(x - \frac{1}{\sqrt{LC}}t) - E_0(x + \frac{1}{\sqrt{LC}}t)\Big)\Big\}e^{-\frac{R}{L}t}, \\ E(t,x) = \Big\{&\frac{1}{2}\sqrt{\frac{L}{C}}\Big(I_0(x - \frac{1}{\sqrt{LC}}t) - I_0(x + \frac{1}{\sqrt{LC}}t)\Big) \\ &+\frac{1}{2}\Big(E_0(x - \frac{1}{\sqrt{LC}}t) + E_0(x + \frac{1}{\sqrt{LC}}t)\Big)\Big\}e^{-\frac{R}{L}t}. \end{aligned}$$

Note that this solution shows that in the case $RC = LG$ the initial data propagate along the line *without distortion* other than exponential decay with increasing time. For this reason a line for which $RC = LG$ is called a *distortionless line*.

Exercise 39 Try to find another way to solve the above Cauchy problem! Can we understand why does an exponential decay in t come in?

5.6 Cauchy problem for linear hyperbolic systems

Let $U = U(t,x)$, $(t,x) \in [0,\infty) \times \mathbb{R}^n$, and let the formal linear differential operator L be defined by

$$LU := A_0(t,x)\partial_t U + \sum_{k=1}^{n} A_k(t,x)\partial_{x_k} U + B(t,x)U.$$

Here $A_0, A_1, \cdots, A_n$ and B are complex $m \times m$ matrices depending on t, x. The matrix A_0 is assumed to be *positive definite*, uniformly with respect to t and to x. The matrices $A_k,\ \ k = 1, \cdots, n$, are assumed to be *Hermitian*.

We assume $A_0, A_1, \cdots, A_n \in \mathcal{B}^1,\ \ B \in \mathcal{B}^0$ (for the notations see the end of Section 3.6). Let

$$a_0 := \min_{V,t,x;|V|=1} A_0(t,x)V \cdot \overline{V} > 0,\ a_1 := \max_{V,t,x,j;|V|=1} |A_j(t,x)V \cdot \overline{V}| > 0,\ \ \alpha := \frac{a_0}{n a_1}.$$

Let $\Omega^\alpha(t_0)$ be the truncated cone $\Omega^\alpha(t_0) := \{(t,x) : x \in K(0, \frac{T_0 - t}{\alpha}),\ \ 0 \le t \le t_0,\ \ T_0 \ge t_0\}$. As in the proof to Theorem 3.5 the boundary $\partial\,\Omega^\alpha(t_0)$ consists of three parts. The lateral surface M is of most interest. The cone $\Omega^\alpha(t_0)$, in particular α, has been chosen in such a way that for integrals of the type

$$\int\limits_{\Omega^\alpha(t_0)} LU \cdot \overline{U} \cdots \qquad \text{the terms} \quad \int\limits_M \cdots,$$

arising through partial integration, are pointwise nonnegative. In such a case M is called *space-like* for L.
We introduce the following notation:

$$|U(t)|_G := \Big(\int\limits_G A_0(t,x)U(t,x) \cdot \overline{U(t,x)} dx \Big)^{1/2}.$$

Theorem 5.6 (Domain of dependence inequality)
Let $U \in C^1(\Omega^\alpha(t_0))$ be a solution to

$$LU = F \in C^0(\Omega^\alpha(T_0)), \quad U(0,x) = U_0(x) \in C^0\Big(K\Big(0, \frac{T_0}{\alpha}\Big)\Big).$$

Then there exists a positive constant $C = C(\|A_0, \partial_t A_0, \nabla_x A_k, B\|_{C^0(\Omega^\alpha(T_0))})$ such that for all $t \in [0, t_0],\ t_0 \in (0, T_0)$, it holds

$$|U(t)|_{K(0,\ \frac{T_0 - t}{\alpha})} \le C\Big(|U_0|_{K(0,\ \frac{T_0}{\alpha})} + \Big(\int\limits_0^{T_0} |F(r)|^2_{K(0,\ \frac{T_0 - r}{\alpha})} dr \Big)^{1/2} \Big) e^{ct}.$$

Proof: We have

$$Re\ LU \cdot \overline{U} = Re(A_0 \partial_t U \cdot \overline{U} + \sum_{k=1}^n A_k \partial_{x_k} U \cdot \overline{U} + BU \cdot \overline{U}) = Re\ F \cdot \overline{U}.$$

This implies

$$Re\ \Big(\frac{1}{2}\,\partial_t (A_0 U \cdot \overline{U}) - \frac{1}{2}\,(\partial_t A_0) U \cdot \overline{U} + \frac{1}{2}\ \sum_{k=1}^n\ \partial_{x_k} (A_k U \cdot \overline{U})$$
$$- \frac{1}{2}\ \sum_{k=1}^n\ (\partial_{x_k} A_k) U \cdot \overline{U} + BU \cdot \overline{U} \Big) = Re\ F \cdot \overline{U}.$$

Let $H := \partial_t A_0 + \sum\limits_{k=1}^{n} \partial_{x_k} A_k - 2B$. Then we obtain by integration over $\Omega^\alpha(t_0)$

$$\int\limits_{\partial\Omega^\alpha(t_0)} (\vec{n}_t A_0 U \cdot \overline{U} + \sum_{k=1}^{n} \vec{n}_k A_k U \cdot \overline{U}) = \int\limits_{\Omega^\alpha(t_0)} (Re\ HU \cdot \overline{U} + 2\ Re\ F \cdot \overline{U}),$$

where $\vec{n}$ denotes the exterior normal vector to $\partial\Omega^\alpha(t_0)$.
The lateral surface M of $\Omega^\alpha(t_0)$ can be parametrized as follows:

$$M = \Big\{(t,x) :\ t = \gamma(x) := T_0 - \alpha|x|,\ x \in K\Big(0, \frac{T_0}{\alpha}\Big),\ \ t \in [0, t_0]\Big\}.$$

The normal vector $\vec{n}$ is given by $\vec{n} = \frac{1}{\sqrt{1+\alpha^2}}\,(1, \frac{\alpha x}{|x|})$. Thus we conclude

$$\begin{aligned}
&\int\limits_{K(0,\,\frac{T_0-t_0}{\alpha})} A_0(t_0,x)U(t_0,x) \cdot \overline{U(t_0,x)}\,dx - \int\limits_{K(0,\,\frac{T_0}{\alpha})} A_0(0,x)U_0(x) \cdot \overline{U_0(x)}\,dx \\
&+ \frac{1}{\sqrt{1+\alpha^2}} \int\limits_M \Big(A_0(t,x)U(t,x) \cdot \overline{U(t,x)} - \sum_{k=1}^{n} (\partial_{x_k}\gamma) A_k(t,x)U(t,x) \cdot \overline{U(t,x)}\Big) \\
&= \int\limits_{\Omega^\alpha(t_0)} (Re\ HU \cdot \overline{U} + 2\ Re\ F \cdot \overline{U}).
\end{aligned}$$

By the definition of γ and $\alpha := \frac{a_0}{na_1}$ we get

$$\left|\sum_{k=1}^{n} (\partial_{x_k}\gamma) A_k(t,x)U(t,x) \cdot \overline{U(t,x)}\right| \le a_0\, U \cdot \overline{U} \le A_0\, U \cdot \overline{U}.$$

Hence, $\int\limits_M \cdots$ is non-negative. The statement of the theorem follows from the application of Gronwall's inequality to

$$\begin{aligned}
|U(t)|^2_{K(0,\frac{T_0-t}{\alpha})} &\le |U_0|^2_{K(0,\frac{T_0}{\alpha})} + \int\limits_0^t \int\limits_{K(0,\frac{T_0-r}{\alpha})} |H\ U \cdot \overline{U}| dx dr \\
&+2 \int\limits_0^t \int\limits_{K(0,\frac{T_0-r}{\alpha})} |F||U|(r,x) dx dr \\
&\le |U_0|^2_{K(0,\frac{T_0}{\alpha})} + C \int\limits_0^t |U(r)|^2_{K(0,\frac{T_0-r}{\alpha})} dr + C \int\limits_0^t |F(r)|^2_{K(0,\frac{T_0-r}{\alpha})} dr.
\end{aligned}$$

In this way we understood how to derive the *domain of dependence inequality.* □

Exercise 40 Study Gronwall's lemma and try to prove it.

From the last theorem we will conclude a corollary with the corresponding estimates for higher order derivatives. Let us introduce the notation

$$|U(t)|_{s,G} := \Big(\sum_{|\alpha|\le s} |\nabla^\alpha U(t)|_G^2\Big)^{1/2} \qquad \text{with} \quad s \in \mathbb{N}_0.$$

Corollary 5.1 *Let $A_0,\ A_k,\ B \in \mathcal{B}^s$, $s \ge 1$, and let $U \in C^{s+1}(\Omega^\alpha(t_0))$ be a solution to*

$$LU \ = \ F \in C^s(\Omega^\alpha(T_0)), \qquad U(0,x) = U_0(x) \in C^s\Big(K\Big(0, \frac{T_0}{\alpha}\Big)\Big).$$

Then there exists a positive constant $C = C(\|A_0, A_k, B\|_{C^s(\Omega^\alpha(T_0))})$ such that for all $t \in [0, t_0]$, $t_0 \in (0, T_0)$, it holds

$$|U(t)|_{s,K(0,\frac{T_0-t}{\alpha})} \le C\Big(|U_0|_{s,K(0,\frac{T_0}{\alpha})} + \Big(\int_0^{T_0} |F(r)|^2_{s,K(0,\ \frac{T_0-r}{\alpha})} dr\Big)^{1/2}\Big)e^{Ct}.$$

Proof: First we differentiate the system

$$A_0(t,x)\partial_t U + \sum_{k=1}^n A_k(t,x)\partial_{x_k} U + B(t,x)U = F(t,x)$$

with respect to $x_k,\ k = 1, \cdots, n$. Using this system we can express $\partial_t U$ in terms of ∇U and U. Setting $V := (U, \nabla U)$ we obtain the following system for V:

$$\tilde{A}_0(t,x)\partial_t V + \sum_{k=1}^n \tilde{A}_k(t,x)\partial_{x_k} V + \tilde{B}(t,x)\, V = F_1(t,x), \ \ V(0,x) = V_0(x),$$

where $\tilde{B}$ is a matrix composed of $B,\ \nabla A_0,\ \nabla B$ and $F_1 := (F, \nabla F)$. Let $s = 1$. Then due to the assumptions $\tilde{B}, \tilde{F} \in \mathcal{B}^0,\ \tilde{A}_0,\ \tilde{A}_k \in \mathcal{B}^1$. Hence, Theorem 5.6 is applicable and $V \in C^1(\Omega^\alpha(T_0))$ or $U \in C^2(\Omega^\alpha(T_0))$ should be supposed. In an analogous way one proves the assertion for $s > 1$. □

The statement of Theorem 5.6 will imply uniqueness of classical solutions to

$$A_0(t,x)\partial_t U + \sum_{k=1}^n A_k(t,x)\partial_{x_k} U + B(t,x)U = 0, \ \ U(0,x) = U_0(x).$$

Question: What about an existence result?

Answer: An answer will be given by the next result.

Theorem 5.7 *Let $s \in \mathbb{N},\ s > \frac{n}{2} + 1$. Moreover, we assume $A_0,\ A_k,\ B \in \mathcal{B}^{s+1}$ and $U_0 \in H^s$. Then there exists (a unique) classical solution $U \in C^1([0,\infty) \times \mathbb{R}^n)$ possessing the additional regularity $U \in C^0([0,\infty), H^s) \cap C^1([0,\infty), H^{s-1})$.*

Sketch of the proof: The existence will be proved in four steps. We will only sketch each of the steps.

Step 1 First we assume that the *coefficients of L are analytic.* For a fixed but arbitrary positive T_0 we *approximate* U_0 in $K(0, \frac{T_0}{\beta}),\ \beta < \alpha$, by a *sequence* $\{U_0^k\}$ *of polynomials* in $H^s(K(0, \frac{T_0}{\beta}))$. By the *Cauchy-Kovalevsky theorem* (cf. with Theorem 2.1 generalized to the system case) there is a local in time solution U_k to $LU = 0,\quad U(0,x) = U_0^k(x)$ in $\Omega^\beta(T_0) \cap \{(t,x) : t \le \delta\}$ for some $\delta \in (0, T_0]$. Using Corollary 5.1 it follows that U_k converges in $H^s(K(0, \frac{T_0-t}{\beta})),\ 0 \le t \le \delta$. Let $U(t)$ be the limit in $H^s(K(0, \frac{T_0-t}{\beta}))$. Then one can show that U solves $LU = 0,\ U(0,x) = U_0(x)$ in $\Omega^\beta(T_0) \cap \{(t,x) : t \le \delta\}$. Now we may consider a new initial value problem on $t = \delta$ and obtain an extension of U into $\Omega^\beta(T_0) \cap \{(t,x) : t \le \delta + \delta_1\}$. Step by step we have a solution $U \in H^s(K(0, \frac{T_0-t}{\beta}))$ for $t \in [0, T_0)$.

Step 2 Let $A_0,\ A_k,\ B \in \mathcal{B}^{s+1}$ and let $U_0 \in H^{s+1}$. We *approximate $A_0,\ A_k,\ B$ by analytic functions $A_0^l,\ A_k^l,\ B^l$* respectively, uniformly with respect to all derivatives up to order $s+1$ in $\Omega^\beta(T_0)$ and such that the lateral surface M^β is space-like for each operator

$$L_l := A_0^l(t,x)\partial_t + \sum_{k=1}^n A_k^l(t,x)\partial_{x_k} + B^l.$$

It should be clear that if the lateral surface M^β is space-like for the operator L, then it is space-like for good approximations L_l of the operator L, too. The Cauchy problem $L_lU = 0,\ U(0,x) = U_0(x)$ can be solved as in Step 1. We conclude

$$U_l \in C^0\Big([0,T], H^{s+1}\Big(K\Big(0, \frac{T_0 - T}{\beta}\Big)\Big)\Big) \cap C^1\Big([0,T],\ H^s\Big(K\Big(0, \frac{T_0 - T}{\beta}\Big)\Big)\Big),$$

where $T \le t_0 < T_0$. Then we can conclude that $\{U_l\}$ converges in $H^s(K(0, \frac{T_0-t}{\beta}))$ to some $U \in H^s(K(0, \frac{T_0-t}{\beta}))$. Here we can use the *compact imbedding of H^{s+1} into H^s*. With the same arguments as in Step 1 we get

$$\begin{aligned}
&LU = 0,\ U(0,x) = U_0(x) \quad \text{in} \quad \Omega^\beta(T_0),\\
&U \in C^0\Big([0,T],\ H^s\Big(K\Big(0,\ \frac{T_0 - T}{\beta}\Big)\Big)\Big) \cap C^1\Big([0,T],\ H^{s-1}\Big(K\Big(0,\ \frac{T_0 - t}{\beta}\Big)\Big)\Big), \quad 0 < T \le t_0 < T_0.
\end{aligned}$$

Step 3 Let $A_0, \ A_k, \ B \in \mathcal{B}^{s+1}$ and let $U_0 \in H^s$. There is a *sequence* $\{U_0^l\} \subset H^{s+1}$ *approximating* $U_0 \in H^s$. According to Step 2 there are solutions U_l of $LU = 0, \ U(0,x) = U_0^l(x)$ in $\Omega^\beta(T_0)$ and they satisfy

$$\sup_{t\in[0,t_0]} \|U_k - U_j\|_{H^s(K(0,\frac{T_0-t}{\beta}))} \le C \, \|U_0^k - U_0^j\|_{H^s} \, .$$

This implies the existence of a solution U belonging to
$C^0([0,T], H^s(K(0, \ \frac{T_0-T}{\beta}))) \cap C^1([0,T], H^{s-1}(K(0, \frac{T_0-T}{\beta})))$ in $\Omega^\beta(T_0)$.

Step 4 The coefficients are assumed to be uniformly bounded. Therefore, for every truncated cone which is congruent to $\Omega^\beta(T_0)$ and which is obtained by translation at any place in $[0,\infty) \times \mathbb{R}^n$ we can find a local in time solution. Using the uniqueness properties for overlapping cones we can construct by this procedure a global solution $u \in C^1([0,\infty)\times\mathbb{R}^n)$. Finally, one can show that this solution belongs to $C^0([0,\infty), H^s) \cap C^1([0,\infty), H^{s-1})$. □

5.7 Cauchy problem for quasi-linear hyperbolic systems

In the sketch of the proof to Theorem 4.1 we needed a local (in time) existence result for the Cauchy problem for quasi-linear hyperbolic systems. In this section we will formulate such a result from [43] and formulate the auxiliary lemmas which proofs form the core of the proof of the main result.

Let us devote to the quasi-linear Cauchy problem

$$A_0(U)\partial_t U + \sum_{k=1}^{n} A_k(U)\partial_{x_k} U + B(U)U = 0, \quad U(0,x) \ = \ U_0(x).$$

As in the previous section we suppose that A_0, A_k, B are complex $m \times m$ matrices and C^∞ functions of their arguments $U \in \mathbb{C}^m$. Moreover, $A_k(U)$ are Hermitian and $A_0(U)$ is positive definite, uniformly on each compact set with respect to U.

By κ_s we shall denote the Sobolev constant characterizing the continuous embedding of H^s into the space of uniformly bounded continuous functions if $s > \frac{n}{2}$. In this section we use the notation

$$|U|_{s,T} := \sup_{0\le t\le T} \|U(t,\cdot)\|_{H^s} \text{ for } U \in L^\infty([0,T], H^s), \ s \in \mathbb{N}_0.$$

Theorem 5.8 *Let $U_0 \in H^s$ with $s > \frac{n}{2} + 1$. Let $g_1 := \kappa_s \|U_0\|_{H^s}$ and $g_2 > g_1$ arbitrary but fixed. Then there is a unique classical solution $u \in \mathcal{B}^1([0,T] \times \mathbb{R}^n)$ with*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^n} |U(t,x)| \le g_2, \quad U \in C^0([0,T], H^s) \cap C^1([0,T], H^{s-1}).$$

Here T depends on g_2 and $\|U_0\|_{H^s}$.

Question: Which statements should be proved to conclude Theorem 5.8?
Answer: The proof bases on the following statements:

• A lemma about uniqueness of solutions from the larger class $C^0([0,T],H^1)\cap C^1([0,T],L^2)\cap L^\infty([0,T],H^s)$.

• We define the sequence of data $U_0^l := J_{\varepsilon_l}U_0$, where J_{ε_l} denotes the convolution with the Friedrich's mollifier. Thus the datum U_0^l belongs to H^∞. Then let U_{l+1} be defined by iteration as the solution of the linear Cauchy problem

$$A_0(U_l)\partial_t U_{l+1} + \sum_{k=1}^n A_k(U_l)\partial_{x_k}U_{l+1} + B(U_l)U_{l+1} = 0, \quad U_{l+1}(0,x) = U_0^{l+1}.$$

This is a linear hyperbolic system with positive definite matrix A_0 and Hermitian matrices A_k. Due to Theorem 5.7 there exists a well-defined solution

$$U_{l+1} \in C^0([0,T],H^m)\cap C^1([0,T],H^{m-1})\cap C^\infty([0,\infty)\times\mathbb{R}^n) \text{ for all } m\in\mathbb{N}_0.$$

Lemma 5.1 *There are positive constants R, L, T_* depending on g_2 and $\|U_0\|_{H^s}$ such that*

$$|U_l|_{s,T_*} \le R, \quad |\partial_t U_l|_{s-1,T_*} \le L, \quad |U_l(t,x)| \le g_2 \text{ for all } (t,x)\in[0,T_*]\times\mathbb{R}^n.$$

• The previous lemma gives the boundedness of the sequence $\{U_l\}$ in H^s. In this step we will prove convergence of subsequences in low norms.

Lemma 5.2 *There are nonnegative real numbers T, α, β_k with $\alpha\in(0,1)$, $\sum_{k=1}^\infty \beta_k < \infty$, and $0<T<T_*$ such that*

$$|U_{k+1}-U_k|_{0,T} \le \alpha|U_k - U_{k-1}|_{0,T} + \beta_k.$$

• A consequence of the last lemma is that $\{U_l\}$ converges to U in $C([0,T],L^2)$.
• In this step we explain the behavior of U in high norms, namely
Lemma 5.3 *The limit U satisfies $U\in L^\infty([0,T],H^s)$ with $|U|_{s,T}\le R$.*
• We can show that the whole sequence $U_l(t,\cdot)$ converges *in a weak sense* in H^s to U for all $t\in[0,T]$, that is, $U_l \rightharpoonup U$ in H^s.
• Let $C_w([0,T],E)$, $\mathrm{Lip}([0,T],E)$, respectively, be the space of functions which are valued in a Banach space E and which are *weakly continuous, Lipschitz continuous* with respect to t. Then the following statement holds:
Lemma 5.4 *The limit U belongs to $C_w([0,T],H^s)\cap Lip([0,T],H^{s-1})$.*
• In order to prove Theorem 5.8 it remains to show $U\in C([0,T],H^s)\cap C^1([0,T],H^{s-1})$ which implies $U\in\mathcal{B}^1([0,T]\times\mathbb{R}^n)$ by *Sobolev's imbedding theorem.*
• It is sufficient to prove $U\in C([0,T],H^s)$. From the starting system we conclude $U\in C^1([0,T],H^{s-1})$.
• It is sufficient to prove the *continuity* of U *on the right* of $t\in[0,T)$. A transformation

$V(t,\cdot) := U(T-t,\cdot)$ and repeating all the steps to the transformed system brings the *continuity* of U *on the left* of $t \in (0,T]$.
• It is sufficient to prove the *continuity* of U *on the right* of $t=0$.
For this purpose we introduce the norm

$$\|U(\cdot)\|_{s,A_0(t)} := \sum_{|\alpha|\le s} \int_{\mathbb{R}^n} A_0(t)\nabla^\alpha U(x) \cdot \overline{\nabla^\alpha U(x)}dx,\ A_0(t) = A_0(U(t,\cdot)),$$

which is equivalent to the H^s-norm. Then one can show

$$\limsup_{t\to+0} \|U(t,\cdot)\|^2_{s,A_0(t)} = \limsup_{t\to+0} \|U(t,\cdot)\|^2_{s,A_0(0)}.$$

Using now $U \in C_w([0,T],H^s)$ it remains to show

$$\limsup_{t\to+0} \|U(t,\cdot)\|^2_{s,A_0(t)} \le \|U_0\|^2_{s,A_0(0)}.$$

• The last inequality follows from the following statement:
Lemma 5.5 *There is a positive constant such that*

$$\|U(t,\cdot)\|^2_{s,A_0(t)} \le \|U_0\|^2_{s,A_0(0)} + ct \text{ for all } t \in [0,T].$$

6 Diffusion phenomenon

6.1 Parabolic structure of solutions to damped waves

We begin this section with an exercise.
Exercise 41 Let us consider the mixed problem

$$\varepsilon^2 u_{tt} - u_{xx} + u_t = 0,\ \ u(0,x,\varepsilon) = \varphi(x),\ \ u_t(0,x,\varepsilon) = \psi(x),\ x \in (0,L),$$
$$u(t,0,\varepsilon) = u(t,L,\varepsilon) = 0,$$

with sufficiently smooth data φ and ψ. We assume that the compatibility conditions are satisfied. Let $u = u(t,x,\varepsilon)$ be the unique solution of this problem (without explaining the precise regularity). Prove, that for every fixed (t,x) the following relation holds: $\lim_{\varepsilon\to 0} u(t,x,\varepsilon) = w(t,x)$, where $w = w(t,x)$ solves the mixed problem

$$w_t - w_{xx} = 0,\ \ w(0,x) = \varphi(x),\ x \in (0,L),\ w(t,0) = w(t,L) = 0.$$

<u>Question:</u> To what does this exercise hint?

Answer: From time to time the mathematicians use instead of the heat equation $w_t - w_{xx} = 0$ which solutions possess an *infinite speed of propagation* the damped wave equation $\varepsilon^2 u_{tt} - u_{xx} + u_t = 0$, $\varepsilon^2 > 0$ is small. Those solutions have a *finite speed of propagation*. The speed depends on ε. In the last exercise we only studied a pointwise behavior. It would be better to have a corresponding result $\lim_{\varepsilon \to 0} u(t, x, \varepsilon) = w(t, x)$ in suitable norms.

The main result of this section is a remarkable relation between solutions of the heat and of the damped wave equation.
Let us consider the Cauchy problem for the heat equation

$$w_t - \Delta w = 0, \quad w(0, x) = \varphi(x).$$

Applying the partial Fourier transformation yields for $\hat{w}(t, \xi) = F_{x \to \xi}(w(t, \cdot))(\xi)$ the Cauchy problem

$$\hat{w}_t + |\xi|^2 \hat{w} = 0, \quad \hat{w}(0, \xi) = \hat{\varphi}(\xi).$$

Its solution is

$$\hat{w}(t, \xi) = e^{-|\xi|^2 t} \hat{\varphi}(\xi).$$

After application of inverse Fourier transformation we obtain

$$w(t, x) = F^{-1}_{\xi \to x} \left(e^{-|\xi|^2 t} \hat{\varphi}(\xi) \right).$$

Lemma 6.1 *We study the Cauchy problem*

$$w_t - \Delta w = 0, \quad w(0, x) = \varphi(x).$$

Then we have the following estimates for the derivatives $\partial_t^k \partial_x^\alpha w(t, \cdot)$ of the solution w:

$$\|\partial_t^k \partial_x^\alpha w(t, \cdot)\|_{L^2} \leq C_{k,\alpha} (1 + t)^{-k - \frac{|\alpha|}{2}} \|\varphi\|_{H^{2k+|\alpha|}}.$$

Proof: Using the properties of the Fourier transformation it holds

$$\partial_t^k \partial_x^\alpha w(t, x) = F^{-1}_{\xi \to x} \left((-1)^k i^{|\alpha|} |\xi|^{2k} \xi^\alpha e^{-|\xi|^2 t} \hat{\varphi}(\xi) \right).$$

From Parseval's inequality we get for $t \geq 1$

$$\begin{aligned} \|\partial_t^k \partial_x^\alpha w(t, \cdot)\|^2_{L^2} &= \left\| |\xi|^{2k} \xi^\alpha e^{-|\xi|^2 t} \hat{\varphi}(\xi) \right\|^2_{L^2} \leq \left\| |\xi|^{2k+|\alpha|} e^{-|\xi|^2 t} \hat{\varphi}(\xi) \right\|^2_{L^2} \\ &= \left\| \frac{|\xi|^{2k+|\alpha|} t^{k + \frac{|\alpha|}{2}}}{t^{k + \frac{|\alpha|}{2}}} \, e^{-|\xi|^2 t} \hat{\varphi}(\xi) \right\|^2_{L^2}. \end{aligned}$$

The term

$$|\xi|^{2k+|\alpha|}t^{k+\frac{|\alpha|}{2}}e^{-|\xi|^2t} = \left(|\xi|^2t\right)^{k+\frac{|\alpha|}{2}}e^{-|\xi|^2t}$$

is uniformly bounded for all k and α by a constant $C_{k,\alpha}$. Consequently, ($C_{k,\alpha}$ is a universal constant.)

$$\|\partial_t^k\partial_x^\alpha w(t,\cdot)\|_{L^2}^2 \le C_{k,\alpha}t^{-2k-|\alpha|}\|\hat\varphi\|_{L^2}^2 = C_{k,\alpha}t^{-2k-|\alpha|}\|\varphi\|_{L^2}^2,$$

$$\|\partial_t^k\partial_x^\alpha w(t,\cdot)\|_{L^2} \le C_{k,\alpha}t^{-k-\frac{|\alpha|}{2}}\|\varphi\|_{L^2} \le C_{k,\alpha}t^{-k-\frac{|\alpha|}{2}}\|\varphi\|_{H^{2k+|\alpha|}}.$$

For small time $t < 1$ we take account of the regularity of f and conclude the estimate

$$\|\partial_t^k\partial_x^\alpha w(t,\cdot)\|_{L^2} \le C\|\varphi\|_{H^{2k+|\alpha|}}.$$

From the last two inequalities we derive the statement of the lemma. □

Exercise 42 Prove for the solutions to the Cauchy problem for the damped wave equation

$$u_{tt} - \Delta u + u_t = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x)$$

the a-priori estimate

$$\|u(t,\cdot)\|_{L^2} \le C\left(\|\varphi\|_{L^2} + \|\psi\|_{H^{-1}}\right).$$

Let us devote to the Cauchy problems

$$\begin{cases} u_{tt} - \Delta u + u_t = 0 \\ u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x) \end{cases} \quad \text{and} \quad \begin{cases} w_t - \Delta w = 0 \\ w(0,x) = \varphi(x) + \psi(x). \end{cases}$$

From Lemma 6.1 and from Exercise 42 we have under the assumption φ, $\psi \in L^2$ the a-priori estimates

$$\|w(t,\cdot)\|_{L^2} \le C\|(\varphi,\psi)\|_{L^2}, \quad \|u(t,\cdot)\|_{L^2} \le C\|(\varphi,\psi)\|_{L^2}.$$

We will show that the norm of the difference $\|w(t,\cdot)-u(t,\cdot)\|_{L^2}$ has a better a-priori estimate. For this reason we introduce a cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi(s) = 1$ for $|s| \le \frac{\sigma}{2} << 1$ and $\chi(s) = 0$ for $|s| \ge \sigma$. Then we have the following remarkable result:

Theorem 6.1 *For the difference of solutions to the above Cauchy problems the following estimate holds:*

$$\left\|F^{-1}_{\xi\to x}\left(\chi(\xi)\left(\hat u(t,\xi) - \hat w(t,\xi)\right)\right)\right\|_{L^2} \le C(1+t)^{-1}\|(\varphi,\psi)\|_{L^2}.$$

Proof: For the solution $u = u(t,x)$ we use the representation from Section 3.4.2 for small frequencies $|\xi| < \frac{1}{2}$:

$$\hat{u}(t,\xi) = e^{-\frac{1}{2}t}\left[\left(\frac{1}{2}\hat{\varphi}(\xi) - \frac{\frac{1}{2}\hat{\varphi}(\xi)+\hat{\psi}(\xi)}{\sqrt{1-4|\xi|^2}}\right)e^{-\frac{1}{2}\sqrt{1-4|\xi|^2}\,t} + \left(\frac{1}{2}\hat{\varphi}(\xi) + \frac{\frac{1}{2}\hat{\varphi}(\xi)+\hat{\psi}(\xi)}{\sqrt{1-4|\xi|^2}}\right)e^{\frac{1}{2}\sqrt{1-4|\xi|^2}\,t}\right].$$

For $w = w(t,x)$ we have the representation of solutions

$$\hat{w}(t,\xi) = e^{-|\xi|^2 t}(\hat{\varphi}(\xi)+\hat{\psi}(\xi)).$$

Now we use the asymptotic behavior $\sqrt{1+s} = 1+\frac{s}{2}-\frac{s^2}{8}+O(s^3)$ and $\frac{1}{\sqrt{1+s}} = 1-\frac{s}{2}+O(s^2)$ for $s \to 0$ to obtain the representations

$$\sqrt{1-4|\xi|^2} = 1-2|\xi|^2-2|\xi|^4+O(|\xi|^6) \qquad \text{and} \qquad \frac{1}{\sqrt{1-4|\xi|^2}} = 1+2|\xi|^2+O(|\xi|^4)$$

for $\xi \to 0$. With these representations we conclude

$$\begin{aligned}
&\left\|F^{-1}_{\xi\to x}\left(\chi(\xi)\left(\hat{u}(t,\xi)-\hat{w}(t,\xi)\right)\right)\right\|_{L^2} \\
&= \|\chi(\xi)\left(\hat{u}(t,\xi)-\hat{w}(t,\xi)\right)\|_{L^2} \\
&= \Bigg\|\chi(\xi)\Bigg[\left(\frac{1}{2}\hat{\varphi}(\xi) - \left(\frac{1}{2}\hat{\varphi}(\xi)+\hat{\psi}(\xi)\right) + \left(\frac{1}{2}\hat{\varphi}(\xi)+\hat{\psi}(\xi)\right)O(|\xi|^2)\right)e^{-\frac{1}{2}t+O(|\xi|^2)t}e^{-\frac{1}{2}t} \\
&\qquad + \left(\frac{1}{2}\hat{\varphi}(\xi) + \left(\frac{1}{2}\hat{\varphi}(\xi)+\hat{\psi}(\xi)\right) + \left(\hat{\varphi}(\xi)+2\hat{\psi}(\xi)\right)|\xi|^2 \right. \\
&\qquad \left. + \left(\frac{1}{2}\hat{\varphi}(\xi)+\hat{\psi}(\xi)\right)O(|\xi|^4)\right)e^{\frac{1}{2}t-|\xi|^2t-|\xi|^4t+O(|\xi|^6)t}e^{-\frac{1}{2}t} - e^{-|\xi|^2t}(\hat{\varphi}(\xi)+\hat{\psi}(\xi))\Bigg]\Bigg\|_{L^2}.
\end{aligned}$$

On the one hand we have

$$\left\|\chi(\xi)\left(-\hat{\psi}(\xi) + \left(\frac{1}{2}\hat{\varphi}(\xi)+\hat{\psi}(\xi)\right)O(|\xi|^2)\right)e^{(-1+O(|\xi|^2))t}\right\|_{L^2} \le Ce^{-ct}\|(\varphi,\psi)\|_{L^2}$$

with a constant $c < 1$ depending on the support of χ. On the other hand we have

$$\begin{aligned}
&\Bigg\|\chi(\xi)\Bigg[\left(\hat{\varphi}(\xi)+\hat{\psi}(\xi) + \left(\hat{\varphi}(\xi)+2\hat{\psi}(\xi)\right)|\xi|^2 + \left(\frac{1}{2}\hat{\varphi}(\xi)+\hat{\psi}(\xi)\right)O(|\xi|^4)\right) \\
&\qquad \cdot e^{-|\xi|^2t-|\xi|^4t+O(|\xi|^6)t} - e^{-|\xi|^2t}(\hat{\varphi}(\xi)+\hat{\psi}(\xi))\Bigg]\Bigg\|_{L^2} \\
&\le \left\|\chi(\xi)\left(\hat{\varphi}(\xi)+\hat{\psi}(\xi)\right)\left(e^{-|\xi|^2t-|\xi|^4t+O(|\xi|^6)t} - e^{-|\xi|^2t}\right)\right\|_{L^2} \\
&\qquad + \left\|\chi(\xi)\left(\left(\hat{\varphi}(\xi)+2\hat{\psi}(\xi)\right)|\xi|^2 + \left(\frac{1}{2}\hat{\varphi}(\xi)+\hat{\psi}(\xi)\right)O(|\xi|^4)\right)e^{-|\xi|^2t+O(|\xi|^4)t}\right\|_{L^2}.
\end{aligned}$$

We denote the summands in the right-hand side by J_1 and by J_2 and assume $t \geq 1$. We obtain the estimates

$$\begin{aligned} J_1 &= \left\| \chi(\xi) \left(\hat{\varphi}(\xi) + \hat{\psi}(\xi) \right) (-|\xi|^4 t + O(|\xi|^6)t) e^{-|\xi|^2 t} \underbrace{\int_0^1 e^{(-|\xi|^4 t + O(|\xi|^6)t)s} ds}_{\leq 1} \right\|_{L^2} \\ &\leq C \left\| \chi(\xi) \left(\hat{\varphi}(\xi) + \hat{\psi}(\xi) \right) \frac{|\xi|^4 t^2}{t} e^{-|\xi|^2 t} \right\|_{L^2} \leq C\, t^{-1} \|(\varphi, \psi)\|_{L^2} \end{aligned}$$

and

$$J_2 \leq C \left\| \chi(\xi) \left(|\hat{\varphi}(\xi)| + |\hat{\psi}(\xi)| \right) \frac{|\xi|^2 t}{t} e^{-c|\xi|^2 t} \right\|_{L^2} \leq C\, t^{-1} \|(\varphi, \psi)\|_{L^2}.$$

For $t < 1$ and for $k = 1, 2$ we conclude

$$J_k \leq C \, \|(\varphi, \psi)\|_{L^2}.$$

Summarizing all derived estimates gives

$$J_k \leq C(1+t)^{-1} \|(\varphi, \psi)\|_{L^2}.$$

The proof is complete. □

<u>Question:</u> Why do we consider the diffusion phenomenon only for small frequencies?
<u>Answer:</u> For large frequencies we have

$$\|F^{-1}_{\xi \to x} \left((1 - \chi(\xi)) \hat{w}(t, \cdot) \right) \|_{L^2} \leq C e^{-C_2 t} \|(\varphi, \psi)\|_{L^2}$$

and

$$\|F^{-1}_{\xi \to x} \left((1 - \chi(\xi)) \hat{u}(t, \cdot) \right) \|_{L^2} \leq C e^{-C_1 t} \|(\varphi, \psi)\|_{L^2}.$$

Thus we have already exponential decay, this is already optimal, there is no reason to expect to get a better decay than exponential decay.
<u>Question:</u> How can we interpret the result from Theorem 6.1?
<u>Answer:</u> Besides the result from Theorem 6.1 we have the estimates

$$\|F^{-1}_{\xi \to x} \left(\chi(\xi) \hat{w}(t, \cdot) \right) \|_{L^2} \leq \|(\varphi, \psi)\|_{L^2}$$

and

$$\|F^{-1}_{\xi \to x} \left(\chi(\xi) \hat{u}(t, \cdot) \right) \|_{L^2} \leq C \|(\varphi, \psi)\|_{L^2}.$$

Hence, the solution to the Cauchy problem

$$u_{tt} - \Delta u + u_t = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x)$$

has asymptotically a *parabolic structure* from the point of view of estimates for the solution itself. *Such kind of results are called diffusion phenomena.*

Without proof we will introduce another result related to this topic. We consider the Cauchy problems

$$\begin{cases} u_{tt} - u_{xx} + u_t = 0 \\ u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x) \end{cases} \quad \text{and} \quad \begin{cases} w_t - w_{xx} = 0 \\ w(0,x) = \varphi(x) + \psi(x) \end{cases}$$

for $(x,t) \in \mathbb{R} \times [0,\infty)$.

Theorem 6.2 ([35])
The following estimate holds for $t \geq t_0 > 0$:

$$\left\| (u-w)(t,\cdot) - \frac{1}{2} e^{-\frac{t}{2}} \left(\varphi(\cdot+t) + \varphi(\cdot-t)\right) \right\|_{L^2} \leq C\, t^{-1} \|(\varphi,\psi)\|_{L^2}.$$

Remark: From d'Alembert's representation of solution from Section 3.2.1 we know that $\frac{1}{2}\left(\varphi(x+t) + \varphi(x-t)\right)$ is a solution to the free wave equation.

Question: What is the difference between the statements of Theorems 6.1 and 6.2?

Answer: In Theorem 6.1 we derived an estimate for $t \geq t_0 > 0$ in the form

$$\|(u-w)(t,\cdot)\|_{L^2} \leq C\, t^{-1} \|(\varphi,\psi)\|_{L^2}.$$

The estimate from Theorem 6.2 is more precise. It explains the profile of the difference $u - w$ with respect to the spatial variables in a better way.

6.2 Parabolic structure of solutions to damped plates

The calculations of this section were carried out by Ms. Nguyen Thi Thu Huong during her stay at Bergakademie Freiberg from May to July 2007.

We only care for the small frequencies in the phase space, because for large frequencies, both of the solutions of the damped plate or the solution of the plate equation $w_{tt} + (-\triangle)^2 w = 0$ has an exponential decay rate. The proof of diffusion phenomena for small frequencies in the damped plate equation case is exactly the same as in the damped wave equation case.

Theorem 6.3: *Consider the two Cauchy problems*

$$u_{tt} + (-\triangle)^2 u + u_t = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),$$
$$w_{tt} + (-\triangle)^2 u = 0,\ w(0,x) = \varphi(x) + \psi(x).$$

For the difference of solutions to the above Cauchy problems the following estimate holds:

$$\|F^{-1}_{\xi\to x}(\chi(\xi)(\hat{u}(t,\xi)-\hat{w}(t,\xi)))\|_{L^2} \le C(1+t)^{-1}\|(\varphi,\psi)\|_{L^2}.$$

Proof: We have the representation of $\hat{u}(t,\xi)$ for $|\xi| < \frac{1}{\sqrt{2}}$:

$$\hat{u}(t,\xi) = e^{-\frac{t}{2}}\Big[\Big(\frac{1}{2}\hat{\varphi}(\xi) - \frac{\frac{1}{2}\hat{\varphi}(\xi)+\hat{\psi}(\xi)}{\sqrt{1-4|\xi|^4}}\Big)e^{-\sqrt{\frac{1}{4}-|\xi|^4}t} + \Big(\frac{1}{2}\hat{\varphi}(\xi) + \frac{\frac{1}{2}\hat{\varphi}(\xi)+\hat{\psi}(\xi)}{\sqrt{1-4|\xi|^4}}\Big)e^{\sqrt{\frac{1}{4}-|\xi|^4}t}\Big].$$

Moreover, $\hat{w}(t,\xi) = e^{-|\xi|^4 t}(\hat{\varphi}(\xi)+\hat{\psi}(\xi))$. Then we conclude by using

$$\sqrt{1-4|\xi|^4} = 1-2|\xi|^4 - 2|\xi|^8 + O(|\xi|^{12}),\quad \frac{1}{\sqrt{1-4|\xi|^4}} = 1+2|\xi|^4 + O(|\xi|^8)$$

$$\begin{aligned}
&\|F^{-1}_{\xi\to x}(\chi(\xi)(\hat{u}(t,\xi)-\hat{w}(t,\xi)))\|_{L_2} = \|\chi(\xi)(\hat{u}(t,\xi)-\hat{w}(t,\xi))\|_{L_2}\\
&\quad = \Big\|\chi(\xi)\Big[\Big(\frac{1}{2}\hat{\varphi}(\xi) - \Big(\frac{1}{2}\hat{\varphi}(\xi)+\hat{\psi}(\xi)\Big) + \Big(\frac{1}{2}\hat{\varphi}(\xi)+\hat{\psi}(\xi)\Big)O(|\xi|^4)\Big)e^{-\frac{t}{2}+O(|\xi|^4)t}e^{-\frac{t}{2}}\\
&\quad + \Big(\frac{1}{2}\hat{\varphi}(\xi) + \Big(\frac{1}{2}\hat{\varphi}(\xi)+\hat{\psi}(\xi)\Big) + \Big(\hat{\varphi}(\xi)+2\hat{\psi}(\xi)\Big)|\xi|^4 + \Big(\frac{1}{2}\hat{\varphi}(\xi)+\hat{\psi}(\xi)\Big)O(|\xi|^8)\Big)\\
&\quad \times e^{\frac{t}{2}-|\xi|^4 t-|\xi|^8 t+O(|\xi|^{12})t}e^{-\frac{t}{2}} - e^{-|\xi|^4 t}(\hat{\varphi}(\xi)+\hat{\psi}(\xi))\Big]\Big\|_{L_2}.
\end{aligned}$$

The first part is estimated by

$$\Big\|\chi(\xi)\Big[\Big(-\hat{\psi}(\xi) + \Big(\frac{1}{2}\hat{\varphi}(\xi)+\hat{\psi}(\xi)\Big)O(|\xi|^4)\Big)e^{-t+O(|\xi|^4)t}\Big\|_{L_2} \le Ce^{-\delta t}\|(\varphi,\psi)\|_{L_2},$$

where $\delta < 1$. The second part is estimated by

$$\begin{aligned}
&\Big\|\chi(\xi)\Big[\Big(\hat{\varphi}(\xi)+\hat{\psi}(\xi) + (\hat{\varphi}(\xi)+2\hat{\psi}(\xi))|\xi|^4 + \Big(\frac{1}{2}\hat{\varphi}(\xi)+\hat{\psi}(\xi)\Big)O(|\xi|^8)\Big)\\
&\quad \times e^{-|\xi|^4 t-|\xi|^8 t+O(|\xi|^{12})t} - e^{-|\xi|^4 t}(\hat{\varphi}(\xi)+\hat{\psi}(\xi))\Big]\Big\|_{L_2}\\
&\le \Big\|\chi(\xi)(\hat{\varphi}(\xi)+\hat{\psi}(\xi))(e^{-|\xi|^4 t-|\xi|^8 t+O(|\xi|^{12})t} - e^{-|\xi|^4 t})\Big\|_{L_2}\\
&+ \Big\|\chi(\xi)\Big((\hat{\varphi}(\xi)+2\hat{\psi}(\xi))|\xi|^4 + \Big(\frac{1}{2}\hat{\varphi}(\xi)+\hat{\psi}(\xi)\Big)O(|\xi|^8)\Big)e^{-|\xi|^4 t+O(|\xi|^8)t}\Big\|_{L_2}.
\end{aligned}$$

First we devote to $t \ge 1$. By using Taylor's expansion of e^t we have

$$e^{-|\xi|^4 t-|\xi|^8 t+O(|\xi|^{12})t} - e^{-|\xi|^4 t} = e^{-|\xi|^4 t}(e^{-|\xi|^8 t+O(|\xi|^{12})t}-1) = e^{-|\xi|^4 t}(-|\xi|^8 t + O(|\xi|^{12})t)$$
$$|e^{-|\xi|^4 t-|\xi|^8 t+O(|\xi|^{12})t} - e^{-|\xi|^4 t}| \le Ce^{-|\xi|^4 t}|\xi|^8 t \le Ct^{-1}.$$

Hence,

$$\begin{aligned}
&\Big\|\chi(\xi)(\hat\varphi(\xi)+\hat\psi(\xi))(e^{-|\xi|^4t-|\xi|^8t+O(|\xi|^{12})t}-e^{-|\xi|^4t})\Big\|_{L_2}\le Ct^{-1}\|(\varphi,\psi)\|_{L_2},\\
&\Big\|\chi(\xi)\Big((\hat\varphi(\xi)+2\hat\psi(\xi))|\xi|^4+\Big(\frac{1}{2}\hat\varphi(\xi)+\hat\psi(\xi)\Big)O(|\xi|^8)\Big)e^{-|\xi|^4t+O(|\xi|^8)t}\Big\|_{L_2}\\
&\quad\le C\|\chi(\xi)\Big(|\hat\varphi(\xi)|+|\hat\psi(\xi)|\Big)|\xi|^4e^{-|\xi|^4t}\|_{L_2}\le Ct^{-1}\|(\varphi,\psi)\|_{L_2}.
\end{aligned}$$

For $t<1$ all the exponential functions $e^{-|\xi|^kt}$ are bounded. Thus we conclude

$$\Big\|\chi(\xi)\Big((\hat\varphi(\xi)+2\hat\psi(\xi))|\xi|^4+\Big(\frac{1}{2}\hat\varphi(\xi)+\hat\psi(\xi))O(|\xi|^8\Big)\Big)e^{-|\xi|^4t+O(|\xi|^8)t}\Big\|_{L_2}\le C\|(\varphi,\psi)\|_{L_2},$$

$$\Big\|\chi(\xi)(\hat\varphi(\xi)+\hat\psi(\xi))(e^{-|\xi|^4t-|\xi|^8t+O(|\xi|^{12})t}-e^{-|\xi|^4t})\Big\|_{L_2}\le C\|(\varphi,\psi)\|_{L_2}.$$

This completes the proof. □

6.3 Diffusion phenomena in thermo-elasticity

Models of thermo-elasticity are typical examples for *hyperbolic-parabolic coupled systems*. They describe the behavior of elastic, heat conductive materials. There are different possibilities to model the heat conductive behavior (see next section). For the solutions of those systems one can observe from time to time the *diffusion phenomenon*. For this reason let us study the linear model of thermo-elasticity of type 1 in 1-d case (type 1 models are the classical models):

$$\begin{cases} u_{tt}-a^2u_{xx}+\gamma_1\theta_x=0,\\ \theta_t-b^2\theta_{xx}+\gamma_2u_{tx}=0,\\ u(0,x)=u_0(x),\ u_t(0,x)=u_1(x),\ \theta(0,x)=\theta_0(x),\end{cases}$$

where a,b,γ_1 and γ_2 are constants satisfying $a,b,\gamma_1\gamma_2>0$. The functions $u=u(t,x)$ and $\theta=\theta(t,x)$ denote the displacement and the difference of temperature (temperature of the medium minus a constant reference temperature of the neighborhood of the medium). This system is a hyperbolic-parabolic coupled systems. The equation for the displacement u remembers to the wave equation. The equation for the difference of temperature θ remembers to the heat equation. If we are interested in several questions as *well-posedness of the Cauchy problem*, *propagation of singularities* or *energy decay estimates or estimates for the solution*, then every time it is interesting which part determines the result, *the hyperbolic part or the parabolic part*.

Now let us devote to the above model. By denoting $u_\pm=u_t\pm au_x$ and $U=(u_+,u_-,\theta)^T$ the above Cauchy problem is equivalent to

$$\begin{cases} \partial_tU+A_1\partial_xU-A_2\partial_x^2U=0,\\ U(0,x)=U_0(x)=(u_1+au_0',u_1-au_0',\theta_0)^T\end{cases}$$

with matrices

$$A_1 = \begin{pmatrix} -a & 0 & \gamma_1 \\ 0 & a & \gamma_1 \\ \frac{\gamma_2}{2} & \frac{\gamma_2}{2} & 0 \end{pmatrix} \qquad \text{and} \qquad A_2 = \operatorname{diag}(0,0,b^2).$$

After some *diagonalization technique* we are able to derive L^p-L^q decay estimates for the energy of solutions (cf. with Section 8.2), in the energy we include the *elastic* and the *kinetic* of u and θ *itself*, that are, estimates of the type

$$\|(u_x, u_t, \theta)(t,\cdot)\|_{L^q} \le C(1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|(u_0', u_1, \theta_0)\|_{H^{2,p}}$$

for $1 \le p \le 2 \le q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.
After some steps of diagonalization procedure we can guess the reference system of partial differential equations which solutions we will compare with the solutions of

$$\begin{cases} \partial_t U + A_1 \partial_x U - A_2 \partial_x^2 U = 0, \\ U(0,x) = U_0(x) = (u_1 + a u_0', u_1 - a u_0', \theta_0)^T. \end{cases}$$

This is the following uncoupled system of heat equations with convection terms

$$\begin{cases} \partial_t W_1 - \dfrac{a^2 b^2}{a^2 + \gamma_1\gamma_2}\, \partial_x^2 W_1 = 0, \\ \partial_t W_2 - \dfrac{b^2 \gamma_1 \gamma_2}{2(a^2 + \gamma_1\gamma_2)}\, \partial_x^2 W_2 + \sqrt{a^2 + \gamma_1\gamma_2}\, \partial_x W_2 = 0, \\ \partial_t W_3 - \dfrac{b^2 \gamma_1 \gamma_2}{2(a^2 + \gamma_1\gamma_2)}\, \partial_x^2 W_3 - \sqrt{a^2 + \gamma_1\gamma_2}\, \partial_x W_3 = 0, \\ W(0,x) = E(D) U_0(x) \end{cases}$$

with a suitable elliptic operator $E(D)$.
Then we have the following estimates:

$$\left\| F^{-1}_{\xi \to x} \left(\chi(\xi) \left(\hat{U}(t,\xi) - R(I - i\xi K_1) \hat{W}(t,\xi) \right) \right) \right\|_{L^q} \le C(1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right) - \frac{1}{2}} \|U_0\|_{L^p}$$

for $1 \le p \le 2 \le q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Here R and K_1 are known (from the diagonalization procedure) constant matrices, and $\chi \in C_0^\infty(\mathbb{R})$ localizes to small frequencies.
Finally, taking into consideration

$$\left\| F^{-1}_{\xi \to x} \left(\chi(\xi) \hat{W}(t,\xi) \right) \right\|_{L^q} \le C(1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|U_0\|_{L^p},$$

we conclude that the difference decays faster. The decay will be determined by the behavior for the small frequencies. Hence, the asymptotic profile of the solution to the classical model of thermo-elasticity is determined by the solution of the Cauchy problem for W. Thus, the

Cauchy problem for the solution to the classical model of thermo-elasticity possesses from the point of energy decay a *parabolic structure*.

Example: If we set $p = q = 2$ in the above estimates, then we have

$$\|U(t,\cdot)\|_{L^2} \le C\|U_0(\cdot)\|_{L^2}, \quad \|W(t,\cdot)\|_{L^2} \le C\|U_0(\cdot)\|_{L^2},$$

but for the difference we have

$$\left\| F^{-1}_{\xi\to x}\left(\chi(\xi)\left(\hat{U}(t,\xi) - R(I - i\xi K_1)\hat{W}(t,\xi)\right)\right)\right\|_{L^2} \le C(1+t)^{-\frac{1}{2}}\|U_0\|_{L^2}.$$

One can show that from the point of view of well-posedness of the Cauchy problem or of propagation of singularities the classical model of thermo-elasticity has a *hyperbolic structure*.

6.4 Further activities of M.Reissig and collaborators

In the previous section we introduced models of linear thermo-elasticity of type 1 in the 1-d case:

$$\begin{cases} u_{tt} - a^2u_{xx} + \gamma_1\theta_x = 0, \\ \theta_t - b^2\theta_{xx} + \gamma_2 u_{tx} = 0, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ \theta(0,x) = \theta_0(x), \end{cases}$$

In this model one uses *Fourier's law* in the energy balance. There exist other ideas to introduce different models in thermo-elasticity. Using thermal waves we obtain models of type 2. In 1-d they have the form

$$\begin{cases} u_{tt} - a^2u_{xx} + \gamma_1\theta_x = 0, \\ \theta_{tt} - b^2\theta_{xx} + \gamma_2 u_{ttx} = 0, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ \theta(0,x) = \theta_0(x),\ \theta_t(0,x) = \theta_1(x). \end{cases}$$

There exist also models of type 3. In these models the constitutive assumption on the heat flux vector is related to thermal waves with an additional dissipation term. Those models are

$$\begin{cases} u_{tt} - a^2u_{xx} + \gamma_1\theta_x = 0, \\ \theta_{tt} - b^2\theta_{xx} - \delta\theta_{txx} + \gamma_2 u_{ttx} = 0, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ \theta(0,x) = \theta_0(x),\ \theta_t(0,x) = \theta_1(x). \end{cases}$$

Finally, also the second sound model

$$\begin{cases} u_{tt} - a^2u_{xx} + \gamma_1\theta_x = 0, \\ \theta_t + \alpha q_x + \gamma_2 u_{tx} = 0, \\ \beta q_t + q + b^2\theta_x = 0, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ \theta(0,x) = \theta_0(x),\ q(0,x) = q_0(x), \end{cases}$$

is of interest. In a recent joint project with K. Jachmann (Freiberg) such models with mass or dissipation term are studied ([20]). Typical questions are those for well-posedness of Cauchy problems, for L^p-L^q decay estimates, in particular, which models show a hyperbolic profile or a parabolic profile (cf. with Section 8.2) (see [48]). More results for the diffusion phenomenon are obtained. Finally, results about the propagation of singularities are of interest. Models of thermo-elasticity with time-dependent coefficients are analyzed in [49]. Some results for nonlinear models are proved in [50]. There the propagation of *mild singularities* are studied for the semi-linear model of type 1 in 3-d case

$$\begin{cases} U_{tt} + \mu\nabla \times \nabla \times U - \tau\nabla \,\mathrm{div} U + \gamma\nabla\theta = f(U, \theta), \\ \theta_t - \kappa \triangle \theta + \gamma \,\mathrm{div} U_t = g(U, \theta), \\ U(0, x) = U_0(x),\, U_t(0, x) = U_1(x),\, \theta(0, x) = \theta_0(x). \end{cases}$$

7 Some comments to mixed problems for hyperbolic equations

In Section 4 we studied the Cauchy problem

$$\begin{cases} u_{tt} - \triangle u = f(u_t, \nabla u, \nabla u_t, \nabla^2 u) \ \text{ in } \ \mathbb{R}^n \times [0, \infty), \\ u(0, x) = \varphi(x), \quad u_t(0, x) = \psi(x) \ \text{ for } \ x \in \mathbb{R}^n. \end{cases} \tag{7.1}$$

Under reasonable assumptions we can obtain the *global existence of small data solutions*. In this section we will present some ideas if the geometry of $\mathbb{R}^n$ will be replaced by different geometries.

7.1 Interior domains

If $G \subset \mathbb{R}^n$ is a bounded domain, then one should remember *Fourier's method* to study the linear mixed problem

$$\begin{aligned} & u_{tt} - \triangle u = 0 \quad \text{in} \quad G \times [0, \infty), \\ & u(0, x) = \varphi(x),\ u_t(0, x) = \psi(x) \ \text{ for } \ x \in G, \\ & \text{either} \quad u(t, x) = 0 \quad \text{or} \quad \frac{\partial u}{\partial n}\,(t, x) = 0 \quad \text{for} \quad (t, x) \in [0, \infty) \times \partial G. \end{aligned}$$

We assume that the *compatibility conditions* between initial and boundary conditions are satisfied. Thus we have to apply methods from spectral theory (eigenvalues, eigenfunctions, Sturm-Liouville theory in higher dimensions, Galerkin procedure).

For wave equations with boundary conditions of Dirichlet-, Neumann-, or Robin type there is no decay at all.

Exercise 43 Let us assume that the boundary ∂G of an interior domain G consists of two parts S_1 and S_2. A function $u = u(t, x)$ satisfies $u(t, x) = 0$ for $x \in S_1$ and $\frac{\partial u}{\partial n}\,(t, x) = 0$ for

$x \in S_2$. Show the conservation of energy and a uniqueness result for u as a classical solution to

$$u_{tt} - \triangle u = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x).$$

Exercise 44 Show, that if $u = u(t,x)$ is a solution of $u_{tt} - \triangle u = 0$ in $G \times [0,\infty)$ and $\frac{\partial u}{\partial n} + \alpha(x)u = 0$ for $(t,x) \in [0,\infty) \times \partial G$, where $\alpha(x) \geq 0$ and α continuous on ∂G, then the following energy conservation holds:

$$\begin{aligned} &\int\limits_G (|\nabla u(t,x)|^2 + u_t(t,x)^2)dx + \int\limits_{\partial G} \alpha(x)u(t,x)^2 ds \\ = &\int\limits_G (|\nabla u(0,x)|^2 + u_t(0,x)^2)dx + \int\limits_{\partial G} \alpha(x)u(0,x)^2 ds. \end{aligned}$$

Now let us devote to the nonlinear mixed problem

$$\begin{cases} u_{tt} - \triangle u = f(x,u,\nabla u,\nabla^2 u), \;\; u(0,x) = \varepsilon\,\varphi(x), \;\; u_t(0,x) = \varepsilon\,\varphi(x), \\ \text{either} \quad u(t,x) = 0 \quad \text{or} \quad \frac{\partial u}{\partial n}\,(t,x) = 0 \quad \text{for} \quad (t,x) \in [0,\infty) \times \partial G. \end{cases} \tag{7.2}$$

Theorem 7.1 *Let $G_3 := \{x \in \mathbb{R}^3 : |x| \in (1,2)\}$. For every $\alpha \in \mathbb{N}$ there are smooth nonlinearities $f = f(x,u,\nabla u,\nabla^2 u)$, $f(x,w) = O(|w|^{\alpha+1})$ uniformly in x near $w = 0$, such that there is no global C^2-solution to the mixed problem (7.2) with radially symmetric data φ and ψ in G_3, $\varepsilon > 0$ is small, no matter how smooth the data φ and ψ are or how small ε is or how large α is. Namely, there are positive constants $C = C(\varphi,\psi)$ and $\varepsilon_0 = \varepsilon_0(\varphi,\psi)$ such that the solution develops a singularity as t approaches $C\,\varepsilon^{-\alpha}$ provided $\varepsilon < \varepsilon_0$.*

Remarks: The existence of a local solution in time follows with the general local existence theorem from [59]. If we replace the boundary conditions by so-called *boundary conditions of dissipative type* (they produce a decay), then one can show the *global existence of small data solutions.*

Some remarks to the proof for Theorem 7.1:

The main idea is to use the assumption for the data (radially symmetric) that after some transformations the mixed problem (7.2) can be transferred to the 1-d mixed problem

$$\begin{cases} v_{tt} - (K(v_r))_r = 0, \quad v(0,r) = r\,\varepsilon\,\varphi(r), \quad v_t(0,r) = r\,\varepsilon\,\psi(r), \\ \text{either} \quad v(t,1) = v(t,2) = 0 \qquad \text{or} \qquad v_r(t,1) = v_r(t,2) = 0. \end{cases} \tag{7.3}$$

From [26] it follows that v develops a singularity in the second derivatives at time $T = C\varepsilon^{-\alpha}$. This gives the desired result for u. This result is about the formation of singularities for bounded domains. The authors studied in correspondence with (7.3) the mixed problem

$$\begin{cases} u_{tt} - (K(u_x))_x = 0, \quad u(0,x) = \varepsilon\,\varphi(x)\;, \quad u_t(0,x) = \varepsilon\,\psi(x), \\ \text{either} \qquad u(t,0) = u(t,L) = 0 \quad \text{or} \quad u_x(t,0) = u_x(t,L) = 0. \end{cases} \tag{7.4}$$

Let $K'(0) = 1$ and $\alpha \in \mathbb{N}$ be the first integer with $K^{(\alpha+1)}(0) \neq 0$. Then in the case of Dirichlet condition there are positive constants $C = C(\varphi, \psi)$ and ε_0 such that a C^2-solution develops singularities at time $T = C\,\varepsilon^{-\alpha}$ provided $\varepsilon < \varepsilon_0$. If K is an odd function the same conclusion holds in the case of Neumann condition. □

Remark: If we replace the Dirichlet or Neumann boundary condition by the *boundary condition of dissipative type*

$$Ku_x(t,0) - \tau\, u_t(t,0) = 0, \quad u_t(t,L) = 0, \quad \tau > 0,$$

then one can prove the *global existence of small data solutions*. Moreover, a *dissipation term* cu_t , $c > 0$, added to the left-hand side of the differential equation for u is an effective tool for proving the *global existence of small data solutions.*

The formation of singularities can be seen directly from the example

$$u(t,x) := \frac{\sqrt[\alpha]{\frac{4}{\alpha^2} + \frac{2}{\alpha}}}{\sqrt[\alpha]{(t-T_0)^2}}, \quad x \in G, \quad T_0 > 0 \text{ is given.}$$

It solves $u_{tt} - \Delta u = u^{\alpha+1}$ with homogeneous Neumann boundary condition and with small data if T_0 is large. The solution tends to infinity if t approaches T_0.

Finally, we prove a more general blow-up result for solutions to semi-linear mixed problems.

Theorem 7.2 *For $\alpha \in \mathbb{N}$ let u be a C^2-solution to*

$$u_{tt} - \Delta u = u^{\alpha+1},\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),\ \frac{\partial u}{\partial n}(t,x) = 0 \quad on \quad \partial G x[0,\infty).$$

If $\int\limits_G \varphi(x)dx =: A > 0, \int\limits_G \psi(x)dx =: B > 0$ and $\frac{1}{2}B^2 - \frac{M}{\alpha+2}A^{\alpha+2} > 0$, then the solution blows-up in finite time.

Proof: The function $v(t) := \int\limits_G u(t,x)dx$ solves

$$v''(t) = \int\limits_G u_{tt}(t,x)dx = \int\limits_G (\Delta u(t,x) + u^{\alpha+1}(t,x))dx = \int\limits_G u^{\alpha+1}(t,x)dx$$
$$\geq Mv(t)^{\alpha+1} \ (M = M(\text{meas } G, \alpha) \text{ appears in the above assumption}).$$

Here we used the Neumann boundary condition and Hölder's inequality. At least for small t we have $v'(t) > 0$ and $v(t) > 0$. From the last inequality it follows

$$\frac{1}{2}v'(t)^2 - \frac{M}{\alpha+2}v(t)^{\alpha+2} \geq \frac{1}{2}B^2 - \frac{M}{\alpha+2}A^{\alpha+2} =: C > 0.$$

Consequently,

$$v'(t) \geq \sqrt{2\frac{M}{\alpha+2}v(t)^{\alpha+2} + C}.$$

Now let us define the reference function $w = w(t)$ satisfying

$$w'(t) = \sqrt{2\frac{M}{\alpha+2}w(t)^{\alpha+2} + C}.$$

Then it holds for all $t \in [0, \infty)$

$$t = \int_0^{w(t)} \frac{ds}{\sqrt{2\frac{M}{\alpha+2}s^{\alpha+2} + C}} \leq \int_0^{\infty} \frac{ds}{\sqrt{2\frac{M}{\alpha+2}s^{\alpha+2} + C}} < \infty.$$

Thus, the solution $w = w(t)$ cannot exist on the interval $[0, T]$. Using $v'(t) \geq w'(t)$ the same holds for v. □

7.2 Exterior domains

Now we devote to exterior domains G with smooth boundary ∂G. General results for fully nonlinear wave equations have been obtained in the papers [57] and [58]. In the following we assume that $\mathbb{R}^n \backslash G$ is convex. This assumption can be weakened to a *non-trapping* condition. As an essential tool to prove general results one needs a *decay result for the local energy*

$$E_R(u)(t) := \int_{G \cap B_R(0)} (|u_t(t,x)|^2 + |\nabla_x u(t,x)|^2)dx,$$

where $B_R(0)$ denotes the ball of radius R around the origin. Such local energy decay can be combined with cut-off techniques and results for the Cauchy problem to a global $L^p - L^q$ decay estimate for the energy of solutions basing on the exterior domain G.

There exist at least two different ways to derive a decay result for *the local energy*. One can use the *generalized Fourier transformation* $\mathcal{F}_+ : L^2(G) \to L^2(\mathbb{R}^n)$, $\mathcal{F}_+$ being unitary, with the property

$$\mathcal{F}_+(g(A)u)(\xi) = g(|\xi|^2)(\mathcal{F}_+ u)(\xi)$$

for functions $g(A)$ of A defined by the spectral theorem for the self-adjoint operator A (see [31] or [66]). An application to wave problems was given in [24] and references therein.

Another way is the approach by using the *Laplace transformation*. As an example we consider the mixed problem

$$u_{tt} - \triangle u = 0, \quad u(0,x) = 0, \quad u_t(0,x) = \psi(x), \quad u(t,x) = 0 \text{ on } [0,\infty) \times \partial G. \tag{7.5}$$

We apply the *Laplace transformation* with respect to t, discuss the resolvent of the resulting stationary equations and use the information just obtained after the inverse transformation. We obtain by using the rules of Laplace transformation $\quad (Lu(f)(s) = \int\limits_0^\infty e^{-ist}u(t)dt)$

$$L(u_{tt} - \triangle u) = -s^2 L(u)(s) - \triangle\, L(u)(s) = \psi(x), \quad (\triangle + s^2)L(u)(s,x) = -\psi(x),$$

respectively. Hence, $L(u)(s,x) = -R(s^2)\psi(x)$, where $R(s^2)$ denotes the resolvent $(\triangle + s^2)^{-1}$. At first $R(s^2)$ is only defined for $s^2 \in \mathbb{C} \setminus [0,\infty)$. Under this assumption to s^2 the solution u is given by

$$u(t,x) = L^{-1}(-R(s^2)\psi)(t,x) := \frac{1}{2\pi} \int\limits_{-\infty - ic}^{\infty - ic} e^{ist}(-R(s^2)\psi)(s,x)ds,$$

where $c > 0$ is arbitrary. The asymptotic behavior of u for $t \to \infty$ can be understood if the behavior of $R(s^2)$ near $s = 0$ and for $|s| \to \infty$ is known. In the papers [63], [64] the following properties for $R(s^2)$ are proved:

- $R(s^2)$ can be holomorphically extended to $s^2 \in [0,\infty)$ as an operator

$$R(s^2) : L^2_{\text{comp}}(\mathbb{R}^n) \longrightarrow H^2_{\text{loc}}(G) = \{u \in W^{2,2}_{\text{loc}}(G) : \\ e^{|x|}\nabla_x^\beta u(x) \in L^2(G) \quad \text{for} \quad |\beta| \le 2\};$$

- $\|R(s^2)\|_{L(L^2_{\text{comp}}(\mathbb{R}^n) \to L^2_{\text{loc}}(\mathbb{R}^n))} \sim |s|^{-1}$ for $|s| \to \infty$ if $\mathbb{R}^n \setminus G$ is convex (G is non-trapping);

- $R(s^2)$ can be developed in a Laurent series.

All this together leads to a decay rate for the local energy $E_R(u)(t)$ of the solution to (7.5).

At the end we want to cite (without mention the regularity of the solution) a result from [42] for a damped mixed problem. For damped problems the assumption *"G is non-trapping"* is not needed.

Theorem 7.3 *Let $\mathbb{R}^3 \setminus G$ be a star-shaped interior domain. Then we have the global existence of small data solutions to the following nonlinear mixed problem for the damped wave equation:*

$$u_{tt} - \triangle u + u_t = f(u, u_t, \nabla u, \nabla u_t, \nabla^2 u),$$
$$u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x), \quad u(t,x) = 0 \quad \text{on} \quad [0,\infty) \times \partial G,$$

where

$$f(w) = O(|w|^3) \quad \text{near} \quad w = 0, \quad f \text{ is smooth.}$$

7.3 Wave guides

In Section 1.3 we introduced domains Ω which are called *wave guides*. Such a geometry is $\Omega = \{(x,y) \in \mathbb{R}^{n+m} : (x,y) \in \mathbb{R}^n \times G,\ G \subset \mathbb{R}^m\}$, where G is an interior domain. If we are interested in the mixed Klein-Gordon problem

$$u_{tt} - \triangle u + m^2 u = f(u, u_t, \nabla u, \nabla u_t, \nabla^2 u) \text{ in } [0,\infty) \times \Omega,$$
$$u(0,x,y) = \varphi(x,y),\ u_t(0,x,y) = \psi(x,y) \text{ in } \Omega,\ u(t,x,y) = 0 \text{ on } [0,\infty) \times \partial\Omega,$$

then it is reasonable to combine methods for interior domains (abstract spectral theory) with methods for the whole space (Fourier transformation). This is done in [32]. From the fact that $x \in \mathbb{R}^n$ we can expect $L^p - L^q$ decay estimates for the solutions to Cauchy problems for the classical Klein-Gordon equation from Section 3.3. Then a similar approach which was presented in the sketch of the proof to Theorem 4.1 leads to the *global existence of small data solutions*.

8 Modern topics of research

8.1 Wave equations with low regularity of coefficients

8.1.1 C^κ property with respect to t

Let us start with the Cauchy problem

$$u_{tt} - a(t)u_{xx} = 0,\ \ u(0,x) = \varphi(x),\ \ u_t(0,x) = \psi(x),$$

where $a \in C^\kappa[0,T],\ \kappa \in (0,1)$. From [2] we have the following result:

Theorem 8.1 *If $a \in C^\kappa[0,T]$, then this Cauchy problem is well-posed in Gevrey classes G^s for $s < \frac{1}{1-\kappa}$. To $\varphi,\ \psi \in G^s$ we have a uniquely determined solution $u \in C^2([0,T], G^s)$.*

We can use different definitions for G^s (by the behavior of derivatives on compact subsets, by the behavior of Fourier transform). If

- $s = \frac{1}{1-\kappa}$, then we should be able to prove local existence in t;
- $s > \frac{1}{1-\kappa}$, then there is no well-posedness in G^s.

The paper [40] is concerned with the strictly hyperbolic Cauchy problem

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t,x)u_{x_k x_l} \ + \text{ lower order terms} = \ f(t,x), u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),$$

with coefficients depending Hölderian on t and Gevrey on x. It was proved well-posedness in Gevrey spaces G^s. Here G^s stands for a scale of Banach spaces. One should understand

- how to define the Gevrey space with respect to x, maybe some suitable dependence on t is reasonable, thus scales of Gevrey spaces appear;
- the difference between $s = \frac{1}{1-\kappa}$ and $s < \frac{1}{1-\kappa}$, in the first case the solution should exist locally, in the second case globally in t if we constructed the right scale of Gevrey spaces.

Remark: In the proof of Theorem 8.1 we use instead of $a \in C^\kappa[0,T]$ the condition $\int_0^{T-\tau} |a(t+\tau) - a(t)|dt \le A\,\tau^\kappa$ for $\tau \in [0, T/2]$. But then the solution belongs only to $H^{2,1}([0,T], G^s)$.

8.1.2 Lip-property with respect to t

Let us suppose $a \in C^1[0,T]$, $a(t) \ge C > 0$, in the strictly hyperbolic Cauchy problem

$$u_{tt} - a(t)u_{xx} = 0, \;\; u(0,x) = \varphi(x), \; u_t(0,x) = \psi(x).$$

Using the energy method and Gronwall's Lemma one can prove immediately the well-posedness in Sobolev spaces H^s, that is, if $\varphi \in H^{s+1}(\mathbb{R}^n)$, $\psi \in H^s(\mathbb{R}^n)$, then there exists a uniquely determined solution $u \in C([0,T], H^{s+1}) \cap C^1([0,T], H^s)$ $(s \in N_0)$.

Exercise 45 Prove this statement!

A more precise result is given in [38].

Theorem 8.2 (cf. with Theorem 3.18)

Let us consider the Cauchy problem

$$u_{tt} - \sum_{k,m=1}^{n} a_{km}(t,x)u_{x_k x_m} = 0, \quad u(0,x) = \varphi(x), \; u_t(0,x) = \psi(x).$$

If the coefficients $a_{km} \in C([0,T], \mathcal{B}^l) \cap C^1([0,T], \mathcal{B}^0)$, $l \in \mathbb{N}$, $l \ge 1$, and $\varphi \in H^{l+1}$, $\psi \in H^l$, then there exists a uniquely determined solution $u \in C([0,T], H^{l+1}) \cap C^1([0,T], H^l)$. Moreover, the energy inequality $E_p(u)(t) \le C_p E_p(u)(0)$ holds for $0 \le p \le l$, where $E_p(u)$ denotes the energy of p'th order of the solution u.

By $\mathcal{B}^\infty$ we denote the space of infinitely differentiable functions having bounded derivatives on $\mathbb{R}^n$. Its topology is generated by the family of norms of spaces $\mathcal{B}^l$, $l \in \mathbb{N}$, consisting of functions with bounded derivatives up to order l.

Remark: For our starting problem we can suppose instead of $a \in C^1[0,T]$ the condition $\int_0^{T-\tau} |a(t+\tau) - a(t)|dt \le A\,\tau$ for $\tau \in [0, T/2]$. Then we have the same statement as in Theorem 8.2. The only difference is that the solution belongs to $C([0,T], H^{s+1}) \cap H^{1,2}([0,T], H^s) \cap H^{2,1}([0,T], H^{s-1})$.

Exercise 46 Use the literature to get information about whether one can weaken the assumptions for a_{kl} from Theorem 8.2 to show the energy estimates $E_k(u)(t) \leq C_k E_k(u)(0)$ for $0 \leq k \leq s$.

All results from this section imply that no *loss of derivatives* or *loss of regularity* appears, that is, the energy $E_k(u)(t)$ of k-th order can be estimated by the energy $E_k(u)(0)$ of k-th order.

Let us recall some standard arguments:

- If the coefficients have more regularity $C^1([0,T], B^\infty)$, and the data φ and ψ are from H^∞, then the Cauchy problem is H^∞ well-posed, that is, there exists a uniquely determined solution from $C^2([0,T], H^\infty)$.
  This result follows from the energy inequality.
- Together with the domain of dependence property from H^∞ well-posedness we conclude C^∞ well-posedness, that is, to arbitrary data φ and ψ from C^∞ there exists a uniquely determined solution from $C^2([0,T], C^\infty)$.
  This result follows from the energy inequality and the domain of dependence property.

Results for domain of dependence property:

Theorem 8.3 ([2])
Let us consider the strictly hyperbolic Cauchy problem

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t) u_{x_k x_l} = f(t,x),\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x).$$

The coefficients $a_{kl} = a_{lk}$ are real and belong to $L^1(0,T)$. Moreover, $\sum_{k,l=1}^{n} a_{kl}(t)\xi_k\xi_l \geq \lambda_0|\xi|^2$ with $\lambda_0 > 0$. If $u \in H^{2,1}([0,T], \mathcal{A}')$ is a solution for given $\varphi, \psi \in \mathcal{A}'$ and $f \in L^1([0,T], \mathcal{A}')$, then from $\varphi \equiv \psi \equiv f \equiv 0$ for $|x - x_0| < \rho$ it follows that $u \equiv 0$ on the set $\{(t,x) \in [0,T] \times \mathbb{R}^n : |x - x_0| < \rho - \int_0^t \sqrt{|a(s)|}ds\}$. Here $|a(t)|$ denotes the Euclidean matrix norm, $\mathcal{A}'$ denotes the space of analytic functionals.

8.1.3 Finite loss of derivatives

In this section we are interested in weakening the Lip-property for the coefficients $a_{kl} = a_{kl}(t)$ in such a way, that we can prove energy inequalities of the form $E_{s-s_0}(u)(t) \leq C E_s(u)(0)$, where $s_0 > 0$. The value s_0 describes the so-called **loss of derivatives**.

Global condition The next idea goes back to [2]. The authors supposed the so-called LogLip-property, that is, the coefficients a_{kl} satisfy

$$|a_{kl}(t_1) - a_{kl}(t_2)| \leq C|t_1 - t_2|\,|\ln|t_1 - t_2||\quad \text{for all}\quad t_1, t_2 \in [0,T],\ t_1 \neq t_2.$$

More precisely, the authors used the condition

$$\int_0^{T-\tau} |a_{kl}(t+\tau) - a_{kl}(t)| dt \le C\ \tau(|\ln\ \tau| + 1) \quad \text{for} \quad \tau \in (0, T/2]\ .$$

Under this condition well-posedness in C^∞ was proved.

There is the following *classification of LogLip-behavior with respect to the related loss of derivatives.*

Theorem 8.4 *Let us consider the Cauchy problem*

$$u_{tt} - a(t)u_{xx} = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x).$$

We suppose $|a(t_1) - a(t_2)| \le C|t_1 - t_2|\ |\ln|t_1 - t_2||^\gamma$ *for all* $t_1, t_2 \in [0,T], t_1 \ne t_2$*. Then the energy estimates* $E_{s-s_0}(u)(t) \le C\ E_s(u)(0)$ *hold, where*

- $s_0 = 0$ *if* $\gamma = 0$,
- s_0 *is arbitrary small and positive if* $\gamma \in (0,1)$,
- s_0 *is positive if* $\gamma = 1$,
- *there is no positive constant* s_0 *if* $\gamma > 1$ *(infinite loss of derivatives).*

Proof: The statement for $\gamma = 0$ can be found in [2]. The counter-example from [6] implies the statement for $\gamma > 1$. □

Problem 1: Prove the above statement for $\gamma \in (0,1)$!

Problem 2: The results of [6] show that $\gamma = 1$ gives a finite loss of derivatives. Do we have a concrete example which shows that the solution has really a finite loss of derivatives?

We already mentioned the paper [6]. In this paper the authors studied strictly hyperbolic Cauchy problems with coefficients of the principal part depending LogLip on spatial and time variables.

- If the principal part is as in (2.1), then the authors derive energy estimates depending on a suitable low energy of the data and of the right-hand side.
- If the principal part is as in (2.1) but with coefficients which are B^∞ in x and LogLip in t, then the energy estimates depending on arbitrary high energy of the data and of the right-hand side.
- In all these energy estimates which exist for $t \in [0, T^*]$, where T^* is a suitable positive constant independent of the regularity of the data and right-hand side, the loss of derivatives depends on t.

Local condition A second possibility to weaken the Lip-property with respect to t goes back to [3]. Under the assumptions

$$a \in C[0,T] \cap C^1(0,T],\ |ta'(t)| \le C \quad \text{for} \quad t \in (0,T], \tag{8.6}$$

the authors proved a C^∞ well-posedness result for $u_{tt} - a(t)u_{xx} = 0$, $u(0,x) = \varphi(x)$, $u_t(0,x) = \psi(x)$ (even for more general Cauchy problems). They observed the effect of a *finite loss of derivatives.*

Remark: Let us compare the local condition with the global one. If $a = a(t) \in \mathrm{LogLip}[0,T]$, then the coefficient may have an irregular behavior (in comparison with the Lip-property) on the whole interval $[0,T]$. In (8.6) the coefficient has an irregular behavior only at $t=0$. Away from $t=0$ it belongs to C^1. Coefficients satisfying (8.6) don't fulfil the non-local condition

$$\int_0^{T-\tau} |a(t+\tau) - a(t)|dt \le C\tau(|\ln\tau| + 1) \quad \text{for} \quad \tau \in (0,T/2].$$

8.1.4 A refined classification of oscillating behavior

Let us suppose more regularity for a, let us say, $a \in L^\infty[0,T] \cap C^2(0,T]$. The higher regularity allows us to introduce a refined classification of oscillations.

Definition 8.1 *Let us assume additionally the condition*

$$|a^{(k)}(t)| \le C_k \left(\frac{1}{t}\left(\ln\frac{1}{t}\right)^\gamma\right)^k, \quad \text{for } k = 1,2. \tag{8.7}$$

We say, that the **oscillating behavior** *of a is*

- **very slow** *if $\gamma = 0$,*
- **slow** *if $\gamma \in (0,1)$,*
- **fast** *if $\gamma = 1$,*
- **very fast** *if condition (8.7) is not satisfied for $\gamma = 1$.*

Example: If $a = a(t) = 2 + \sin\left(\ln\frac{1}{t}\right)^\alpha$, then the oscillations produced by the sin term are very slow (slow, fast, very fast) if $\alpha \le 1$ ($\alpha \in (1,2), \alpha = 2, \alpha > 2$).

Now we are going to prove the next result yielding a *connection between the type of oscillations and the loss of derivatives* which appears. The proof uses ideas from the papers [4] and [15]. The main goal is the *construction of WKB-solutions.* We will sketch our *approach,* *which is a universal one* in the sense, that it can be used to study more *general models from non-Lipschitz theory, weakly hyperbolic theory and the theory of $L^p - L^q$ decay estimates.*

Theorem 8.5 *Let us consider*

$$u_{tt} - a(t)u_{xx} = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x),$$

where $a = a(t)$ satisfies the condition (8.7), and the data φ , ψ belong to H^{s+1}, H^s respectively. Then the following energy inequality holds:

$$E(u)(t)\,|_{H^{s-s_0}} \leq C(T)E(u)(0)\,|_{H^s} \quad \textit{for all } \; t \in (0,T], \tag{8.8}$$

where

- $s_0 = 0$ *if* $\gamma = 0$,
- s_0 *is an arbitrary small positive constant if* $\gamma \in (0,1)$,
- s_0 *is a positive constant if* $\gamma = 1$,
- *there does not exist a positive constant* s_0 *satisfying (8.8) if* $\gamma > 1$, *that is, we have an infinite loss of derivatives.*

Proof: The proof will be divided into several steps. Important tools of the proof are introduced in [72] to study weakly hyperbolic Cauchy problems. Without loss of generality we can suppose that T is small. After partial Fourier transformation we obtain

$$v_{tt} + a(t)\xi^2 v = 0, \;\; v(0,\xi) = \hat{\varphi}(\xi), \;\; v_t(0,\xi) = \hat{\psi}(\xi). \tag{8.9}$$

1. step: Zones

We divide the phase space $\{(t,\xi) \in [0,T] \times \mathbb{R} : |\xi| \geq M\}$ into two zones by using the function $t = t_\xi$ which solves $t_\xi \langle\xi\rangle = N(\ln\langle\xi\rangle)^\gamma$.

Exercise 47 Explain why it is sufficient do consider large frequencies $\{\xi : |\xi| \geq M\}$, M is large.

The constant N is determined later. Then the *pseudo-differential zone* $Z_{pd}(N)$, *hyperbolic zone* $Z_{hyp}(N)$, respectively, is defined by

$$Z_{pd}(N) = \{(t,\xi) : t \leq t_\xi\}, \quad Z_{hyp}(N) = \{(t,\xi) : t \geq t_\xi\}.$$

2. step: Symbols

To given real numbers $m_1, m_2 \geq 0, \; r \leq 2$, we define

$$S_r\{m_1,m_2\} = \{d = d(t,\xi) \in L^\infty_{loc}([0,T] \times \mathbb{R}) :$$
$$|D_t^k D_\xi^\alpha d(t,\xi)| \leq C_{k,\alpha} \langle\xi\rangle^{m_1 - |\alpha|} \left(\frac{1}{t}\left(\ln \frac{1}{t}\right)^\gamma\right)^{m_2+k}, \; k \leq r, \; (t,\xi) \in Z_{hyp}(N)\}.$$

These classes of symbols are only defined in $Z_{hyp}(N)$.

Properties:

- $S_{r+1}\{m_1, m_2\} \subset S_r\{m_1, m_2\}$;
- $S_r\{m_1 - p, m_2\} \subset S_r\{m_1, m_2\}$ for all $p \geq 0$;
- $S_r\{m_1 - p, m_2 + p\} \subset S_r\{m_1, m_2\}$ for all $p \geq 0$, this follows from the definition of $Z_{hyp}(N)$;
- if $a \in S_r\{m_1, m_2\}$ and $b \in S_r\{k_1, k_2\}$, then $a\, b \in S_r\{m_1 + k_1, m_2 + k_2\}$;
- if $a \in S_r\{m_1, m_2\}$, then $D_t a \in S_{r-1}\{m_1, m_2 + 1\}$, and $D_\xi^\alpha a \in S_r\{m_1 - |\alpha|, m_2\}$.

Exercise 48 Show these properties!

3. step: Considerations in $Z_{pd}(N)$

Setting $V = (\xi v, D_t v)^T$ the equation from (8.9) can be transformed to the system of first order

$$D_t V = \begin{pmatrix} 0 & \xi \\ a(t)\xi & 0 \end{pmatrix} V =: A(t,\xi)V. \tag{8.10}$$

We are interested in the fundamental solution $X = X(t, r, \xi)$ to (8.10) with $X(r, r, \xi) = I$ (identity matrix). Using the *matrizant* we can write X in an explicit way by

$$X(t, r, \xi) = I + \sum_{k=1}^{\infty} i^k \int_r^t A(t_1, \xi) \int_r^{t_1} A(t_2, \xi) \cdots \int_r^{t_{k-1}} A(t_k, \xi) dt_k \cdots dt_1.$$

Exercise 49 Prove this representation! Do you know such kind of representations from the course "Functional Analysis"?

The norm $\|A(t,\xi)\|$ can be estimated by $C\langle\xi\rangle$. Consequently, $\int_0^{t_\xi} \|A(s,\xi)\| ds \leq C t_\xi \langle\xi\rangle = C_N (\ln\langle\xi\rangle)^\gamma$. The solution of the Cauchy problem to (8.10) with $V(0,\xi) = V_0(\xi)$ can be represented in the form $V(t,\xi) = X(t, 0, \xi)V_0(\xi)$. Using $\|X(t, 0, \xi)\| \leq \exp(\int_0^t \|A(s,\xi)\| ds) \leq \exp(C_N(\ln\langle\xi\rangle)^\gamma)$ the next result follows.

Lemma 8.1 *The solution to (8.10) with Cauchy condition $V(0,\xi) = V_0(\xi)$ satisfies in $Z_{pd}(N)$ the energy estimate*

$$|V(t,\xi)| \leq \exp(C_N(\ln\langle\xi\rangle)^\gamma)|V_0(\xi)|.$$

Remark: In $Z_{pd}(N)$ we are near to the line $t = 0$, where the derivative of the coefficient $a = a(t)$ has an irregular behavior. It is *not a good idea to use the hyperbolic energy* $(\sqrt{a(t)}\xi\, v, D_t v)$ there because of the "bad" behavior of $a' = a'(t)$. To avoid this fact we introduce the microenergy $(\xi\, v, D_t v)$.

4. step: Two steps of diagonalization procedure

Substituting $V := (\sqrt{a(t)}\xi v, D_t v)^T$ (hyperbolic microenergy) brings the system of first order

$$D_t V - \begin{pmatrix} 0 & \sqrt{a(t)}\xi \\ \sqrt{a(t)}\xi & 0 \end{pmatrix} V - \frac{D_t a}{2a} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V = 0. \tag{8.11}$$

The first matrix belongs to the symbol class $S_2\{1,0\}$, the second one belongs to $S_1\{0,1\}$. Setting $V_0 := MV$, $M = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, this system can be transformed to the first order system

$$D_t V_0 - M \begin{pmatrix} 0 & \sqrt{a(t)}\xi \\ \sqrt{a(t)}\xi & 0 \end{pmatrix} M^{-1} V_0 - M\frac{D_t a}{2a} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M^{-1} V_0 = 0,$$

$$D_t V_0 - \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} V_0 - \frac{D_t a}{4a} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V_0 = 0,$$

where $\tau_{1/2} := \mp\sqrt{a(t)}\xi + \frac{1}{4}\,\frac{D_t a}{a}$. Thus we can write this system in the form $D_t V_0 - \mathcal{D} V_0 - R_0 V_0 = 0$, where

$$\mathcal{D} := \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \in S_1\{1,0\}\ ;\ R_0 = \frac{1}{4}\,\frac{D_t a}{a} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in S_1\{0,1\}.$$

This step of diagonalization is the

- *diagonalization of our starting system (8.11) modulo* $R_0 \in S_1\{0,1\}$.

Let us set

$$\mathcal{N}^{(1)} := -\frac{1}{4}\,\frac{D_t a}{a} \begin{pmatrix} 0 & \frac{1}{\tau_1 - \tau_2} \\ \frac{1}{\tau_2 - \tau_1} & 0 \end{pmatrix} = \frac{D_t a}{8a^{3/2}\xi} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the matrix $N_1 := I + \mathcal{N}^{(1)}$ is invertible in $Z_{hyp}(N)$ for sufficiently large N. This follows from the definition of $Z_{hyp}(N)$, from

$$\|N_1 - I\| = \|\mathcal{N}^{(1)}\| \le C_a \frac{1}{t|\xi|} \left(\ln\frac{1}{t}\right)^{\gamma} \le \frac{C_a}{N} \left(\frac{\ln\frac{1}{t}}{\ln\langle\xi\rangle}\right)^{\gamma} \le \frac{C_a}{N} \le \frac{1}{2} \quad \text{if } N \text{ is large},$$

and from $\ln\langle\xi\rangle - \ln\frac{1}{t} \ge \ln N + \ln(\ln\langle\xi\rangle)^{\gamma}$. We observe that on the one hand $\mathcal{D}N_1 - N_1\mathcal{D} = -R_0$ and on the other hand $(D_t - \mathcal{D} - R_0)N_1 = N_1(D_t - \mathcal{D} - R_1)$, where $R_1 := -N_1^{-1}(D_t\mathcal{N}^{(1)} - R_0\mathcal{N}^{(1)})$. Taking account of $\mathcal{N}^{(1)} \in S_1\{-1,1\}$, $N_1 \in S_1\{0,0\}$ and $R_1 \in S_0\{-1,2\}$ the transformation $V_0 =: N_1 V_1$ gives the following first order system:

$$D_t V_1 - \mathcal{D} V_1 - R_1 V_1 = 0,\ \mathcal{D} \in S_1\{1,0\},\ R_1 \in S_0\{-1,2\}.$$

The second step of diagonalization is the

- *diagonalization of our starting system (8.11) modulo* $R_1 \in S_0\{-1,2\}$.

5. step: Representation of solution of the Cauchy problem

Now let us devote to the Cauchy problem

$$D_t V_1 - \mathcal{D} V_1 - R_1 V_1 = 0,\ V_1(t_\xi,\xi) = V_{1,0}(\xi) := N_1^{-1}(t_\xi,\xi) M\, V(t_\xi,\xi). \tag{8.12}$$

If we have a solution $V_1 = V_1(t,\xi)$ in $Z_{hyp}(N)$, then $V = V(t,\xi) = M^{-1}N_1(t,\xi)V_1(t,\xi)$ solves (8.11) with given $V(t_\xi,\xi)$ on $t = t_\xi$.
The matrix-valued function

$$E_2(t,r,\xi) := \begin{pmatrix} \exp\left(i\int\limits_r^t (-\sqrt{a(s)}\xi + \frac{D_s a(s)}{4a(s)}) ds \right) & 0 \\ 0 & \exp\left(i\int\limits_r^t (\sqrt{a(s)}\xi + \frac{D_s a(s)}{4a(s)}) ds \right) \end{pmatrix}$$

solves the Cauchy problem $(D_t - \mathcal{D})E(t,r,\xi) = 0,\ E(r,r,\xi) = I$. We define the matrix-valued function $H = H(t,r,\xi),\ t,r \geq t_\xi$, by

$$H(t,r,\xi) := E_2(r,t,\xi) R_1(t,\xi) E_2(t,r,\xi).$$

Using the fact that $\int\limits_r^t \frac{\partial_s a(s)}{4a(s)}\, ds = \ln\ a(s)^{1/4}\,|_r^t$ (this integral depends only on a, but is independent of the influence of a') the function H satisfies in $Z_{hyp}(N)$ the estimate

$$\|H(t,r,\xi)\| \ \leq\ \frac{C}{\langle\xi\rangle}\ \left(\frac{1}{t}\left(\ln\ \frac{1}{t}\right)^\gamma \right)^2. \tag{8.13}$$

Finally, we define the matrix-valued function $Q = Q(t,r,\xi)$ is defined by

$$Q(t,r,\xi) := \sum_{k=1}^\infty i^k \int\limits_r^t H(t_1,r,\xi)dt_1 \int\limits_r^{t_1} H(t_2,r,\xi)dt_2 \cdots \int\limits_r^{t_{k-1}} H(t_k,r,\xi)dt_k.$$

The reason for introducing the function Q is that

$$V_1 = V_1(t,\xi) := E_2(t,t_\xi,\xi)(I + Q(t,t_\xi,\xi))V_{1,0}(\xi)$$

represents a solution to (8.12).

6. step: Basic estimate in $Z_{hyp}(N)$

Using (8.13) and the estimate $\int_{t_\xi}^{t} \|H(s,t_\xi,\xi)\|ds \le C_N(\ln\langle\xi\rangle)^\gamma$ we get from the representation for Q immediately

$$\|Q(t,t_\xi,\xi)\| \le \exp\Big(\int_{t_\xi}^{t} \|H(s,t_\xi,\xi)\|ds\Big) \le \exp(C_N(\ln\langle\xi\rangle)^\gamma). \tag{8.14}$$

Summarizing the statements from the previous steps gives together with (8.14) the next result.

Lemma 8.2 *The solution to (8.11) with Cauchy condition on $t = t_\xi$ satisfies in $Z_{hyp}(N)$ the energy estimate*

$$|V(t,\xi)| \le C\ \exp(C_N(\ln\langle\xi\rangle)^\gamma)|V(t_\xi,\xi)|.$$

7. step: Conclusions

From Lemmas 8.1 and 8.2 we conclude

Lemma 8.3 *The solution $v = v(t,\xi)$ to*

$$v_{tt} + a(t)\xi^2 v = 0, \quad v(0,\xi) = \hat{\varphi}(\xi), \quad v_t(0,\xi) = \hat{\psi}(\xi)$$

satisfies the a-priori estimate

$$\left|\begin{pmatrix} \xi\, v(t,\xi) \\ v_t(t,\xi) \end{pmatrix}\right| \le C\ \exp(C_N(\ln\langle\xi\rangle)^\gamma)\left|\begin{pmatrix} \xi\,\hat{\varphi}(\xi) \\ \hat{\psi}(\xi) \end{pmatrix}\right|$$

for all $(t,\xi) \in [0,T] \times \mathbb{R}$.

Exercise 50 Show that the statement of Lemma 8.3 proves the statements of Theorem 8.5 for $\gamma \in [0,1]$.

The statement for $\gamma > 1$ follows from Theorem 8.6 (see next section) if we choose in this theorem $\omega(t) = \ln^q \frac{C(q)}{t}$ with $q \ge 2$. $\square$

Remarks:

- From Theorems 8.3 and 8.5 we conclude the C^∞ well-posedness of the Cauchy problem

  $$u_{tt} - a(t)u_{xx} = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x),$$

  under the assumptions $a \in L^\infty[0,T] \cap C^2(0,T]$ and (8.7) for $\gamma \in [0,1]$.

- Without any new problems all the results can be generalized to

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t) u_{x_k x_l} = 0, \quad u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),$$

  with corresponding assumptions for $a_{kl} = a_{kl}(t)$.

- If we stop the diagonalization procedure after the first step, then we have to assume in Theorem 8.5 the condition (8.6). This approach was used in [3].

Open Problem 1: In this section we have given a very effective classification of oscillations under the assumption $a \in L^\infty[0,T] \cap C^2(0,T]$. At the moment it does not seem to be clear what kind of oscillations we have if $a \in L^\infty[0,T] \cap C^1(0,T]$ satisfies $|a'(t)| \le C\ \frac{1}{t}\ \left(\ln\frac{1}{t}\right)^\gamma,\ \gamma > 0$. If $\gamma = 0$, we have a finite loss of derivatives. What happens if $\gamma > 0$? To study this problem we have to use in a correct way the low regularity $C^1(0,T]$.

Remark: Let us consider the strictly hyperbolic Cauchy problem

$$u_{tt} + b(t)u_{xt} - a(t)u_{xx} = 0, \quad u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x).$$

Does the existence of a mixed derivative of second order change the classification of oscillations from Definition 8.1? The answer is "Yes!" From the results of [16] we know that there exist coefficients $a, b \in L^\infty[0,T] \cap C^2(0,T]$ satisfying (8.7) with $\gamma > 0$ such that the Cauchy problem is not C^∞ well-posed or there does not exist a finite constant s_0 which satisfies (8.8).

Remark: *Mixing of different non-regular effects*
The survey article [9] gives results if we mix the different non-regular effects of Hölder regularity of $a = a(t)$ on $[0,T]$ and L^p integrability of a weighted derivative on $[0,T]$. Among all these results we mention only that one which guarantees C^∞ well-posedness of

$$u_{tt} - a(t)u_{xx} = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x),$$

namely, if $a = a(t)$ satisfies $t^q \partial_t a \in L^p(0,T)$ for $q + 1/p = 1$.

8.1.5 Hirosawa's counter-example

To end the proof of Theorem 8.5 we cite a result from [4] which explains that *very fast oscillations have a deteriorating influence on C^∞ well-posedness.*

Theorem 8.6 (see [4])
Let $\omega : (0,1/2] \to (0,\infty)$ be a continuous, decreasing function satisfying $\lim \omega(s) = \infty$ for $s \to +0$ and $\omega(s/2) \le c\,\omega(s)$ for all $s \in (0,1/2]$. Then there exists a function $a \in C^\infty(\mathbb{R} \setminus \{0\}) \cap C^0(\mathbb{R})$ with the following properties:

- $1/2 \le a(t) \le 3/2$ *for all* $t \in \mathbb{R}$*;*
- *there exists a suitable positive* T_0 *and to each* p *a positive constant* C_p *such that*

$$|a^{(p)}(t)| \le C_p \omega(t) \Big(\frac{1}{t} \ \ln \ \frac{1}{t}\Big)^p \quad \textit{for all } \ t \in (0, T_0);$$

- *there exist two functions* φ *and* ψ *from* $C^\infty(\mathbb{R})$ *such that the Cauchy problem* $u_{tt} - a(t)u_{xx} = 0, \ \ u(0,x) = \varphi(x), \ \ u_t(0,x) = \psi(x)$*, has no solution in* $C^0([0,r), \ \mathcal{D}'(\mathbb{R}))$ *for all* $r > 0$.

The coefficient $a = a(t)$ possesses the regularity $a \in C^\infty(\mathbb{R} \setminus \{0\})$. To attack the open problem it is valuable to have a counter-example from [15] with lower regularity $a \in C^2(\mathbb{R} \setminus \{0\})$. To understand this counter-example let us devote to the Cauchy problem

$$u_{tt} - b\Big((\ln \frac{1}{t})^q\Big)^2 \triangle u = 0, \ u(1,x) = \varphi(x), \ u_s(1,x) = \psi(x). \tag{8.15}$$

Then the results of [15] imply the next statement.

Theorem 8.7 *Let us suppose that* $b = b(s)$ *is a positive, 1-periodic, non-constant function belonging to* C^2*. If* $q > 2$*, then there exist data* $\varphi, \psi \in C^\infty(\mathbb{R}^n)$ *such that (8.15) has no solution in* $C^2([0,1], \mathcal{D}'(\mathbb{R}^n))$*.*

Proof: The proof bases on the application of Floquet's theory. □

Remark: The idea to apply Floquet's theory to describe an instability behavior for hyperbolic equations with oscillating coefficients goes back to [8]. In [62] the tool of Floquet theory was used to construct special counter-examples for the C^∞ well-posedness of weakly hyperbolic Cauchy problems. This idea was employed in connection to $L^p - L^q$ decay estimates for solutions of wave equations with time-dependent coefficients in [56]. The merit of [15] is the application of Floquet's theory to strictly hyperbolic Cauchy problems with non-Lipschitz coefficients. We underline that the assumed regularity $b \in C^2$ comes from statements of Floquet's theory itself. An attempt to consider non-Lipschitz theory, weakly hyperbolic theory and theory of $L^p - L^q$ decay estimates for solutions of wave equations with a time-dependent coefficient is presented in [44].

8.1.6 Further activities of M.Reissig and collaborators

How to weaken $\mathbf{C^2}$ *regularity to keep the classification of oscillations*

There arises after the results of [3] and [4] the question whether there is something between the conditions ($\gamma \in [0,1]$)

- $a \in L^\infty[0,T] \cap C^1(0,T], \ |t^\gamma a'(t)| \le C \quad \text{for } \ t \in (0,T];$ (8.16)
- $a \in L^\infty[0,T] \cap C^2(0,T], \ |a^{(k)}(t)| \le C_k \left(\frac{1}{t}\left(\ln \frac{1}{t}\right)^\gamma\right)^k \text{ for } t \in (0,T], \ k = 1,2.$ (8.17)

The paper [18] is devoted to the model Cauchy problem

$$u_{tt} - a(t,x) \triangle u = 0, \quad u(T,x) = \varphi(x), \quad u_t(T,x) = \psi(x), \tag{8.18}$$

where $a = a(t,x) \in L^\infty([0,T], B^\infty(\mathbb{R}^n))$ and $a_0 \leq a(t,x)$ with a positive constant a_0.

Definition 8.2 *Definition of admissible space of coefficients*
Let T be a positive small constant, and let $\gamma \in [0,1]$ and $\beta \in [1,2]$ be real numbers. We define the weighted spaces of Hölder differentiable functions $\Lambda_\gamma^\beta = \Lambda_\gamma^\beta(0,T]$ in the following way:

$$\Lambda_\gamma^\beta(0,T] = \{a = a(t,x) \in L^\infty([0,T], B^{(k)}(\mathbb{R}^n)) : \sup_{t\in(0,T]} \|a(t)\|_{B^k(\mathbb{R}^n)}$$
$$+ \sup_{t\in(0,T]} \frac{\|\partial_t a(t)\|_{B^k(\mathbb{R}^n)}}{t^{-1}(\ln t^{-1})^\gamma} + \sup_{t\in(0,T]} \frac{\|\partial_t a\|_{M^{\beta-1}([t,T],B^k(\mathbb{R}^n))}}{(t^{-1}(\ln t^{-1})^\gamma)^\beta} \quad \text{for all } \ k \geq 0\},$$

where $\|F\|_{M^{\beta-1}(I)}$ with a closed interval I is defined by

$$\|F\|_{M^{\beta-1}(I)} = \sup_{s_1,s_2\in I, s_1\neq s_2} \frac{|F(s_1) - F(s_2)|}{|s_1 - s_2|^{\beta-1}}.$$

- If a satisfies (8.16) with $\gamma = 1$, then $a \in \Lambda_0^1$.
- If a satisfies (8.17) with $\gamma \in [0,1]$, then $a \in \Lambda_\gamma^2$.

Definition 8.3 *Space of solutions*
Let σ and γ be non-negative real numbers. We define the exponential-logarithmic scale $H_{\gamma,\sigma}$ by the set of all functions $f \in L^2(\mathbb{R}^n)$ satisfying

$$\|f\|_{H_{\gamma,\sigma}} := \left(\int_{\mathbb{R}^n} |\exp(\sigma(\ln\langle\xi\rangle)^\gamma)\hat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty.$$

In particular, we denote $H_\gamma = \bigcup_{\sigma>0} H_{\gamma,\sigma}$.

Theorem 8.8 *Let $\gamma \in [0,1]$ and $\beta \in (1,2]$. If $a \in \Lambda_\gamma^\beta(0,T]$, then the Cauchy problem (8.18) is well-posed in H_γ on $[0,T]$, that is, there exist positive constants $C_{\gamma,\beta}$, σ and σ' with $\sigma \leq \sigma'$ such that*

$$\|(\nabla u(t), u_t(t))\|_{H_{\gamma,\sigma}} \leq C_{\gamma,\beta} \|(\nabla\varphi, \psi)\|_{H_{\gamma,\sigma'}} \ \text{ for all } t \in [0,T].$$

Remark: In the Cauchy problem (8.18) we prescribe data φ and ψ on the hyperplane $t = T$. It is clear that a unique solution of the backward Cauchy problem (8.18) exists for

$t \in (0, T]$. The statement of Theorem 8.8 tells us that in the case of very slow, slow or fast oscillations ($\gamma \in [0, 1]$), the solution possesses a continuous extension up to $t = 0$.

Open Problem 2: Try to prove the next statement:
If $a = a(t, x) \in \Lambda_\gamma^\beta(0, T]$ with $\gamma > 1$ and $\beta \in (1, 2)$, then these oscillations are very fast oscillations!

The energy inequality from Theorem 8.8 yields the same connection between the type of oscillations and the loss of derivatives as in Theorem 8.5.

Theorem 8.9 *Let us consider the Cauchy problem (8.18), where $a \in \Lambda_\gamma^\beta(0, T]$ with $\gamma \in [0, 1]$ and $\beta \in (1, 2]$. The data φ, ψ belong to H^{s+1}, H^s, respectively. Then the following energy inequality holds:*

$$E(u)(t)\Big|_{H^{s-s_0}} \le C(T)E(u)(0)\Big|_{H^s} \quad \text{for all } t \in [0, T],$$

where

- *$s_0 = 0$ if $\gamma = 0$ (very slow oscillations),*
- *s_0 is an arbitrary small positive constant if $\gamma \in (0, 1)$ (slow oscillations),*
- *s_0 is a positive constant if $\gamma = 1$ (fast oscillations).*

Construction of parametrix

Now we devote to the strictly hyperbolic Cauchy problem

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t, x)u_{x_k x_l} = 0, \quad u(0, x) = \varphi(x),\ u_t(0, x) = \psi(x) \tag{8.19}$$

taking account of the classification of oscillations which is introduced in Definition 8.1. We assume

$$a_{kl} \in C([0, T], \mathcal{B}^\infty(\mathbb{R}^n)) \cap C^\infty((0, T], \mathcal{B}^\infty(\mathbb{R}^n)). \tag{8.20}$$

The *non-Lipschitz behavior of coefficients* is characterized by

$$|D_t^k D_x^\beta a_{kl}(t, x)| \le C_{k,\beta} \left(\tfrac{1}{t} \left(\ln\ \tfrac{1}{t}\right)^\gamma\right)^k \tag{8.21}$$

for all k, β and $(t, x) \in (0, T] \times \mathbb{R}^n$, where T is sufficiently small and $\gamma \ge 0$. The transformation $U = (\langle D_x \rangle u, D_t u)^T$ transfers our starting Cauchy problem (8.19) to a Cauchy problem for $D_t U - AU = F$, where $A = A(t, x, D_x)$ is a matrix-valued pseudo-differential operator. The goal of this section is the *construction of a parametrix* to $D_t - A$.

Definition 8.4 *An operator $E = E(t,s)$, $0 \le s \le t \le T_0$, is said to be a* **parametrix to the operator** $D_t - A$ *if* $D_tE - AE \in L^\infty([0,T_0]^2, \Psi^{-\infty}(\mathbb{R}^n))$. *Here* $\Psi^{-\infty}$ *denotes the space of pseudo-differential operators with symbols from* $S^{-\infty}$ *(see [29]).*

One can prove that E *is a matrix Fourier integral operator*. This is done in [28], where the case $\gamma = 1$ is studied, and in [44].

Theorem 8.10 *The parametrix* $E = E(t,s)$ *to the operator* $D_t - A$ *can be written as* $E(t,s) = M(t)N_1(t)N_2(t)K(t)E_2(t,s)Q_4(t,s)K^\sharp(s)Q_1(t,s)N_2^\sharp(s)N_1^\sharp(s)M^\sharp(s)$, *where* $E_2 = E_2(t,s)$ *is a diagonal Fourier integral operator from Proposition 4.1 of [28]. The operators* $M, M^\sharp, N_1, N_1^\sharp$, N_2 *and* $N_2^\sharp$ *are elliptic pseudo-differential operators with symbols belonging to* $S_N\{0,0\}$ *and whose symbols are constant matrices in* $Z_{pd}(N)$. *The elliptic pseudo-differential operators* $K, K^\sharp$ *are taken from Corollary 4.1 of [28]. Finally* Q_4, Q_1 *are matrix pseudo-differential operators with symbols belonging to* $W^1_\infty([0,T_0]^2, S^0_{1,0}(\mathbb{R}^n))$, $L_\infty([0,T_0]^2, S^{K_0}_{1-\varepsilon,\varepsilon}(\mathbb{R}^n)) \cap W^1_\infty([0,T_0]^2, S^{K_0+1+\varepsilon}_{1-\varepsilon,\varepsilon}(\mathbb{R}^n))$ *for every small* $\varepsilon > 0$, *respectively. Here the constant* K_0 *describes the loss of derivatives coming from the pseudo-differential zone* $Z_{pd}(2N)$ *and from the oscillations sub-zone* $Z_{osc}(2N)$.

From the last result we can conclude the following one.

Theorem 8.11 *Let us consider the strictly hyperbolic Cauchy problem*

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t,x)u_{x_kx_l} = 0, \quad u(0,x) = \varphi(x),\, u_t(0,x) = \psi(x),$$

where the coefficients satisfy the conditions (8.20) and (8.21). The data φ, ψ *belong to* H^{s+1}, H^s, *respectively. Then the following energy inequality holds:*

$$E(u)|_{H^{s-s_0}}(t) \le C(T)E(u)|_{H^s}(0) \text{ for all } t \in (0,T], \tag{8.22}$$

where
- $s_0 = 0$ *if* $\gamma = 0$,
- s_0 *is an arbitrary small positive constant if* $\gamma \in (0,1)$,
- s_0 *is a positive constant if* $\gamma = 1$,
- *there doesn't exist a positive constant* s_0 *satisfying (8.22) if* $\gamma > 1$, *that is, we have an infinite loss of derivatives.*

It seems to be remarkable that we can prove the same relation between types of oscillations and loss of derivatives as in Theorem 8.5.

8.2 $L^p - L^q$ decay estimates

The starting point of our considerations is the so-called *Strichartz' decay estimate* (see [61]) for the energy $E(u)(t) := (\nabla u(t,\cdot), u_t(t,\cdot))|_{L_q}$ basing on the $L^q(\mathbb{R}^n)$-norm

$$E(u)(t)|_{L^q} \le C(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}E(u)(0)|_{W^{N_p,p}} \tag{8.23}$$

on the conjugate line $2 \le q \le \infty$, $1/p + 1/q = 1$ for solutions of the Cauchy problem for classical wave equations, where $N_p > n\Big(\frac{1}{p} - \frac{1}{q}\Big)$. Here $W^{N_p,p}$ denotes the Sobolev spaces of fractional order N_p basing on L^p.

Several years ago the author formulated the following question:

Are we able to prove such $L^p - L^q$ decay estimates for wave equations with variable coefficients and with mass and dissipation but without drift term like

$$u_{tt} - a(t,x) \triangle u + m(t,x)u + b(t,x)u_t = 0.$$

Today we have a good understanding of the influence of time-dependent coefficients on such $L^p - L^q$ decay estimates. In general, the dependence on spatial and time variables causes difficulties. We have only a few results about energy decay (see [39], [14] and references therein). But we have to emphasize that in special situations where the coefficients depend only on the spatial variables a self-adjoint structure of the "elliptic part" allows to apply methods from spectral theory. An additional dependence on the time variable excludes the application of such methods up to now.

8.2.1 Wave equations with weak dissipation

The results of this subsection are taken from [51],[68],[69] and [70].

8.2.1.1 A model case We begin our considerations for the Cauchy problem

$$u_{tt} - \triangle u + b(t)u_t = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x), \tag{8.24}$$

where as usually $b(t) \ge 0$. In this case the energy decays although it is not clear if it decays to 0. In the paper [51] the authors supposed the case of a *weak dissipation*, that is, $b' < 0$, $\lim_{t\to\infty} b(t) = 0$. The goal is to get a complete picture from the free wave ($b(t) \equiv 0$) to the damped wave case ($b(t) \equiv 1$, which does not represent a weak dissipation).

What kind of expectations does a specialist have?

• If b has a very weak influence, then there should be a relation to the free wave equation. Such relations are described by so-called *scattering results* and the corresponding *scattering operator.*

• If b has a weak influence, then the classical energy should decay to 0 and the corresponding $L^p - L^q$ decay estimate is the classical Strichartz decay estimate with an additional term as a time-dependent coefficient coming from the energy decay to 0 itself. Such weak dissipations will be called *non-effective.*

• There exists a critical case which brings a change of the influence of the weak dissipation. This critical case is discussed in [68].

• If b has a stronger influence, then the $L^p - L^q$ decay estimate is similar to that one for the damped wave equation but with another decay function related with the dissipation itself.

Such weak dissipations will be called *effective*.

The critical case will be described by the family of Cauchy problems

$$u_{tt} - \triangle u + \frac{\mu}{1+t} u_t = 0,\ \mu > 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x). \tag{8.25}$$

Using the *theory of special functions* (Bessel functions) a $L^p - L^q$ decay estimate was proved in [68] for the *energy operator* $\mathbb{E}(t) : (\langle D\rangle\varphi, \psi) \in L^{p,r} \to (u_t(t,\cdot), |D|u(t,\cdot)) \in L^q$, where $L^{p,r}$ denotes the Bessel potential space of order r basing on $L^p(\mathbb{R}^n)$.

Theorem 8.12 *The energy operator $\mathbb{E}(t)$ satisfies the $L^p - L^q$ decay estimate*

$$\|\mathbb{E}(t)\|_{L^{p,r}\to L^q} \le C(1+t)^{\max\{-\frac{(n-1)}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\mu}{2}, -n(\frac{1}{p}-\frac{1}{q})-1\}} \tag{8.26}$$

on the conjugate line with $p \in (1,2]$ and $r = n(\frac{1}{p} - \frac{1}{q})$.

Remarks:

• If $p = q = 2$, then the estimate (8.26) yields the $L^2 - L^2$ decay estimate

$$\|\mathbb{E}(t)\|_{L^2\to L^2} \le C(1+t)^{\max\{-\frac{\mu}{2},-1\}}.$$

Thus the norm of the energy operator depends on μ if $\mu \le 2$, otherwise it is independent. Thus in the critical case $\mu = 2$ we obtain a maximal energy decay.

• If $\mu \le 2$, then the $L^p - L^q$ decay estimate generalizes the Strichartz decay estimate (8.23). In this case the dissipation is non-effective.

• If $\mu \ge n+3$, then the $L^p - L^q$ decay estimate (8.26) yields

$$\|\mathbb{E}(t)\|_{L^{p,r}\to L^q} \le C(1+t)^{-n(\frac{1}{p}-\frac{1}{q})-1}.$$

This decay estimate hints that the family of Cauchy problems (8.25) is something intermediate between Cauchy problems with non-effective and effective weak dissipations (cf. with the remarks after Theorem 8.14).

8.2.1.2 Non-effective weak dissipations Let us devote to (8.24) with non-effective dissipations. Due to [51] or [69] such dissipations are characterized by the assumptions

(A1) $\int_0^t b(\tau)d\tau = \infty$;
(A2) $b(t) \ge 0$;
(A3) $|D_t^k b(t)| \le C_k b(t)(\frac{1}{1+t})^k$ for $t \ge 0$ and all positive integers k;
(A4) $\limsup_{t\to\infty} tb(t) < 1$.

Theorem 8.13 *Under the assumptions (A1) to (A4) the energy operator $E(t)$ satisfies the $L^p - L^q$ decay estimate*

$$\|\mathbb{E}(t)\|_{L^{p,r}\to L^q} \le C\lambda(t)^{-1}(1+t)^{-\frac{(n-1)}{2}(\frac{1}{p}-\frac{1}{q})} \tag{8.27}$$

on the conjugate line with $p \in (1,2]$ *and* $r = n(\frac{1}{p} - \frac{1}{q})$. *Here* $\lambda = \lambda(t)$ *is defined as* $\lambda(t) = \exp(\frac{1}{2}\int_0^t b(s)ds)$.

Remarks:

- We conclude from (8.27) the decay estimate (8.26) for $\mu < 1$ (take account of (A4)), although it holds formally for $\mu \leq 2$, too.
- The classical energy decay is described by $\lambda(t)^{-1}$. This function may tend arbitrary slow to 0 as the next example shows:

$b(t) = \frac{\mu}{(e^{[n]}+t)\log(e^{[n]}+t)\cdots\log^{[n]}(e^{[n]}+t)}$ implies $\lambda(t) = (\log^{[n]}(e^{[n]}+t))^{\mu/2}$.

Here $\log^{[n]}$ denotes the n-times iterated logarithm and $e^{[n]}$ denotes the n-times application of e to the power.

Proof: We will only present the main steps of the proof. The x-independence of coefficients allows to *apply the partial Fourier transformation.* Thus we have to study a Cauchy problem for an ordinary differential equation depending on the parameter $|\xi|^2$. The goal is to find a *WKB-representation of its solution.* As usually these representations contain terms as

$$e^{i\phi(t,\xi)}a(t,\xi)w(\xi), \tag{8.28}$$

where $\phi = \phi(t,\xi)$ is the so-called *phase function* and $a = a(t,\xi)$ is the so-called *amplitude function.* One has to show that a *finite number of derivatives of a with respect to* ξ *satisfies symbol like estimates*, that is,

$$|\partial_\xi^\alpha a(t,\xi)| \leq C\langle\xi\rangle^{m-|\alpha|}$$

with a suitable m and for all $\alpha : |\alpha| \leq k$, k depends on the concrete application. The first step to get the WKB-representation we *divide the extended phase space* $\mathbb{R}^n \times (0,\infty)$ (it is essential to include the t-variable into the phase space) into *zones*, into the *dissipative zone* $Z_{diss}(N) = \{(t,\xi) : (1+t)|\xi| \leq N\}$ and the *hyperbolic zone* $Z_{hyp}(N) = \{(t,\xi) : (1+t)|\xi| \geq N\}$. In both zones one has to estimate the fundamental solution. In $Z_{diss}(N)$ this can be done in a more or less straightforward way. More preparations needs the consideration in the hyperbolic zone. Here we need classes of symbols

$$S_N\{m_1,m_2,m_3\} =: \Big\{a = a(t,\xi) : |D_t^k D_\xi^\alpha a(t,\xi)| \leq C_{k,\alpha}|\xi|^{m_1-|\alpha|}b(t)^{m_2}\Big(\frac{1}{1+t}\Big)^{m_3+k} \text{ in } Z_{hyp}(N) \text{ for all } k \text{ and for all } \alpha\Big\}.$$

Then one can start a diagonalization procedure related to these symbol classes to the partial Fourier transformed Cauchy problem (8.24). A *sufficiently large number of steps* guarantees that the amplitudes behave like symbols for a finite number of derivatives with respect to ξ. Some additional techniques lead to the desired WKB-representation consisting of terms as (8.28). Consequently, we arrive at a representation of the solution to (8.24) by the aid of Fourier multipliers.

Now one has to discuss these Fourier multipliers. This will be done after localizing the amplitudes by using of a *dyadic decomposition related to the extended phase space*. The part of the dyadic decomposition which belongs to the dissipative zone generates Fourier multipliers which can be studied by the *Hardy-Littlewood inequality*. The part which belongs to the hyperbolic zone needs some *Littman-type lemmas* (see [33]). These are results for oscillating integrals with localized amplitude away from the origin. The *stationary phase method* together with usual properties of the Fourier transformation leads to a $L^1 - L^\infty$ estimate for Fourier multipliers with localized amplitude. After deriving a $L^2 - L^2$ estimate (here we can use directly the structure of the terms (8.28)) some interpolation gives $L^p - L^q$ estimates on the conjugate line. Gluing all these estimates for Fourier multipliers with localized amplitudes together leads to the desired $L^p - L^q$ decay estimate for the Fourier multiplier, the energy operator itself on the conjugate line for $p \in (1, 2]$. The supposed regularity we need to avoid constants depending on the parameter of the dyadic decomposition in the $L^p - L^q$ decay estimates for the Fourier multipliers with localized amplitudes. □

Comments

• The study of wave equations with non-effective dissipation bases on two completely different ideas: the application of Hardy-Littlewood inequality on the one hand and the proof of Littman-type lemmas for oscillating integrals on the other hand.

• The *energy decay* in the $L^2 - L^2$ estimate appears from the *behavior of the amplitudes in the hyperbolic zone.*

8.2.1.3 Effective weak dissipations Effective dissipations are characterized in [51] or [70] by the assumptions (A1),(A2),(A3) and

(A5) $tb(t) \to \infty$ if $t \to \infty$.

Theorem 8.14 *Under the assumptions (A1),(A2),(A3) and (A5) the energy operator* $\mathbb{E}(t)$ *satisfies the* $L^p - L^q$ *decay estimate*

$$\|\mathbb{E}(t)\|_{L^{p,r} \to L^q} \le C\Big(1 + \int_0^t b(\tau)^{-1} d\tau\Big)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \qquad (8.29)$$

on the conjugate line with $p \in (1, 2]$ *and* $r = n(\frac{1}{p} - \frac{1}{q})$.

Remarks:

• If we choose $\mu \ge n+3$ in (8.26) and $b(t) = \frac{\mu}{1+t}$ in (8.29), then we conclude (8.26). Although this family of dissipations does not satisfy (A5) the last observation hints that this family is something intermediate between families of non-effective and effective dissipations.

• The classical damped wave equation has no weak dissipation but nevertheless an effective dissipation as the corresponding $L^p - L^q$ decay estimate shows.

• Typical examples of effective weak dissipations are the following ones:
$b(t) = \frac{\log^{[n]}(e^{[n]}+t)}{1+t}$; $b(t) = (1+t)^{-\kappa}, \kappa \in (0, 1)$; $b(t) = \frac{1}{\log^{[n]}(e^{[n]}+t)}$.

Proof: We will only discuss some changes to the proof of Theorem 8.13. $L^p - L^q$ decay estimates for the solutions of the damped wave equation were studied in [36] after the construction of explicit representation of the solution. This construction bases on hyperbolic as well as elliptic WKB-constructions. This results from the fact that $b(t) \equiv 1$ represents an effective dissipation (in a more general sense as considered in this subsection).
Consequently, our approach needs more zones, the most essential ones are the dissipative zone (see the previous proof), the *elliptic zone*
$Z_{ell}(t_0, \varepsilon, N) = \{(t,\xi) : \frac{N}{1+t} \leq |\xi| \leq (1-\varepsilon)\frac{b(t)}{2}\} \cap \{t \geq t_0\}$ and the hyperbolic zone $Z_{hyp}(N') = \{(t,\xi) : |\xi| \geq N' b(t)\}$. Of importance is the *separating line* $\Gamma_{sep} = \{(t,\xi) : |\xi| = \frac{b(t)}{2}\}$. In the elliptic and hyperbolic zone we define classes of symbols

$$S_{ell}\{m_1, m_2\} =: \Big\{ a = a(t,\xi) : |D_t^k D_\xi^\alpha a(t,\xi)| \leq C_{k,\alpha} b(t)^{m_1 - |\alpha|} \Big(\frac{1}{1+t}\Big)^{m_2+k} \text{ in } Z_{ell}(t_0,\varepsilon,N) \text{ for all } k \text{ and for all } \alpha \Big\};$$

$$S_{hyp}\{m_1, m_2, m_3\} =: \Big\{ a = a(t,\xi) : |D_t^k D_\xi^\alpha a(t,\xi)| \leq C_{k,\alpha} \langle\xi\rangle_t^{m_1 - |\alpha|} b(t)^{m_2} \Big(\frac{1}{1+t}\Big)^{m_3+k} \Big\} \text{ in } Z_{hyp}(N') \text{ for all } k \text{ and for all } \alpha \Big\},$$

where $\langle\xi\rangle_t = \sqrt{\Big||\xi|^2 - \frac{1}{4}b^2(t)\Big|}$. In both zones we carry out a WKB-construction of elliptic or of hyperbolic type. In the case of effective dissipations the $L^p - L^q$ decay estimates follow from standard properties for Fourier multipliers without using the stationary phase method and from estimates of the fundamental solution by a Gauss function. In the phase function from (8.28) there appears an integral over $\langle\xi\rangle_t$ which would cause new difficulties to prove a Littman-type result if it would be necessary. □

Comment
• The *energy decay* in the $L^2 - L^2$ estimate appears from the *behavior of the amplitudes in a neighborhood of the separating line* Γ_{sep}.

8.2.1.4 Scattering result If we do not suppose in Theorem 8.13 the assumption (A1), then the corresponding $L^2 - L^2$ estimate gives that the classical energy does not tend to 0 if t tends to infinity. This proposes that there is a connection between a solution to (8.24) and a solution to the free wave equation.
We denote by $E := |D|^{-1}L^2 \times L^2$ the energy space. Our goal is to compare in the energy space the solution $(u(t,\cdot), u_t(t,\cdot))$ to (8.24) with the solution $(v(t,\cdot), v_t(t,\cdot))$ to the Cauchy problem

$$v_{tt} - \triangle v = 0,\ v(0,x) = \tilde{\varphi}(x),\ v_t(0,x) = \tilde{\psi}(x). \tag{8.30}$$

For this reason we define the operators $S(t,s) := (u(s), u_t(s))^T \to (u(t), u_t(t))^T$ and $S_0(t,s) := (v(s), v_t(s))^T \to (v(t), v_t(t))^T$ describing the evolution of the solutions to

(8.24), (8.30), respectively. To get a scattering result one defines the operator $W_+ := \lim_{t\to\infty} S_0(0,t)S(t,0)$. In [51] it is shown the norm-convergence
$\|W_+ - S_0(0,t)S(t,0)\|_{L(E\to E)} \to 0$ if $t \to \infty$. By the aid of this operator we can describe the above mentioned connection.

Theorem 8.15 *We assume for the weak dissipation $b = b(t)$ the condition $\int_0^\infty |b(t)|dt < \infty$. Then the above defined operator $W_+ \in L(E \to E)$ is an isomorphism on the energy space such that for the solution u of (8.24) to data (φ, ψ) and the solution v to (8.30) to data $(\tilde{\varphi}, \tilde{\psi}) := W_+(\varphi, \psi)$ the estimate*

$$\|(u(t,\cdot), u_t(t,\cdot)) - (v(t,\cdot), v_t(t,\cdot))\|_E \leq C\|(\varphi(\cdot), \psi(\cdot))\|_E \int_t^\infty |b(\tau)|d\tau \tag{8.31}$$

holds, where the constant C depends only on $\|b\|_{L^1}$.

Remarks:
• We restricted ourselves to the forward Cauchy problem. Without new difficulties one can consider the backward Cauchy problem, too. We have only to suppose $b \in L^1(\mathbb{R})$. Then we get W_-. Using W_+ and W_- one can define the *scattering operator* which is of special interest.
• Besides the above mentioned norm-convergence we can describe by (8.31) how the convergence rate can be estimated. This rate can be arbitrary small how the following examples show:
$b(t) = \frac{1}{(e^{[n]}+t)\log(e^{[n]}+t)\cdots(\log^{[n-1]}(e^{[n]}+t))(\log^{[n]}(e^{[n]}+t))^\gamma}$, $\gamma > 1$, $b(t) = (1+t)^{-\kappa}$, $\kappa > 1$.

8.2.2 Wave equations with time dependent coefficients containing mass and dissipation

8.2.2.1 Wave equations with variable speed of propagation To get a first impression what may happen in the case of variable speed of propagation let us consider

$$u_{tt} - (2 + \sin t) \triangle u = 0, \; u(0,x) = \varphi(x), \; u_t(0,x) = \psi(x). \tag{8.32}$$

Then in [52] the following result is proved.

Theorem 8.16 *There are no constants p, q, r and C and a nonnegative function f defined on $\mathbb{N}$ such that the estimate*

$$\|\mathbb{E}(m)\|_{L^{p,r}\to L^q} \leq Cf(m) \tag{8.33}$$

is fulfilled for all $m \in \mathbb{N}$ while $\log f(m) = o(m)$ as $m \to \infty$.

Remarks:
• The conditions for f are very near to optimal ones. Indeed, due to Gronwall's inequality one can prove the $L^2 - L^2$ estimate (which gives in general a very weak energy estimate)

$$\|\mathbb{E}(t)\|_{L^2\to L^2} \leq C\exp(C_0 t)$$

with suitable nonnegative constants C and C_0. Choosing $t = m$, $m \in \mathbb{N}$, $p = q = 2$, $r = 0$, we get an inequality like (8.33) with $\log f(m) = O(m)$ as $m \to \infty$.

• The above example shows the *deteriorating influence of oscillating behavior of time-dependent coefficients* on $L^p - L^q$ decay estimates.

Proof: Let us sketch how the oscillating behavior of the coefficient implies the *instability of the zero solution*. For this reason we study a family of solutions $\{u_M\}$ to (8.32) with data $\{\varphi_M, \psi_M\}$, $M \in \mathbb{N}$. With a cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi(x) = 1$ when $|x| \leq 1$, and $\chi(x) = 0$ when $|x| \geq 2$ let us choose the initial (known from *geometric optics*)

$$\varphi_M(x) = e^{ix\cdot y}\chi(\tfrac{x}{M^2}), \quad \psi_M = e^{ix\cdot y}\chi(\tfrac{x}{M^2})C_0,$$

with a suitable constant C_0. Using the finite propagation speed of solutions one can construct a dependence domain with size depending on $O(M)$ in such a way that the solution u_M has to possess in this domain the structure $u_M(t,x) = \exp(ix \cdot y)w_M(t)$, where w_M is a solution to $w_{tt} + |y|^2(2 + \sin t)w = 0$. In this way the relation to Floquet's theory should be clear. If we choose $|y|^2 = \lambda_0$ from an *interval of instability* for Hill's equation $w_{tt} + \lambda b(t)w = 0$ with periodic coefficient $b(t) = 2 + \sin t$, then the fundamental matrix has an eigenvalue with modulus larger than one. Thus the effect of instability comes in. □

Very few explanations about Floquet theory

(see e.g. [10], [34])

• *Liouville transformation* The ordinary differential equation with periodic coefficient $b = b(t)$, λ is a constant, can be transformed to $y_{xx} + (\lambda\gamma^2 + Q(x))y = 0$. Therefore we need b from the class of twice continuously differentiable functions (see also the assumption from Theorem 8.7). Moreover, we assume that b is positive and non-constant.

• One can show that the Floquet theory for $w_{tt} + \lambda b(t)^2 w = 0$ with π-periodic coefficient b is equivalent to Floquet theory for $y_{xx} + (\lambda\gamma^2 + Q(x))y = 0$ with π-periodic coefficient Q and $\int_0^\pi Q(x)dx = 0$.

• The function Q is constant if and only if $b(t) = (\alpha t^2 + 2\beta t + \delta)^{-1}$.

• We consider *Hill's equation in normal form* $y_{xx} + (\lambda\gamma^2 + Q(x))y = 0$ with periodic and twice differentiable Q.

• *Theorem about oscillations*
There exist two sequences $\{\lambda_k\}_{k\geq 0}$ *and* $\{\mu_k\}_{k\geq 1}$ *with* $\lambda_0 < \mu_1 \leq \mu_2 < \lambda_1 \leq \lambda_2 < \mu_3 \leq \mu_4 < \lambda_3 \leq \lambda_4 < \cdots$ *and the following properties for the solutions:*
The solutions are stable in the intervals $(\lambda_0, \mu_1), (\mu_2, \lambda_1), (\lambda_2, \mu_3), (\mu_4, \lambda_3), \cdots$. *In general, the solutions are unstable in the endpoints of the interval, this is every time the case in* λ_0. *In the intervals* (μ_1, μ_2), (λ_1, λ_2), (μ_3, μ_4) *there exist unstable solutions.*

• Intervals of stability or instability cannot degenerate into a point.

• The interval of instability $(-\infty, \lambda_0]$ always exists. This is the only interval of instability if and only if $Q(x)$ is constant.

Lemma ([10], [34], [62])
Let the coefficient $b = b(t)$ be 1-periodic, non-constant, smooth and positive. Then there exists a positive λ_0 such that the corresponding fundamental matrix $X(1,0)$ to the equation $w_{tt} + \lambda b(t)^2 w = 0$ has eigenvalues μ_0 and μ_0^{-1} with $|\mu_0| > 1$.

Remark: This lemma guarantees the existence of *unstable solutions*, these are solutions which are unbounded for $t \to \infty$. Let us recall the ordinary differential equation $w_{tt}+\lambda w = 0$, where λ is a constant. Then there exist unstable solutions for $\lambda \leq 0$, but for $\lambda > 0$ all solutions are stable, thus they remain bounded for t to ∞. If we consider $w_{tt} + \lambda b(t)^2 w = 0$, then under suitable assumptions we have *unstable solutions for positive values* λ. This is the core of the above lemma, in some sense of Floquet theory.

Let us continue to explain our results. The next question is to understand if a "slower" oscillating behavior may lead to some kind of instability behavior. For this reason let us consider the Cauchy problem

$$u_{tt} - (2 + \sin((\log(t+30))^\alpha)) \triangle u = 0, \; u(0,x) = \varphi(x), \; u_t(0,x) = \psi(x), \tag{8.34}$$

with $\alpha > 0$. Then the following statement can be concluded from the results of [47].

Theorem 8.17 *Consider the Cauchy problem (8.34). Then there exists a constant C such that the following $L^p - L^q$ estimate holds:*

$$\|\mathbb{E}(t)\|_{L^{p,r} \to L^q} \leq C(1+t)^{s_0 - \frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})} \tag{8.35}$$

on the conjugate line with $p \in (1,2]$, $r = n(\frac{1}{p} - \frac{1}{q})$ and
- *$s_0 = 0$ if $\alpha \leq 1$;*
- *$s_0 = \varepsilon$ if $\alpha \in (1,2)$ for all sufficiently small positive ε;*
- *s_0 is a fixed positive constant if $\alpha = 2$;*
- *there does not exist a positive constant s_0 such that (8.35) is satisfied if $\alpha > 2$.*

Comment
- The results for the above family of Cauchy problems (8.34) explain the sensitivity of oscillating behavior of coefficients on $L^p - L^q$ decay estimates. The constant s_0 describes how the decay rate differs from the classical Strichartz' decay rate for the wave operator.
- From the last statement we understand the change of possible $L^p - L^q$ decay estimates from the case $\alpha \leq 2$ to the case $\alpha > 2$. But we can give a more precise description of the new quality which comes in for $\alpha > 2$. For this reason we assume that the statement for $\alpha = 2$ remains valid for $\alpha > 2$. Then one expects for the family of forward Cauchy problems

$$u_{tt} - (2 + \sin((\log(t+30))^\alpha)) \triangle u = 0, \; u(t_0,x) = \varphi(x), \; u_t(t_0,x) = \psi(x), \tag{8.36}$$

a $L^p - L^q$ estimate of the energy operator as

$$\|\mathbb{E}(t,t_0)\|_{L^{p,r} \to L^q} \leq C(1+t)^{s_0}$$

with a finite s_0 uniformly for all $t \geq t_0 \geq 0$. But this estimate is a contradiction to the following result from [47] (cf. with Theorem 8.16):

Theorem 8.18 *Let us consider (8.36) with $\alpha > 2$. There are no constants p, q, r, s and C_1, C_2 such that for all initial times t_0 and for all initial data $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$, the following $L^p - L^q$ estimate holds for all $t \geq t_0$:*

$$E(u)(t)\big|_{L^q} \leq C_1 \exp(C_2(\log(t+e))^s) E(u)(t_0)\big|_{L^{p,r}},$$

where $s < \alpha - 1$. Here the (non-standard) energy $E(u)(t)\big|_{L^{p,r}}$ is defined by

$$E(u)(t)\big|_{L^{p,r}} := \Big\|\sigma(t)\nabla_x u(t,\cdot)\Big\|_{L^{p,r}} + \Big\|\frac{1}{\sigma(t)^2}\partial_t(u(t,\cdot)\sigma(t))\Big\|_{L^{p,r}}$$

with $\sigma(t) := \sqrt{\frac{\alpha(\log(t+e))^{\alpha-1}}{t+e}}$.

The result from Theorem 8.17 proposes the following classification of oscillating coefficients for the strictly hyperbolic Cauchy problem with bounded coefficient,

$$u_{tt} - a(t) \triangle u = 0, \; u(0,x) = \varphi(x), \; u_t(0,x) = \psi(x). \tag{8.37}$$

Definition 8.5 *Let $a = a(t)$ be a smooth function satisfying*

$$|D_t^k a(t)| \leq C_k \Big(\frac{1}{t}\Big(\log t\Big)^\gamma\Big)^k, \quad k \in \mathbb{N}, \; \textit{for large}\, t. \tag{8.38}$$

We say that the oscillations of a are **very slow, slow** *or* **fast** *if $\gamma = 0$, $0 < \gamma < 1$ or $\gamma = 1$ respectively. If (8.38) is not satisfied for $\gamma = 1$, then we say that a has* **very fast oscillations** *(cf. with Definition 8.1).*

If we apply this definition to the coefficient from (8.34), then we see that the oscillations are very slow, slow, fast, very fast if $\alpha \leq 1$, $\alpha \in (1,2)$, $\alpha = 2$, $\alpha > 2$.

One can generalize the statement from Theorem 8.17 to Cauchy problems (8.37) satisfying (8.38). To derive decay estimates of the energy operator one can use the methods to study $L^p - L^q$ decay estimates for wave equations with non-effective dissipation in the sense to couple representation of solutions by Fourier multipliers with Hardy-Littlewood inequality and stationary phase method. We define suitable zones of the phase space, symbol classes in these zones, e.g. such classes of amplitudes satisfying in the hyperbolic zone estimates as

$$|D_t^k D_\xi^\alpha a(t,\xi)| \leq C_{k,\alpha}|\xi|^{m_1-|\alpha|}(\tfrac{1}{t+e^3}(\log(t+e^3))^\gamma)^{m_2+k} \tag{8.39}$$

for all k and α with a real m_1 and a nonnegative m_2. Here one term takes account of the oscillating behavior of a. In the WKB-representations there appear phase functions (see (8.28)) containing terms like $|\xi|\int_0^t \sqrt{a(s)}ds$. The proof of Littman-type lemmas using the

method of stationary phase is more or less standard. The other steps are technical but well understood.
The proof of the statement of Theorem 8.17 for $\alpha > 2$ bases on the application of Floquet's theory, too. The coefficient is in opposite to that one from (8.32) not pure periodic. If one tries to transform it to the periodic case, then additional terms appear. For this reason the method how to prove such *instability results* for non-periodic coefficients should be developed (cf. with [45], where for the first time such an approach was presented).

The specialists of wave equations may ask the following question:
Is it possible to generalize the results for (8.37) to strictly hyperbolic Cauchy problems with bounded coefficients as

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t) u_{x_k x_l} = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x)?$$

This seems to be a more difficult problem, but not from the point of view of the construction of WKB-solutions. This approach can be generalized without new difficulties. The main problem seems to be to prove a Littman-type result. In the oscillating integrals with localized amplitudes there appear the phase functions
$\phi_\pm = \phi_\pm(t,\xi) = i(x \cdot \xi \pm \int_0^t \sum_{k,l=1}^n a_{kl}(s)\xi_k \xi_l ds)$. One has to study the rank of the *Hessian* in *stationary points* of these phase functions. In the recent paper [37] such a Littman-type result is proved.

In [55] a so-called *stabilization condition* for the coefficients a_{kl} and b_k was introduced to derive $L^p - L^q$ decay estimates for the solutions of the strictly hyperbolic Cauchy problem with *increasing in time coefficients* (see the following considerations)

$$u_{tt} + \sum_{k=1}^{n} b_k(t) u_{x_k t} - \sum_{k,l=1}^{n} a_{kl}(t) u_{x_k x_l} = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x).$$

Such a stabilization condition restricts the admissible oscillating behavior of coefficients.
The next step is to study $L^p - L^q$ decay estimates for

$$u_{tt} - \lambda(t)^2 b(t)^2 \triangle u = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x). \tag{8.40}$$

The coefficient consists of two parts, one part $\lambda(t)^2$ describes the *increasing behavior* of the coefficient, the other part $b(t)^2$ describes the *oscillating behavior*. In opposite to the *deteriorating influence of oscillations* an increasing behavior has an *improving* influence. The latter was shown for special examples in [46], [51], and [11]. Thus we expect an *interplay between the influence of both parts*. That it is really so show the next examples. The related hyperbolic energy for the solutions to (8.40) is $E(u)(t)\big|_{L^q} := \|(\lambda(t)\nabla u(t,\cdot), u_t(t,\cdot))\|_{L^q}$.

Examples: The following examples allow $L^p - L^q$ decay estimates for (8.40):
• *Logarithmic growth*: $\lambda(t) = \log(t + e^3)$, $b(t) = 2 + \sin((\log(t + e^3))^\alpha)$ for $\alpha \le 2$ (cf. with (8.34) and Theorem 8.17);
• *Potential growth*: $\lambda(t) = (1 + t)^\beta$, $\beta > 0$, $b(t)$ as in the previous example;
• *Exponential growth*: $\lambda(t) = \exp(t^\beta), \beta \ge 1/2$, $b(t) = 2 + \sin(t + e^3)$;
• *Superexponential growth*: $\lambda(t) = \exp(\exp(t^\beta)), \beta > 0$, $b(t)$ as in the previous example.

If we apply the main result from [45] to special situations then we see that the following examples do not allow $L^p - L^q$ decay estimates for (8.40):
• *Potential growth*: $\lambda(t) = (1+t)^\beta$, $\beta \ge 0$, $b(t)$ is an arbitrary periodic, non-constant, smooth and positive function;
• *Exponential growth*: $\lambda(t) = \exp(t^\beta), \beta < 1/2, b(t)$ is as in the previous example.

The interplay between increasing and oscillating behavior of the coefficients will be described by a classification of oscillations generalizing the proposed classification from Definition 8.5.

Definition 8.6 *Let the oscillating part $b = b(t)$ satisfy with $\Lambda(t) := \int_0^t \lambda(s)ds$ the conditions*

$$|D_t^k b(t)| \le C_k \Big(\frac{\lambda(t)}{\Lambda(t)} \Big(\log \Lambda(t) \Big)^\gamma \Big)^k, \quad k \in \mathbb{N}, \ \text{for large } t. \tag{8.41}$$

We say that the oscillations of b are **very slow, slow** *or* **fast** *if $\gamma = 0$, $0 < \gamma < 1$ or $\gamma = 1$ respectively. If (8.41) is not satisfied for $\gamma = 1$, then we say that b has* **very fast oscillations**.

Remarks:
• If $\lambda(t) \equiv 1$, then we obtain the condition (8.38).
• If $\lambda(t) = \exp(t^\beta), \beta > 0$, and $b(t)$ is an arbitrary periodic, non-constant, smooth and positive function, then the oscillations are very slow, slow, fast, very fast if $\beta \ge 1$, $\beta \in (1/2, 1)$, $\beta = 1/2$, $\beta < 1/2$, respectively.
• If the oscillations are at least fast, then one can expect $L^p - L^q$ decay estimates. If the oscillations are very fast, then one cannot expect such decay estimates. One has to expect statements like those from Theorems 8.15 and 8.17 (see [45]).

The $L^p - L^q$ decay estimates are similar to (8.35). If we assume $\lambda(0) > 0$, then the following estimate holds for the energy operator $\mathbb{E}(t)$ which bases on the hyperbolic energy $E(u)(t)|_{L^q} := \|(\lambda(t)\nabla u(t, \cdot), u_t(t, \cdot))\|_{L^q}$:

$$\|\mathbb{E}(t)\|_{L^{p,r} \to L^q} \le C(1 + \Lambda(t))^{s_0 - \frac{n-1}{2}(\frac{1}{p} - \frac{1}{q})} \tag{8.42}$$

on the conjugate line with $p \in (1, 2]$ and with a suitable r. The necessary regularity is related to $n(\frac{1}{p} - \frac{1}{q})$. The improving influence of the increasing part is reflected to $\Lambda(t)$ in the decay function. The decay rate is unchanged.

Comment
• Up to now we have not studied the exact relation between the type of oscillations and

the constant s_0 appearing in the decay rate (cf. with Theorem 8.16). In [54] the case of fast oscillations is studied. The case of slow oscillations is studied in [52]. Nevertheless a sharp relation between γ in (8.41) and s_0 in (8.42) is open up to now. We have the following conjectures:

Conjecture 1: At most potential growth of λ

$$\|\mathbb{E}(t)\|_{L^{p,r}\to L^q} \le C\sqrt{\lambda(t)}(1+\Lambda(t))^{s_0-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}$$

with $s_0 = 0$, $s_0 = \varepsilon$, s_0 is a finite positive constant, and there does not exist a finite s_0 if $\gamma = 0$, $\gamma \in (0,1), \gamma = 1$, (8.41) is not satisfied for $\gamma = 1$, respectively. The regularity r is related to $n(\frac{1}{p}-\frac{1}{q})$.

Conjecture 2: At least exponential growth of λ

$$\|\mathbb{E}(t)\|_{L^{p,r}\to L^q} \le C\sqrt{\lambda(t)}(1+\Lambda(t))^{s_0-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}$$

with $s_0 = \varepsilon$, s_0 is a finite positive constant, and there does not exist a finite s_0 if $\gamma \in [0,1), \gamma = 1$, (8.41) is not satisfied for $\gamma = 1$, respectively. The regularity r is related to $n(\frac{1}{p}-\frac{1}{q})$.

8.2.2.2 Variable Mass In the introduction we recalled Strichartz' decay estimate (8.23) for the solutions to the wave equation. If we are interested in the Cauchy problem for the Klein-Gordon equation

$$u_{tt} - \triangle u + m^2 u = 0, \;\; u(0,x) = \varphi(x), \; u_t(0,x) = \psi(x), \; m > 0,$$

then $L^p - L^q$ decay estimates for its solutions are proven in [65] or [43]. The method of proof bases on a transformation of the Klein-Gordon equation to a wave equation in $n+1$ variables. If we study the corresponding representation of the solution by Fourier multipliers, then we expect the following $L^p - L^q$ decay estimate for the energy operator $\mathbb{E}(t)$ basing on the energy $E(u)(t)\big|_{L^q} := \|(u(t,\cdot), \nabla u(t,\cdot), u_t(t,\cdot))\|_{L^q}$:

$$\|\mathbb{E}(t)\|_{L^{p,r}\to L^q} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \tag{8.43}$$

on the conjugate line with $p \in (1,2]$ and $r = n(\frac{1}{p}-\frac{1}{q})$.

Comment

• Using Fourier multipliers the term $\langle\xi\rangle_m := \sqrt{|\xi|^2+m^2}$ appears instead of $|\xi|$ in the phase functions (the mass term is included into the phase). Then the stationary phase method gives the better Littman-type estimate with $n/2$ instead of $(n-1)/2$ (cf. (8.43) with (8.23)) because the Hessian has full rank n instead of $n-1$ in stationary points.

The paper [53] is devoted to the Klein-Gordon type model

$$u_{tt} - \lambda(t)^2 b(t)^2 \triangle u + m^2\lambda(t)^2 b(t)^2 u = 0, \;\; u(0,x) = \varphi(x), \; u_t(0,x) = \psi(x), \; m > 0. \tag{8.44}$$

Here λ and b describe the same (increasing or oscillating) behavior as in (8.40). Then the mass term has an essential influence on the interplay between $\lambda(t)$ and $b(t)$ as the following examples show:

Examples:

• If $\lambda(t) \equiv 1$ and $b(t)$ is a periodic, non-constant, smooth and positive function, then we can prove a similar result to Theorem 8.15.

• If $\lambda(t) = (1+t)^l$ and b is as before, then we have a $L^p - L^q$ decay estimate if $l \geq 1$. If we compare this observation with the examples from the previous subsection we see, that now less growth of λ is necessary to compensate the periodic part to get $L^p - L^q$ decay estimates.

Open Problem 3: Up to now we have only a conjecture what happens in the case $\lambda(t) = (1+t)^l$, $l \in (0,1)$ and $b(t)$ supposed as a periodic, non-constant, smooth and positive function.

The above examples propose a change of condition (8.41) describing the interplay between increasing and oscillating part.

Definition 8.7 *Let the oscillating part $b = b(t)$ satisfy with $\Lambda(t) := \int_0^t \lambda(s)ds$ the conditions*

$$|D_t^k b(t)| \leq C_k \Big(\frac{\lambda(t)}{\Lambda(t)^\gamma}\Big)^k, \quad k \in \mathbb{N}, \ \textit{for large } t. \tag{8.45}$$

We say that the oscillations of b are **slow** *or* **fast** *if $\gamma \in (1/2, 1], \gamma = 1/2$ respectively. If (8.45) is not satisfied for $\gamma = 1/2$, then we say that b has* **very fast oscillations**.

Then the results from [53] yield the following statement.

Theorem 8.19 *Consider the Cauchy problem (8.44) under the assumption (8.45). Then there exists a constant C such that the following $L^p - L^q$ estimate holds:*

$$\|\mathbb{E}(t)\|_{L^{p,r}\to L^q} \leq C\sqrt{\lambda(t)}(1+\Lambda(t))^{s_0 - \frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \tag{8.46}$$

on the conjugate line with $p \in (1,2]$, $r = n(\frac{1}{p} - \frac{1}{q})$ and

• *$s_0 = 0$ if $\gamma \in (1/2, 1]$;*

• *s_0 is a fixed positive constant if $\gamma = 1/2$.*

Here we are interested only in estimates of the hyperbolic energy
$E(u)(t)\big|_{L^q} := \|(\lambda(t)\nabla u(t,\cdot), u_t(t,\cdot))\|_{L^q}$.

There appears the term $n/2$ in the decay rate of the estimate (8.46). We call the *mass term effective* if such an improvement of the decay rate in the sense that $n/2$ instead of $(n-1)/2$ appears. The paper [17] is devoted to a family of Klein-Gordon models which allow to study the change from $(n-1)/2$ to $n/2$, that is, the change from models with *non-effective mass term* to models with *effective mass term.*

The family of Cauchy problems is

$$u_{tt} - \lambda(t)^2 b(t)^2 \triangle u + \frac{\lambda(t)^2 b(t)^2}{(e^3 + \Lambda(t))^{2\gamma}(\log(e^3 + \Lambda(t)))^{2\delta}} u = 0,$$
$$u(0,x) = \varphi(x), \ u_t(0,x) = \psi(x),$$

under the condition

$$|D_t^k b(t)| \le C_k \Big(\frac{\lambda(t)}{\Lambda(t)^\beta}\Big(\log\Lambda(t)\Big)^{-\omega}\Big)^k \qquad \text{for large } t,$$

for $\beta \in [0,1]$, $\gamma \ge 0$ and $\delta \ne 0$ only for $\gamma = 1$, where $\omega = 0$ if $\beta \in [0,1)$ and $\omega \in (-\infty,0]$ if $\beta = 1$. For $\gamma = 0$ we have (8.46), for $\gamma = \infty$, due to our arrangement this means no mass, we have (8.42) if $\omega \in (-1,0]$. It seems to be reasonable and we will follow this strategy that the oscillating behavior in the mass term coincides with that one from the main part.

What kind of results do the readers expect?

Cases $\beta \in (\frac{1}{2},1)$; $\beta = 1, \omega < -1$:

• If γ is small, this means we have a bit smaller mass than in the case $\gamma = 0$, then one should expect a Klein-Gordon type $L^p - L^q$ decay estimate.

• If γ is large, this means we have a small mass, then one cannot expect a $L^p - L^q$ decay estimate, *maybe one has a Floquet-type result.*

It should be interesting to describe the change-over from Klein-Gordon to Floquet, to find the critical mass $\gamma_0 = \gamma_0(\beta)$ *and to study the influence of the mass on decay estimates.*

Case $\beta = 1$, $\omega \in (-1,0]$:

• If γ is small, then one should expect a Klein-Gordon type decay rate.

• If γ is large, then one should expect a wave decay rate.

It should be interesting to describe the change-over from Klein-Gordon to wave decay rates, to find the critical mass $\gamma_0 = \gamma_0(\beta)$ *and to study the influence of the mass on decay estimates.*

Case $\beta \in [0,\frac{1}{2})$:

• If we have no mass ($\gamma = \infty$), then one cannot expect a $L^p - L^q$ decay estimate.

It should be interesting to study what happens during the change-over from $\gamma = \infty$ *to* $\gamma = 0$. *Does a Floquet-type effect appear? What is the influence of the mass term?*

In [17] there are answers to all of these questions and the change-over to the critical cases is described.

Final remarks:

• The model with dissipation (related to the Klein-Gordon model (8.44))

$$u_{tt} - \lambda(t)^2 b(t)^2 \triangle u + a\lambda(t)b(t)u_t = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x), \;\; a \ne 0,$$

is studied in [45]. There we introduce the same classification of oscillations which is given in Definition 8.7. We are able to prove a statement like Theorem 8.18 for solutions to this wave model.

• Dissipative wave models with coefficients depending on time and spatial variables, too, are discussed in [14] and [39] (see further references therein). Using the method of weighted energy inequalities (this method differs from our approach) some $L^2 - L^2$ estimates for the energy are proved. Corresponding $L^p - L^q$ decay estimates are still open up to now.

• In the previous sections we discussed the influence of increasing and oscillating parts of coefficients on $L^p - L^q$ decay estimates. The possible deteriorating influence of decreasing parts is studied in [67].

• Some results of this paper can be used to understand a new qualitative behavior of solutions to non-linear models like

$$u_{tt} - \lambda(t)^2 b(t)^2 \triangle u = \lambda(t)^2 b(t)^2 (\nabla u)^2 - (u_t)^2, \;\; u(0,x) = \varphi(x), \; u_t(0,x) = \psi(x).$$

If the corresponding Cauchy problem (8.40) shows a Floquet effect, that is, we can expect results like those from Theorems 8.15 and 8.17, then the non-linear Cauchy problem does not have the property of *existence of small data solutions* (see [71] and [67]) (cf. with the remark from Section 4.1). In opposite to this relation the proof of this property seems to be reasonable in cases for which $L^p - L^q$ decay estimates for solutions to (8.40) are known at least in cases we have *no loss of decay*.

How to get a mark in the PDE course?

• Solve the exercises 1,4,8,11,13,16,20,22,24,25,27,30,31,34,37,38,40,41,43,44,45,

• prepare a pdf-file and send it up to the end of April 07 to
reissig@math.tu-freiberg.de
Then I fix a mark for everybody.

Participants:

Dao Phan Vu, Do Van Hiep, Le Chi Ngoc, Le Cuong, Nguyen Quoc Hung, Nguyen Thi Thu Huong, Nguyen Manh Hung, Nguyen Nhu Thang, Tran Quoc Tuan, Tran Thi Thuy.

Marks:

• Nguyen Thi Thu Huong - very good,
• Nguyen Nhu Thang - very good,
• Tran Quoc Tuan - very good,
• Nguyen Manh Hung - very good,
• Le Chi Ngoc - very good,
• Dao Phan Vu - good,
• Do Van Hiep - good,
• Tran Thi Thuy - satisfactory

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References

[1] E. Becker, *Gas Dynamics*, Academic Press, New York, 1968.

[2] F. Colombini, E. De Giorgi and S. Spagnolo, *Sur les equations hyperboliques avec des coefficients qui ne dependent que du temps*, Ann. Scuola Norm. Sup. Pisa, **6** (1979) 3, 511-559.

[3] F. Colombini, D. Del Santo and T. Kinoshita, *Well-posedness of the Cauchy problem for a hyperbolic equation with non- Lipschitz coefficients*, Ann. Scuola Norm. Sup. Pisa, **1** (2002), 327-358.

[4] F. Colombini, D. Del Santo and M. Reissig, *On the optimal regularity of coefficients in hyperbolic Cauchy problems*, Bulletin des Sciences Mathematiques, **127** (2003) 4, 328-347.

[5] F. Colombini, E. Janelli and S. Spagnolo, *Hyperbolic equations and classes of infinitely differentiable functions*, Annali di Matematica pura ed applicata, **143** (1986), 187-195.

[6] F. Colombini and N. Lerner, *Hyperbolic operators with non-Lipschitz coefficients*, Duke Math. J., **77** (1995) 3, 657-698.

[7] F. Colombini and S. Spagnolo, *Second order hyperbolic equations with coefficients real analytic in space variables and discontinuous in time*, Journal D'Analyse Mathematique, **38** (1980), 1-33.

[8] F. Colombini and S. Spagnolo, *Hyperbolic equations with coefficients rapidly oscillating in time: a result of nonstability*, J. Diff. Equ., **52** (1984) 1, 24-38.

[9] D. Del Santo, *Some results on the Cauchy problem for hyperbolic operators with non-regular coefficients*, preprint.

[10] M.S.P. Eastham, *The spectral theory of periodic differential equations*, Scottish Academic Press, Edinburgh and London, 1973.

[11] A. Galstian, $L_p - L_q$ *decay estimates for the wave equations with exponentially growing speed of propagation*, Applicable Analysis, **82** (2003) 3, 197-214.

[12] P. Garabedian, *Partial Differential Equations*, John Wiley & Sons, New York, 1964.

[13] M.G. Grillakis, *Regularity and asymptotic behavior of the wave equation with a critical nonlinearity*, Ann. of Math., **132** (1990), 485-509.

[14] F. Hirosawa, *Energy decay for degenerate hyperbolic equations of Klein-Gordon type with dissipative term*, Funkcialaj Ekvacioj **43** (2000) 1, 163-191.

[15] F. Hirosawa, *On the Cauchy problem for second order strictly hyperbolic equations with non-regular coefficients*, Math. Nachr., **256** (2003), 29-47.

[16] F. Hirosawa and M. Reissig, *About the optimality of oscillations in non-Lipschitz coefficients for strictly hyperbolic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci (5) (2004), 589-608.

[17] F. Hirosawa and M. Reissig M, *From wave- to Klein-Gordon type decay rates*, in: Schulze et.al, Nonlinear Hyperbolic Equations, Spectral Theory, and Wavelet Transformations, Advances in PDE, Operator Theory, Advances and Applications, Birkhäuser, vol. **145** (2003), 95-155.

[18] F. Hirosawa and M. Reissig, *Well-posedness in Sobolev spaces for second order strictly hyperbolic equations with non-differentiable oscillating coefficients*, Annals of Global Analysis and Geometry, **25** (2004), 99-119.

[19] V. Ivrii and V. Petkov, *Necessary conditions for well-posedness of the Cauchy problem for non-strictly hyperbolic equations* (in Russian), Usp. Mat. Nauk, **29** (1974) 5, 3-70.

[20] K. Jachmann, *$L^p - L^q$ decay estimates for linear type-2 models of thermoelasticity with lower order terms*, Diplomarbeit, Freiberg, 2005.

[21] K. Jörgens, *Das Anfangswertproblem im Großen für eine Klasse nichtlinearer Wellengleichungen*, Math. Zeitschrift, **77** (1961), 295-308.

[22] F. John, *Blow-up of solutions of nonlinear wave equations in three space dimensions,* Manuscripta Math., **28** (1979), 235-265.

[23] F. John, *Blow-up for quasilinear wave equations in three space dimensions*, Comm. Pure Appl. Math., **34** (1981), 29-51.

[24] Ch. Kerler, *Differenzierbarkeit im Bild und Abbildungseigenschaften verallgemeinerter Fouriertransformationen bei variablen Koeffiizienten im Außengebiet und Anwendungen auf Gleichungen vom Kirchhoff-Typ*, Phd thesis, Konstanz, 1998.

[25] S. Klainerman, *Uniform decay estimates and the Lorentz invariance of the classical wave equation*, Comm. Pure Appl. Math., **38** (1985), 321-332.

[26] S. Klainerman and A. Majda, *Formation of singularities for wave equations including the nonlinear vibrating string*, Comm. Pure Appl. Math., **33** (1980), 241-263.

[27] S. Klainerman and G. Ponce, *Global, small amplitude solutions to nonlinear evolution equations*, Comm. Pure Appl. Math., **36** (1983), 133-141.

[28] A. Kubo and M. Reissig, *Construction of parametrix for hyperbolic equations with fast oscillations in non-Lipschitz coefficients*, Communications in PDE, **28** (2003) 7&8, 1471-1502.

[29] H. Kumano-go, *Pseudodifferential operators*, MIT Press, Cambridge, Mass.-London, 1981.

[30] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics*, Pergamon Press, Elmsford, New York, 1959.

[31] R. Leis, *Initial boundary value problems in mathematical physics*, Teubner-Verlag Stuttgart, John Wiley & Sons, Chichester et. al., 1986.

[32] P.H. Lesky and R. Racke, *Nonlinear wave equations in infinite wave guides*, Communications in PDE, **28** (2003) 7&8, .

[33] W. Littman, *Fourier transformations of surface carried measures and differentiability of surface averages*, Bull. Amer. Math. Soc., **69** (1963), 766-770.

[34] W. Magnus und St. Winkler, *Hill's equation*, Interscience Publishers, New York-London-Sydney, 1966.

[35] P. Marcati and K. Nishihara, *The $L^p - L^q$ estimates of solutions to one-dimensional wave equations and their application to the compressible flow through porous media*, J. Diff. Equ., **191** (2003), 445-469.

[36] A. Matsumura, *On the asymptotic behavior of solutions of semi-linear wave equations*, Publ. RIMS Kyoto Univ., **12** (1976), 169-189.

[37] T. Matsuyama and M. Reissig, *Stabilization and $L^p - L^q$ decay estimates*, Asymptotic Analysis, to appear.

[38] S. Mizohata, *The theory of partial differential equations*, University Press, Cambridge, 1973.

[39] K. Mochizuki and H. Nakazawa, *Energy decay and asymptotic behavior of solutions to the wave equations with linear dissipation*, Publ. RIMS Kyoto Univ., **32** (1996), 401-414.

[40] T. Nishitani, *Sur les equations hyperboliques a coefficients Hölderiens en t et de classe de Gevrey en x*, Bull. Sc. math., **107** (1983), 113-138.

[41] I.G. Petrovski, *Partial Differential Equations*, W.B. Saunders, Philadelphia, 1967.

[42] R. Racke, *Non-homogeneous, non-linear damped wave equations in unbounded domains*, Math. Meth. Appl. Sci., **13** (1990), 481-491.

[43] R. Racke, *Lectures on Nonlinear Evolution Equations*, Aspects of Mathematics, vol. **19**, Vieweg (1992).

[44] M. Reissig, *A refined diagonalization procedure to handle fast oscillations in degenerate hyperbolic problems*, to appear.

[45] M. Reissig, *Klein-Gordon type decay rates for wave equations with a time-dependent dissipation*, Adv. Math. Sci. Appl., **11** (2001) 2, 859-891.

[46] M. Reissig, *On $L_p - L_q$ estimates for solutions of a special weakly hyperbolic equation*, Ed. Li Ta-Tsien, Nonlinear Evolution Equations and Infinite-Dimensional Dynamical Systems, 153-164, World Scientific (1997).

[47] M. Reissig and J. Smith, *$L_p - L_q$ estimate for wave equation with bounded time-dependent coefficient*, Hokkaido Mathematical Journal, **34** (2005) 3, 541-586.

[48] M. Reissig and Y.G. Wang, *Cauchy problems for linear thermoelastic systems of type III in one space variable*, Math. Meth. Appl. Sc., **28** (2005), 1359-1381.

[49] M. Reissig and Y.G. Wang, *Parabolic type decay rates for 1-d-thermoelastic systems with time-dependent coefficients*, Monatshefte für Mathematik, **138** (2003), 239-259.

[50] M. Reissig and Y.G. Wang, *Propagation of mild singularities in higher dimensional thermoelasticity*, J. Math. Analysis and Application, **240** (1999), 398-415.

[51] M. Reissig and J. Wirth, *Wave equations with monotone weak dissipation*, Fakultät für Mathematik und Informatik, TU Bergakademie Freiberg, Preprint 2003-02, 71 A4, ISSN 1433-9307.

[52] M. Reissig and K. Yagdjian, *About the influence of oscillations on Strichartz-type decay estimates*, Rend. Sem. Mat. Univ. Pol. Torino, **58** (2000) 3, 375-388.

[53] M. Reissig and K. Yagdjian, *Klein-Gordon type decay rates for wave equations with time-dependent coefficients*, Banach Center Publications, vol.**52** (2000), 189-212.

[54] M. Reissig and K. Yagdjian, *$L_p - L_q$ decay estimates for hyperbolic equations with oscillations in the coefficients*, Chin. Ann. of Math., Ser. B, **21** (2000) 2, 153-164.

[55] M. Reissig and K. Yagdjian, *$L_p - L_q$ estimates for the solutions of strictly hyperbolic equations of second order with increasing in time coefficients*, Math. Nachr., **214** (2000), 71-104.

[56] M. Reissig and K. Yagdjian, *One application of Floquet's theory to $L_p - L_q$ estimates for hyperbolic equations with very fast oscillations*, MMAS, **22** (1999), 937-951.

[57] Y. Shibata, *On the global existence of classical solutions of second order fully nonlinear hyperbolic equations with first order dissipationin the exterior domain*, Tsukuba J. Math., **7** (1983) 1, 1-68.

[58] Y. Shibata and Y. Tsutsumi, *On a global existence theorem of small amplitude solutions for nonlinear wave equations in an exterior domain*, Math. Zeitschrift, **191** (1986), 165-199.

[59] Y. Shibata and Y. Tsutsumi, *Local existence of solution for the initial boundary value problem of fully nonlinear wave equation*, Nonlinear Analysis, **11** (1987), 335-365.

[60] C.D. Sogge, *Lectures on nonlinear wave equations*, Monographs in Analysis, **vol. 2**, International Press, Boston, 1995.

[61] R. Strichartz, *A priori estimates for the wave-equation and some applications*, J. Funct. Anal., **5** (1970), 218-235.

[62] S. Tarama, *On the second order hyperbolic equations degenerating in the infinite order-example-*, Math. Japonica, **42** (1995), 523-534.

[63] B.R. Vainberg, *On the short wave asymptotic behaviour of solutions of stationary problems and the asymptotic behaviour as $t \to \infty$ of solutions of non-stationary problems*, Russian Math. Surveys, **30** (1975), 1-58.

[64] B.R. Vainberg, *Asymptotic methods in equations of mathematical physics*, Gordon and Breach Science Publishers, New York et. al., 1989.

[65] W. v.Wahl, *L^p-decay rates for homogeneous wave-equations*, Math. Zeitschrift, **120** (1971), 93-106.

[66] C.H. Wilcox, *Scattering theory for the d'Alembert equation in exterior domains*, Lecture Notes in Mathematics **442**, 1975.

[67] J. Wirth, *About the solvability behavior for special classes of non-linear hyperbolic equations*, Nonl. Anal., **52** (2003), 421-431.

[68] J. Wirth, *Solution representations for a wave equation with weak dissipation*, Math. Meth. Appl. Sci., **27** (2004), 101-124.

[69] J. Wirth, *Wave equations with time-dependent dissipation I. Non-effective dissipation*, J. Differential Equations, **222** (2006), 487-514.

[70] J. Wirth, *Wave equations with time-dependent dissipation II. Effective dissipation*, J. Differential Equations, **232** (2007), 74-103.

[71] K. Yagdjian, *Parametric resonance and nonexistence of global solution to non-linear wave equations*, J. Math. Anal. Appl., **260** (2001) 1, 251-268.

[72] K. Yagdjian, *The Cauchy Problem for Hyperbolic Operators. Multiple Characteristics, Micro-Local Approach*, Akademie-Verlag, Berlin, 1997.