Hyperbolic-parabolic singular perturbation for Kirchhoff equations

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Modern Aspects in Phase Space Analysis
Freiberg – 14 February 2012
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Outline

1. Kirchhoff equations
   - Non-dissipative Kirchhoff equations
   - Dissipative Kirchhoff equations

2. Singular Perturbation: problem and subproblems
   - Statement of the problem
   - Existence, uniqueness, decay estimates

3. Singular Perturbation: results
   - Global-in-time error estimates
   - Decay-error estimates
Kirchhoff Equation (concrete form)

\[ u_{tt} - m \left( \int_\Omega |\nabla u|^2 \, dx \right) \Delta u = 0 \]

+ Boundary conditions
+ Initial conditions

where

- \( \Omega \subseteq \mathbb{R}^n \) is an open set,
- \( m : [0, +\infty) \to [0, +\infty) \) is a continuous function.

Main features

- Quasilinear hyperbolic equation.
- Integro-differential equation.
- Nonlocal equation.
The name of the game


Original Kirchhoff Equation (dimension one)

\[ u_{tt} - \left( a + b \int_0^L u_x^2 \, dx \right) u_{xx} = 0 \]

where

- \([0, L]\) is an interval,
- \(a\) and \(b\) are positive constants, and \(m(\sigma) = a + b\sigma\).

This equation was introduced as a model for the small transversal vibrations of a pre-stretched string. It was derived from the “true” system by means of some mathematical simplifications.
From linear to quasilinear equations

Linear hyperbolic equation:

\[ u_{tt} - c \Delta u = 0 \]

Quasilinear Kirchhoff equation and its linearization:

\[ u_{tt} - m \left( \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = 0 \]

Note: the propagation speed depends on \( t \) only.

Quasilinear (local) hyperbolic equation and its linearization:

\[ u_{tt} - m(u_{x_1},...,u_{x_n}) \Delta u = 0 \]

Note: the propagation speed depends on \( x \) and \( t \).
Non-dissipative Kirchhoff equation (abstract setting)

Kirchhoff Equation (abstract form)

\[ u'' + m \left( |A^{1/2}u|^2 \right) Au = 0 \]
\[ u(0) = u_0, \quad u'(0) = u_1 \]

where

- \( H \) is a (separable) Hilbert space,
- \( A : D(A) \to H \) is a self-adjoint linear nonnegative operator with dense domain \( D(A) \subseteq H \),
- \( |A^{1/2}u|^2 = \langle Au, u \rangle \),
- \( m : [0, +\infty) \to [0, +\infty) \) is a continuous function.
Abstract vs concrete setting

The concrete equation can be put in abstract form by setting

\[ H := L^2(\Omega), \quad Au = -\Delta u. \]

The domain \( D(A) \) depends on the boundary conditions. Under reasonable boundary conditions we have that

\[ |A^{1/2}u|^2 = \langle Au, u \rangle = -\int_\Omega u \cdot \Delta u \, dx = \int_\Omega |\nabla u|^2 \, dx. \]

Advantages of the abstract setting:

1. unified treatment of different boundary conditions,
2. more general operators are allowed,
3. one has to write simpler and shorter equations.
Dissipative Kirchhoff Equation (abstract form)

\[ \varepsilon u'' + b(t)u' + m \left( |A^{1/2}u|^2 \right) Au = 0 \]

\[ u(0) = u_0, \quad u'(0) = u_1 \]

As before

- \( H \) is a (separable) Hilbert space,
- \( A : D(A) \to H \) is a self-adjoint linear nonnegative operator with dense domain \( D(A) \subseteq H \),
- \( |A^{1/2}u|^2 = \langle Au, u \rangle \),
- \( m : [0, +\infty) \to [0, +\infty) \) is a continuous function,

and moreover

- \( \varepsilon > 0 \) is a (small) positive parameter,
- \( b : [0, +\infty) \to (0, +\infty) \) is a continuous function.
Notation: Degenerate vs Nondegenerate Equations

\[ \varepsilon u'' + b(t)u' + m \left( |A^{1/2}u|^2 \right) Au = 0 \]
\[ u(0) = u_0, \quad u'(0) = u_1 \]

Taking into account the **nonlinearity**, the equation is said to be

- **non-degenerate if** \( m(\sigma) \geq \mu > 0 \)
- **mildly degenerate if** \( m(\sigma) \geq 0 \) but \( m \left( |A^{1/2}u_0|^2 \right) \neq 0 \)
- **really degenerate if** \( m(\sigma) \geq 0 \) and \( m \left( |A^{1/2}u_0|^2 \right) = 0 \)

**Nondegenerate model cases:** \( m(\sigma) \equiv 1, \quad m(\sigma) = 1 + \sigma \)

**Degenerate model case:** \( m(\sigma) = \sigma^\gamma \) \( (\gamma \geq 1 \text{ or } \gamma \in (0, 1)) \)
Non-dissipative Kirchhoff equations
Singular Perturbation: problem and subproblems
Singular Perturbation: results

Kirchhoff equations

Notation: Standard vs Weak Dissipation

\[ \varepsilon u'' + b(t)u' + m \left( |A^{1/2}u|^2 \right) Au = 0 \]

Taking into account the **dissipative term**, we have

- **Constant dissipation** if \( b(t) \) is a positive constant
- **Weak dissipation** if \( b(t) \to 0 \) as \( t \to +\infty \)

- The case where \( b(t) \geq \delta > 0 \) and \( b'(t) \) is bounded is more or less equivalent to constant dissipation.
- Model weak dissipation: \( b(t) = (1 + t)^{-p} \) for some \( p > 0 \).
- “Strong dissipation” usually refers to dissipative terms of the form \( A^\alpha u' \), with \( \alpha > 0 \) or even \( \alpha \geq 1/2 \).
Kirkhoff equations

Singular Perturbation: problem and subproblems
Singular Perturbation: results

Non-dissipative Kirchhoff equations
Dissipative Kirchhoff equations

**Notation: Coercive vs Non-coercive Operators**

\[ \varepsilon u'' + b(t)u' + m\left(|A^{1/2}u|^{2}\right)Au = 0 \]

Taking into account the **linear operator** \( A \), we have a

- coercive operator if
  \[ |A^{1/2}u|^2 \geq \nu |u|^2 \text{ for some } \nu > 0 \]

- non-coercive operator if
  \[ \langle Au, u \rangle \geq 0 \]

- In the concrete case of the Dirichlet-Laplacian, coercivity holds true when \( \Omega \) is a (regular enough) **bounded** domain.
- Coercivity is usually needed in order to deduce that
  \[ |Au|^2 \geq \nu |A^{1/2}u|^2 \text{ (hence a nontrivial kernel is less dangerous than a vanishing sequence of eigenvalues).} \]
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The limit problem

Let us start again with the second order (hyperbolic) problem

\[ \varepsilon u''_\varepsilon + b(t)u'_\varepsilon + m \left( |A^{1/2}u_\varepsilon|^2 \right) Au_\varepsilon = 0 \]

\[ u_\varepsilon(0) = u_0, \quad u'_\varepsilon(0) = u_1 \]

The formal limit as \( \varepsilon \to 0^+ \) is the first order (parabolic) problem

\[ b(t)u' + m \left( |A^{1/2}u|^2 \right) Au = 0 \]

\[ u(0) = u_0 \]

Passing from a second order problem to a first order problem entails the **loss of one initial condition**.
The corrector

In order to keep into account the boundary layer due to the loss of one initial condition, one defines the corrector $\theta_\varepsilon(t)$ as the solution of the second order problem

$$
\varepsilon \theta''_\varepsilon + b(t)\theta'_\varepsilon = 0
$$

$$
\theta_\varepsilon(0) = 0, \quad \theta'_\varepsilon(0) = u_1 + m \left( |A^{1/2}u_0|^2 \right) Au_0
$$

Remarks

- The equation solved by the corrector is a **linear and homogeneous ODE** (hence an explicit expression for $\theta_\varepsilon(t)$ can be easily found).

- $\theta'_\varepsilon(0) = u'_\varepsilon(0) - u'(0)$ (and in particular $\theta_\varepsilon(t) \equiv 0$ when there is no loss of initial conditions).
The remainders

Let

- $u_\varepsilon(t)$ be the solution of the second order problem,
- $u(t)$ be the solution of the limit problem,
- $\theta_\varepsilon(t)$ be the corrector.

Then one defines $\rho_\varepsilon(t)$ and $r_\varepsilon(t)$ in such a way that

$$u_\varepsilon(t) = u(t) + \rho_\varepsilon(t) = u(t) + \theta_\varepsilon(t) + r_\varepsilon(t)$$

The singular perturbation problem consists in proving that $\rho_\varepsilon(t) \to 0$ or $r_\varepsilon(t) \to 0$ in some sense as $\varepsilon \to 0^+$. 

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Dissipative Kirchhoff equations
Six Subproblems

The singular perturbation problem gives rise to (at least) six subproblems.

1. Parabolic problem: global existence and decay estimates.
2. Local existence for the hyperbolic problem and local-in-time error estimates.
4. Hyperbolic problem: decay estimates (behavior of $u(t)$ and $u_\varepsilon(t)$ as $t \to +\infty$).
5. Singular perturbation problem: global-in-time error estimates (behavior of $u_\varepsilon(t)$ as $\varepsilon \to 0^+$).
6. Singular perturbation problem: decay-error estimates (behavior of $u_\varepsilon(t)$ as $t \to +\infty$ and $\varepsilon \to 0^+$).
### State of the art

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**Legenda:**

- NDG nondegenerate case $m(\sigma) \geq \mu > 0$
- MDG1 mildly degenerate case $m(\sigma) = \sigma^\gamma$ with $\gamma \geq 1$
- MDG2 mildly degenerate case $m(\sigma) = \sigma^\gamma$ with $\gamma \in (0, 1)$
- MDG3 mildly degenerate case with any $m(\sigma) \geq 0$
The parabolic problem – Existence results

\[ b(t)u' + m (|A^{1/2}u|^2) Au = 0, \quad u(0) = u_0 \]

For application to the singular perturbation problem, we limit ourselves to initial data \( u_0 \in D(A) \) with \( m (|A^{1/2}u_0|^2) \neq 0 \) (mildly degenerate case).

1. The problem admits a unique global-in-time solution.
2. The solution can always be written as a reparametrization of the solution of the heat equation \( v' + Av = 0, \quad v(0) = u_0 \).
3. \( u(t) \in D(A^\alpha) \) for every \( \alpha \geq 0 \) and every \( t > 0 \).
4. The time regularity of \( u(t) \) depends in the expected way from the regularity of \( m \) and \( b \).
The parabolic problem – Decay estimates

Let us consider the simplest case where $H = \mathbb{R}$ and $A = \text{Identity}$. Up to constants the problem reduces to an ODE.

$$b(t)y' + m(y^2) y = 0, \quad y(0) = 1$$

The basic idea in that $y(t)$ describes the decay of $u(t)$ also in the general case!

1. If the operator $A$ is coercive, then $A^\alpha u(t) \sim y(t)$ as $t \to +\infty$ (estimates from below and from above).
2. If the operator $A$ is noncoercive, then $y(t)$ is the optimal estimate from below. The estimate from above is in general worse, and all intermediate rates are actually attained for suitable choices of initial data.
Local-in-time existence for Kirchhoff equations has long been studied in the nondissipative case. The theory is now quite complete, the leitmotiv being that the regularity required on initial data does depend on the regularity of $m$.

From the point of view of local existence there is no difference between:

- nondissipative, dissipative, or weakly dissipative equations,
- nondegenerate or mildly degenerate equations.
The hyperbolic problem – Global existence and decay

\[ \varepsilon u''_{\varepsilon} + b(t)u'_{\varepsilon} + m \left( |A^{1/2}u_{\varepsilon}|^2 \right) Au_{\varepsilon} = 0, \]
\[ u_{\varepsilon}(0) = u_0, \quad u'_{\varepsilon}(0) = u_1 \]

The problem admits a global-in-time solution (which decays as \( t \to +\infty \) as the solution of the parabolic problem) provided that

- \((u_0, u_1) \in D(A) \times D(A^{1/2})\) with \( m \left( |A^{1/2}u_0|^2 \right) \neq 0\),
- \(\varepsilon\) is small enough,
- \(b(t) = (1 + t)^{-p}\) for some \(p \in [0, 1]\) (but only special nonlinearities are allowed when \(p > 0\)).

No one knows what happens when the assumptions are not satisfied. Global-in-time existence is the big open problem in the field of Kirchhoff equation.
Hyperbolic vs parabolic behavior – Linear case

Let us consider for simplicity the linear equation \((a, b, c \text{ are positive constants})\).

\[ a u'' + \frac{b}{(1 + t)^p} u' + c A u = 0 \]

1. When \(p < 1\) one can neglect inertia, and the solution is asymptotic to a solution of the corresponding parabolic problem.
2. When \(p > 1\) one can neglect dissipation, and the solution is asymptotic to a solution of the corresponding hyperbolic problem (hence it does not tend to 0).
3. When \(p = 1\) one has parabolic or hyperbolic behavior depending on \(a/b\) (with parabolic behavior when the ratio is small).
Kirchhoff equations

Hyperbolic vs parabolic behavior – Kirchhoff case

\[ \varepsilon u'' + \frac{1}{(1 + t)^p} u' + m |A^{1/2} u|^{2} A u = 0 \]

The Kirchhoff case behaves exactly as the linear one, with the case \( p = 1 \) which always falls in the parabolic regime when \( \varepsilon \) is small enough. In particular

1. solutions of the parabolic problem always decay to 0 as \( t \to +\infty \), faster and faster as \( p \) grows,

2. solutions of the hyperbolic problem decay to 0 as \( t \to +\infty \) if and only if \( p \in [0, 1] \).

We can never expect a global-in-time convergence \( u_{\varepsilon}(t) \to u(t) \) when \( p > 1 \).
The linear case – Statement of the problem

In the linear case where $b(t) \equiv 1$ and $m(\sigma) \equiv 1$ the singular perturbation problem becomes

\[
\varepsilon u''_\varepsilon(t) + u'_\varepsilon(t) + Au_\varepsilon(t) = 0 \\
\varepsilon u_\varepsilon(0) = u_0, \quad u'_\varepsilon(0) = u_1
\]

\[
\sim \rightarrow \\
\begin{align*}
    u'(t) + Au(t) &= 0 \\
    u(0) &= u_0
\end{align*}
\]

In this case the corrector turns out to be

\[
\theta_\varepsilon(t) = \varepsilon(u_1 + Au_0)(1 - e^{-t/\varepsilon})
\]

As usual, the problem consists in estimating

\[
\rho_\varepsilon(t) := u_\varepsilon(t) - u(t) \quad \text{and} \quad r_\varepsilon(t) := u_\varepsilon(t) - u(t) - \theta_\varepsilon(t)
\]

as $\varepsilon \rightarrow 0^+$. 
The linear case – Basic convergence result

Let us consider the same assumptions on initial data as in the existence results for Kirchhoff equations, namely

\[(u_0, u_1) \in D(A) \times D(A^{1/2})\]

Then there exists a function \(\omega(\varepsilon) \to 0\) such that

\[
|\rho_\varepsilon(t)|^2 + |A^{1/2}\rho_\varepsilon(t)|^2 + |A\rho_\varepsilon(t)|^2 + |r'_\varepsilon(t)|^2 \leq \omega(\varepsilon)
\]

\[
\int_0^{+\infty} |A^{1/2}r'_\varepsilon(t)|^2 dt \leq \omega(\varepsilon)
\]

\(\omega(\varepsilon)\) is independent of \(t\) (global-in-time error estimates).

Minimal assumptions on initial data.

Convergence in the same space of initial data.

No prescribed convergence rate.
The linear case – Better convergence result

Let us consider slightly better initial data, namely

\[(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})\]

Then there exists a constant \(C\) such that

\[\left| \rho_\varepsilon(t) \right|^2 + \left| A^{1/2} \rho_\varepsilon(t) \right|^2 + \varepsilon \left| r'_\varepsilon(t) \right|^2 \leq C\varepsilon^2\]

\[\int_0^{+\infty} \left| r'_\varepsilon(t) \right|^2 dt \leq C\varepsilon^2\]

>): C is independent of \(t\) (global-in-time error estimates).

We have a fixed (polynomial) convergence rate.

Intermediate assumptions on initial data yield intermediate convergence rates.

No convergence rate for \(A\rho_\varepsilon(t)\) and \(A^{1/2}r'_\varepsilon(t)\).
Let us consider even better initial data, namely

\[ (u_0, u_1) \in D(A^2) \times D(A) \]

Then there exists a constant \( C \) such that

\[
|A \rho_{\varepsilon}(t)|^2 + |r'_{\varepsilon}(t)|^2 + \varepsilon |A^{1/2} r'_{\varepsilon}(t)|^2 \leq C \varepsilon^2
\]

\[
\int_0^{+\infty} |A^{1/2} r'_{\varepsilon}(t)|^2 \, dt \leq C \varepsilon^2
\]

\( C \) is independent of \( t \) (global-in-time error estimates).

We have a fixed (polynomial) convergence rate for all quantities up to \( A \rho_{\varepsilon} \) and \( A^{1/2} r'_{\varepsilon} \).

Intermediate assumptions on initial data yield intermediate convergence rates.
The linear case – Comments

The remainder $r_\varepsilon(t)$ is good for estimates involving derivatives, while $\rho_\varepsilon(t)$ is better suited for estimates without derivatives.

The space $D(A^{3/2}) \times D(A^{1/2})$ is optimal when looking for estimates such as $|A^{1/2}\rho_\varepsilon(t)|^2 \leq C\varepsilon^2$ (and $\varepsilon^2$ is the best possible convergence rate).

The minimal convergence result requires initial data in $D(A) \times D(A^{1/2})$, hence with “gap 1/2” (typical of hyperbolic problems). Convergence results with polynomial rate do require data in $D(A^{3/2}) \times D(A^{1/2})$, hence with “gap 1” (typical of parabolic problems).
The Kirchhoff case – Statement of the problem

Let us consider the more general equation

\[ \varepsilon u''_\varepsilon + \frac{1}{(1 + t)^p} u'_\varepsilon + m \left( |A^{1/2} u_\varepsilon|^2 \right) Au_\varepsilon = 0 \]

\[ u_\varepsilon(0) = u_0, \quad u'_\varepsilon(0) = u_1 \]

with \( p \in [0, 1] \), and the corresponding first order limit problem.

As in the linear case, the corrector \( \theta_\varepsilon(t) \) can be explicitly computed by integrating the linear ODE it solves.

Once again, the problem consists in estimating

\[ \rho_\varepsilon(t) := u_\varepsilon(t) - u(t) \quad \text{and} \quad r_\varepsilon(t) := u_\varepsilon(t) - u(t) - \theta_\varepsilon(t) \]

as \( \varepsilon \to 0^+ \).
The Kirchhoff case – Results

The basic convergence result (initial data in $D(A) \times D(A^{1/2})$ and convergence rate $\omega(\varepsilon) \to 0$) should be true in general (but it has never been put into paper in full generality).

The further convergence results (requiring more regular initial data, but with the optimal convergence rate $\omega(\varepsilon) = C\varepsilon^2$) hold true
- in the non-degenerate case,
- when the operator $A$ is coercive and $m(\sigma) = \sigma^\gamma$ ($\gamma \geq 1$),
- when the operator $A$ is non-coercive, $m(\sigma) = \sigma^\gamma$ ($\gamma \geq 1$), and $p \leq (\gamma^2 + 1)/(\gamma^2 + 2\gamma - 1)$. 
The Kirchhoff case – Results

The further convergence results (requiring more regular initial data, but with the non-optimal convergence rate $\omega(\varepsilon) = C\varepsilon$) hold true

- in the case with constant dissipation and Lipschitz nonlinearity $m(\sigma) \geq 0$ satisfying a suitable monotonicity assumption,
- in the non-coercive case with $m(\sigma) = \sigma^\gamma$ ($\gamma \geq 1$),
  and $$(\gamma^2 + 1)/(\gamma^2 + 2\gamma - 1) \leq p \leq 1$$

Open problems: prove optimal convergence results when

1. $m(\sigma) = \sigma^\gamma$ with $\gamma \in (0, 1)$,
2. $m(\sigma) = \sigma^\gamma$ with $\gamma \geq 1$ and non-coercive operator $A$,
3. $m(\sigma) \geq 0$ is any Lipschitz continuous nonlinearity.
As the name suggests, decay-error estimates are the meeting point of decay estimates and error estimates. They describe in the same time the behavior of $u_\varepsilon(t) - u(t)$ (hence also of $u_\varepsilon(t)$) as $t \to +\infty$ and as $\varepsilon \to 0^+$. 

The general form of a decay-error estimate is something like

$$|A^\alpha \rho_\varepsilon(t)| \leq \omega(\varepsilon) \gamma(t) \quad \text{or} \quad |A^\alpha r'_\varepsilon(t)| \leq \omega(\varepsilon) \gamma(t)$$

It is fundamental to keep in mind the different roles of $\rho_\varepsilon(t)$ and $r_\varepsilon(t)$. It is not possible to prove decay-error estimates on $|A^\alpha \rho'_\varepsilon(t)|$ or $|A^\alpha r_\varepsilon(t)|$. 

Achtung!
Low-cost decay-error estimates

Error-estimates + Decay-estimates \implies \text{Decay-error estimates}

This follows from the general fact that

\[ \varphi(\varepsilon, t) \leq \omega(\varepsilon) \]
\[ \varphi(\varepsilon, t) \leq \gamma(t) \]

\[ \implies \varphi(\varepsilon, t) \leq \sqrt{\omega(\varepsilon)} \cdot \sqrt{\gamma(t)} \]

These low-cost decay-error estimates are never optimal. Indeed one expects that

- convergence rates (w.r.t. \( \varepsilon \to 0^+ \)) are the same which appear in the local-in-time error estimates,
- decay rates (w.r.t. \( t \to +\infty \)) are the same of solutions of the parabolic problem (or even better).

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Decay-error estimates

Decay-error estimates for Kirchhoff equations

\[ \varepsilon u''_{\varepsilon} + b(t) u'_\varepsilon + m \left( |A^{1/2} u_{\varepsilon}|^2 \right) Au_{\varepsilon} = 0 \]
\[ u_{\varepsilon}(0) = u_0, \quad u'_\varepsilon(0) = u_1 \]

\[ \varepsilon u'' + b(t) u' + m \left( |A^{1/2} u|^2 \right) Au = 0 \]
\[ u(0) = u_0 \]

To the best of our knowledge, decay-error estimates for Kirchhoff equations have been investigated only in two cases:

1. nondegenerate equations \((m(\sigma) \geq \mu > 0)\) with constant or weak dissipation,
2. degenerate equations with \(m(\sigma) = \sigma^\gamma \) \((\gamma \geq 1)\) and constant dissipation.

Decay-error estimates in the degenerate case resisted some years after the existence theory was established, but there is a reason for this . . .
Nondegenerate Kirchhoff equation

Let us consider initial data \((u_0, u_1) \in D(A) \times D(A^{1/2})\).

\[
|\rho_\varepsilon(t)|^2 \leq \omega(\varepsilon)
\]

\[
|A^{1/2}\rho_\varepsilon(t)|^2 \leq \frac{\omega(\varepsilon)}{(1 + t)^{p+1}}
\]

\[
|A\rho_\varepsilon(t)|^2 \leq \frac{\omega(\varepsilon)}{(1 + t)^2(p+1)}
\]

\[
|r'_\varepsilon(t)|^2 \leq \frac{\omega(\varepsilon)}{(1 + t)^2}
\]

\[
\int_0^{+\infty} (1 + t)^p \left( |r'_\varepsilon(t)|^2 + |A^{1/2}\rho_\varepsilon(t)|^2 \right) dt \leq \omega(\varepsilon)
\]

\[
\int_0^{+\infty} (1 + t)^{2p+1} \left( |A^{1/2}r'_\varepsilon(t)|^2 + |A\rho_\varepsilon(t)|^2 \right) dt \leq \omega(\varepsilon)
\]
Nondegenerate Kirchhoff equation

Let us consider initial data \((u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})\).

\[
|\rho_\varepsilon(t)|^2 \leq C_\varepsilon^2
\]

\[
|A^{1/2}\rho_\varepsilon(t)|^2 \leq \frac{C_\varepsilon^2}{(1 + t)^{p+1}}
\]

\[
|A\rho_\varepsilon(t)|^2 \leq \frac{C_\varepsilon}{(1 + t)^{2(p+1)}}
\]

\[
|r'_\varepsilon(t)|^2 \leq \frac{C_\varepsilon}{(1 + t)^2}
\]

\[
\int_0^{+\infty} (1 + t)^p \left( |r'_\varepsilon(t)|^2 + |A^{1/2}\rho_\varepsilon(t)|^2 \right) dt \leq C_\varepsilon^2
\]

\[
\int_0^{+\infty} (1 + t)^{2p+1} \left( |A^{1/2}r'_\varepsilon(t)|^2 + |A\rho_\varepsilon(t)|^2 \right) dt \leq C_\varepsilon
\]
Nondegenerate Kirchhoff equation

Let us consider initial data \((u_0, u_1) \in D(A^2) \times D(A)\).

\[
|\rho(t)|^2 \leq C\varepsilon^2
\]

\[
|A^{1/2}\rho(t)|^2 \leq \frac{C\varepsilon^2}{(1 + t)^{p+1}}
\]

\[
|A\rho(t)|^2 \leq \frac{C\varepsilon^2}{(1 + t)^{2(p+1)}}
\]

\[
|r'(t)|^2 \leq \frac{C\varepsilon^2}{(1 + t)^2}
\]

\[
\int_0^{+\infty} (1 + t)^p \left( |r'(t)|^2 + |A^{1/2}\rho(t)|^2 \right) dt \leq C\varepsilon^2
\]

\[
\int_0^{+\infty} (1 + t)^{2p+1} \left( |A^{1/2}r'(t)|^2 + |A\rho(t)|^2 \right) dt \leq C\varepsilon^2
\]
Nondegenerate Kirchhoff equation – Comments

Decay rates are the same involved in decay estimates for the parabolic and the hyperbolic problems alone.

Convergence rates are the same involved in the local-in-time error estimates.

If the operator $A$ is coercive, we expect analogous results with exponential decay rates.

However, this part of the theory has never been put into paper in full generality.
Degenerate Kirchhoff equation – Counterexample

Let us consider the degenerate case with $m(\sigma) = \sigma^\gamma$

\[
\varepsilon u''_\varepsilon + u'_\varepsilon + |A^{1/2} u_\varepsilon|^{2\gamma} A u_\varepsilon = 0
\]

\[
u_\varepsilon(0) = u_0, \quad u'_\varepsilon(0) = u_1
\]

Let $u_0$ and $u_1$ be eigenvectors of $A$ with respect to eigenvalues $0 < \lambda_1 < \lambda_0$, respectively (note the reverse order!). Then

\[
\liminf_{t \to +\infty} (1 + t)^{1/\gamma} |A^{1/2} \rho_\varepsilon(t)|^2 \geq \frac{1}{(2\gamma \lambda_0)^{1/\gamma}}.
\]

As a consequence, decay-error estimates with the optimal decay rate cannot be true in the degenerate nonlinear case. Fourier components of $u_0$ vs $u_1$ seem to play an important role.
Let $0 < \lambda_0 < \lambda_1$. Let $v_0$ be an eigenvector with respect to $\lambda_0$. Let us assume that

$$
\begin{align*}
u_0 &= \alpha v_0 + (\text{components w.r.t. frequencies } \geq \lambda_1), \\
u_1 &= \beta v_0 + (\text{components w.r.t. frequencies } \geq \lambda_1)
\end{align*}
$$

for some $\alpha \neq 0, \beta \in \mathbb{R}$.

Under the previous assumptions on initial data one can prove decay-error estimates with

- optimal convergence rates $\omega(\varepsilon)$,
- decay rates $\gamma(t)$ with are better than the decay rates of $u(t)$ and $u_\varepsilon(t)$. 

Massimo GOBBINO Dissipative Kirchhoff equations
Degenerate Kirchhoff equation – Result

For initial data \((u_0, u_1) \in D(A^2) \times D(A)\) we have that

\[
|\rho_\varepsilon(t)|^2 + |A^{1/2}\rho_\varepsilon(t)|^2 + |A\rho_\varepsilon(t)|^2 \leq \frac{C\varepsilon^2}{(1 + t)^{\delta + 1/\gamma}}
\]

\[
|r'_\varepsilon(t)|^2 \leq \frac{C\varepsilon^2}{(1 + t)^{\delta + 2 + 1/\gamma}}
\]

 аналогичные оценки не утратят своей силы для менее регулярных начальных данных.

Also integral estimates hold true with an improvement in decay rates.

There is an improvement in decay rates expressed by some \(\delta > 0\) depending on \(\gamma\) and \(\lambda_1/\lambda_0\).
Heuristics – Toy model

Ansatz:
- for $\varepsilon$ small the parabolic equation is a good approximation of the hyperbolic one,
- hence, after the initial layer, $u_\varepsilon(t)$ and $u(t)$ can both be considered as solutions of the parabolic equation,
- after the initial layer the difference $u_\varepsilon - u$ is of order $\omega(\varepsilon)$.

1. Let us consider the simple case $H = \mathbb{R}$, $A = \text{Identity}$, $\varepsilon = 0$.
2. The equation reduces to $u' + m(u^2)u = 0$.
3. Let us consider two solutions $v(t)$ and $w(t)$ of this equation with **two different initial data** $v_0$ and $w_0$.
4. The toy model consists in estimating the difference $v(t) - w(t)$ in terms of $v_0 - w_0$ and a suitable decay rate.
Toy models without improvement in decay rates

**Linear and nondegenerate**

\[ u' + u = 0 \]

\[ v(t) = v_0 e^{-t}, \quad w(t) = w_0 e^{-t}, \quad w(t) - v(t) = (w_0 - v_0) e^{-t} \]

**Nonlinear and nondegenerate**

\[ u' + u + u^3 = 0 \]

\[ v(t) \sim v_0 e^{-t}, \quad w(t) \sim w_0 e^{-t}, \quad w(t) - v(t) \sim (w_0 - v_0) e^{-t} \]

**Linear and degenerate**

\[ u' + \frac{k}{1+t} u = 0 \]

\[ v(t) = \frac{v_0}{(1+t)^k}, \quad w(t) = \frac{w_0}{(1+t)^k}, \quad w(t) - v(t) = \frac{w_0 - v_0}{(1+t)^k} \]
Toy models with improvement in decay rates

Nonlinear and degenerate

\[ u' + u^3 = 0 \]

\[ \nu(t) = \frac{\nu_0}{(1 + 2\nu_0^2 t)^{1/2}}, \quad w(t) = \frac{w_0}{(1 + 2w_0^2 t)^{1/2}}, \]

- Both \( \nu(t) \) and \( w(t) \) decay at infinity as \( (1 + t)^{-1/2} \).
- The limit of \( (1 + t)^{-1/2} \nu(t) \) or \( (1 + t)^{-1/2} w(t) \) does not depend on the initial condition.

The difference \( w(t) - \nu(t) \) decays at infinity as \( (1 + t)^{-3/2} \). Therefore in this case there is an improvement in decay rates.
Conclusions – Open problems

<table>
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<tr>
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<th>$p \in (0, 1], \nu &gt; 0$</th>
<th>$p \in (0, 1], \nu \geq 0$</th>
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<td>5 6</td>
<td>3-4-5-6</td>
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</tbody>
</table>

1. Nondegenerate case: decay-error estimates with improved decay rates.
2. Case $m(\sigma) = \sigma^\gamma$ with $\gamma \geq 1$: decay-error estimates in the weakly dissipative case.
3. Case $m(\sigma) = \sigma^\gamma$ with $\gamma \in (0, 1)$: error and decay-error estimates in all cases.
4. Generic $m(\sigma) \geq 0$: good luck!
THANK YOU FOR YOUR ATTENTION!