Some results on edge intersection hypergraphs

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Abstract. If $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph, its edge intersection hypergraph $EI(\mathcal{H}) = (V, \mathcal{E}^{EI})$ has the edge set $\mathcal{E}^{EI} = \{e_1 \cap e_2 \mid e_1, e_2 \in \mathcal{E} \land e_1 \neq e_2 \land |e_1 \cap e_2| \geq 2\}$. Besides investigating several structural properties, for $n \geq 24$ we prove that there is a 3-regular (and - if $n$ is even - 6-uniform) hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $\lceil \frac{n}{2} \rceil$ hyperedges and $EI(\mathcal{H}) = C_n$.

Keywords. Edge intersection hypergraph

Mathematics Subject Classification 2010: 05C65

1. Introduction and basic definitions

All hypergraphs $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ and (undirected) graphs $G = (V(G), E(G))$ considered in the following may have isolated vertices but no multiple edges or loops.

A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is $k$-uniform if all hyperedges $e \in \mathcal{E}$ have the cardinality $k$. Trivially, any 2-uniform hypergraph $\mathcal{H}$ is a graph. The degree $d(v)$ of a vertex $v \in V$ is the number of hyperedges $e \in \mathcal{E}$ being incident to the vertex $v$. $\mathcal{H}$ is $r$-regular if all vertices $v \in V$ have the same degree $r = d(v)$.

If $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph, its edge intersection hypergraph $EI(\mathcal{H}) = (V, \mathcal{E}^{EI})$ has the edge set $\mathcal{E}^{EI} = \{e_1 \cap e_2 \mid e_1, e_2 \in \mathcal{E} \land e_1 \neq e_2 \land |e_1 \cap e_2| \geq 2\}$. For $k \geq 2$, the $k$-th iteration of the EI-operator is defined to be $EI^k(\mathcal{H}) := EI(EI^{k-1}(\mathcal{H}))$, where $EI^1(\mathcal{H}) := EI(\mathcal{H})$. Moreover, the EI-number $k^{EI}(\mathcal{H})$ is the smallest $k \in \mathbb{N}$ such that $\mathcal{E}(EI^k(\mathcal{H})) = \emptyset$.

Let $e = \{v_1, v_2, \ldots, v_l\} \in \mathcal{E}^{EI}$ be a hyperedge in $EI(\mathcal{H})$. By definition, in $\mathcal{H}$ there exist (at least) two hyperedges $e_1, e_2 \in \mathcal{E}(\mathcal{H})$ both containing all the vertices $v_1, v_2, \ldots, v_l$, more precisely $\{v_1, v_2, \ldots, v_l\} = e_1 \cap e_2$. In this sense, the hyperedges of $EI(\mathcal{H})$ describe sets $\{v_1, v_2, \ldots, v_l\}$ of vertices having a certain, "strong" neighborhood relation in the original hypergraph $\mathcal{H}$.

As an application, we consider a hypergraph $\mathcal{H} = (V, \mathcal{E})$ representing a communication system. The vertices $v_1, v_2, \ldots, v_n \in V$ and the hyperedges $e_1, e_2, \ldots, e_m \in \mathcal{E}$ correspond to $n$ people and to $m$ (independent) communication channels, respectively. A group $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \subseteq V$ of people can communicate in a conference call if and only if their members use one and the same communication channel, i.e. there is a hyperedge $e \in \mathcal{E}$ such that $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \subseteq e$. If we ask whether or not $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ can even communicate in a conference call after the breakdown of an arbitrarily chosen communication channel, then this question is equivalent to the problem of the existence of a hyperedge $e^{EI} \in \mathcal{E}^{EI}$ in the edge intersection hypergraph $EI(\mathcal{H})$ containing all these vertices, i.e. $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \subseteq e^{EI}$.
Note that there is a significant difference to the well-known notions of the \textit{intersection graph} (cf. [6]) or \textit{edge intersection graph} (cf. [9]) \( G = (V(G), E(G)) \) of linear hypergraphs \( H = (V(H), E(H)) \), since there we have \( V(G) = E(H) \).

In [5], [2] and [3] the same notation is used for so-called \textit{edge intersection graphs of paths}, but there the authors consider paths in a given graph \( G \) and the vertices of the resulting edge intersection graph correspond to these paths in the original graph \( G \).

Obviously, for certain hypergraphs \( H \) the edge intersection hypergraph \( EI(H) \) can be 2-uniform; in this case \( EI(H) \) is a simple, undirected graph \( G \). But in contrast to the intersection graphs or edge intersection graphs mentioned above, \( G = EI(H) \) and \( H \) have one and the same vertex set \( V(G) = V(H) \). Therefore we consistently use our notion "edge intersection hypergraph" also when this hypergraph is 2-uniform.

First of all, in Section 2 we investigate structural properties of edge intersection hypergraphs.

In Section 3 we consider 2-uniform edge intersection hypergraphs. In doing so, a natural question arises.

**Problem 1.** Let \( G = (V, E) \) be a graph. What is the minimum cardinality \( |\mathcal{E}| \) of the edge set of a hypergraph \( H = (V, \mathcal{E}) \) with \( EI(H) = G \)?

In order to attack Problem 1, at first it makes sense to investigate simple classes of graphs. Let us consider \( G = C_n \), where \( C_n \) denotes the cycle with \( n \) vertices. Of course, for some small \( n \), hypergraphs \( H = (V, \mathcal{E}) \) with \( EI(H) = C_n \) and minimum cardinality \( |\mathcal{E}| \) can be easily found. So by distinctions of cases it can be proved that for \( n = 3 \) this minimum cardinality is \( n + 1 = 4 \), whereas for \( n \in \{4, 5, 6\} \) the wanted minimum is \( n \) (see the hypergraph \( H^3_n \), which follows Problem 2 at the end of Subsection 3.2).

This situation completely changes for larger \( n \). Then it seems to be difficult to determine this minimum cardinality \( |\mathcal{E}| \) without additional restrictions on the hypergraphs \( H \) being under consideration. So in the range \( 7 \leq n \leq 23 \) only unsatisfying, partial results are known.

In this context, \( k \)-uniformity will be proved to be very useful. As we will see later, multiples of 3 are good candidates for the number \( k \). We will build the hyperedges \( e \in \mathcal{E} \) of \( H \) by combining so-called 3-sections of the vertices of \( V = V(C_n) = V(H) \) (see Subsection 3.2). Therefore in our main result the number \( k \) will be chosen equal to 6. Under the restriction of 6-uniformity, for even \( n \geq 24 \) we will construct a family of 3-regular hypergraphs \( H = (V, \mathcal{E}) \) with \( EI(H) = C_n \) and minimum cardinality \( |\mathcal{E}| = \frac{3n}{2} \).

If \( n \) is odd, then we additionally need one hyperedge \( e \) of cardinality 3; in this case we obtain \( |\mathcal{E}| = \frac{3n+1}{2} \).

### 2. Some structural properties of edge intersection hypergraphs

**Theorem 1.** (i) For each linear hypergraph \( H = (V, \mathcal{E}) \) with \( V \notin \mathcal{E} \) there is a hypergraph \( H' = (V, \mathcal{E}') \) with \( EI(H') = H \).

(ii) Let \( H = (V, \mathcal{E}) \) be a hypergraph containing \( e_1, e_2 \in \mathcal{E} \) with \( |e_1 \cap e_2| \geq 2 \), \( e_1 \nsubseteq e_2 \), \( e_2 \nsubseteq e_1 \), and \( H' = (V, \mathcal{E}') \) be a hypergraph with \( H = EI(H') \). Then there is an \( \tilde{e} \in \mathcal{E} \setminus \{e_1, e_2\} \) with \( e_1 \cap e_2 \subseteq \tilde{e} \).

(iii) Not every hypergraph \( H = (V, \mathcal{E}) \) with \( V \notin \mathcal{E} \) is an edge intersection hypergraph of some hypergraph \( H' = (V, \mathcal{E}') \).
This follows from (ii); a minimal example is $H$.

(ii) There are vertices $v_1 \in e_1 \setminus e_2$ and $v_2 \in e_2 \setminus e_1$ and edges $e'_1, e'_2, e''_2 \in E'$ with $e'_1 \cap e''_2 = e_1$ and $e'_2 \cap e''_2 = e_2$. Clearly

$$\exists e^1 \in \{e'_1, e''_1\} : v_2 \notin e^1 \land \exists e^2 \in \{e'_2, e''_2\} : v_1 \notin e^2.$$ W.l.o.g. let $e^1 = e'_1$, $e^2 = e'_2$. Then

$$\tilde{e} : = e'_1 \cap e'_2 \geq (e'_1 \cap e''_1) \cap (e'_2 \cap e''_2) = e_1 \cap e_2.$$ Hence $\tilde{e} \in E$ and $e_1 \notin \tilde{e}$, $e_2 \notin \tilde{e}$.

(iii) This follows from (ii); a minimal example is $H = (V,E)$ with $V = \{1,2,3,4\}$, $E = \{\{1,2,3\},\{2,3,4\}\}$.

Next we consider relations between edge intersection hypergraphs and several other classes of hypergraphs known from the literature: The competition hypergraph $\mathcal{CH}(D) = (V,\mathcal{E}^C)$ of a digraph $D = (V,A)$ (see [10]) has the edge set

$$\mathcal{E}^C = \{e \subseteq V \mid |e| \geq 2 \land \exists v \in V : e = N_D^-(v)\}.$$ The double competition hypergraph $D\mathcal{CH}(D) = (V,\mathcal{E}^{DC})$ of a digraph $D = (V,A)$ (see [7]) has the edge set

$$\mathcal{E}^{DC} = \{e \subseteq V \mid |e| \geq 2 \land \exists v_1, v_2 \in V : e = N_D^+(v_1) \cap N_D^-(v_2)\}.$$ The niche hypergraph $\mathcal{NH}(D) = (V,\mathcal{E}^N)$ of a digraph $D = (V,A)$ (see [4]) has the edge set

$$\mathcal{E}^N = \{e \subseteq V \mid |e| \geq 2 \land \exists v \in V : e = N_D^-(v) \lor e = N_D^+(v)\}.$$ Further, for technical reasons, we need the common enemy hypergraph $CE\mathcal{H}(D) = (V,\mathcal{E}^{CE})$ of a digraph $D = (V,A)$ with the edge set

$$\mathcal{E}^{CE} = \{e \subseteq V \mid |e| \geq 2 \land \exists v \in V : e = N_D^+(v)\},$$ as well as the hypergraph $\mathcal{H}'(D) = (V,E')$ of a digraph $D = (V,A)$ with the edge set

$$\mathcal{E}' = \{e \subseteq V \mid |e| \geq 2 \land \exists v_1, v_2 \in V : e = N_D^-(v_1) = N_D^+(v_2)\}.$$ The following theorem yields relations between these classes of hypergraphs.

**Theorem 2.** Let $D = (V,A)$ be digraph; then

$$EI(\mathcal{NH}(D)) \cup \mathcal{H}'(D) = D\mathcal{CH}(D) \cup EI(\mathcal{CH}(D)) \cup EI(CE\mathcal{H}(D)).$$

**Proof.** All hypergraphs have the vertex set $V$. For simplifying the logical expressions below, in the following let the symbol $e$ always denote a subset $e \subseteq V$ of cardinality at least 2. Then we obtain

$$e \in (\mathcal{E}^N)^{EI} \cup \mathcal{E}'$$

$$\iff [\exists e_1, e_2 \in \mathcal{E}^C \cup \mathcal{E}^{CE} : e_1 \neq e_2 \land e = e_1 \cap e_2] \lor e \in \mathcal{E}'$$

$$\iff [\exists e_1, e_2 \subseteq V : e_1 \neq e_2 \land e = e_1 \cap e_2 \land (\exists v_1, v_2 \in V : (e_1 = N_D^-(v_1) \land e_2 = N_D^-(v_2)) \lor (e_1 = N_D^+(v_1) \land e_2 = N_D^+(v_2)) \lor (e_1 = N_D^+(v_1) \land e_2 = N_D^+(v_2)) \lor (e_1 = N_D^-(v_1) \land e_2 = N_D^-(v_2)))] \lor$$

$$\exists v_1, v_2 \in V : e = N_D^-(v_1) = N_D^+(v_2)$$
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\[ \iff \exists v_1, v_2 \in V \exists e_1, e_2 \subseteq V : e_1 \neq e_2 \land \left[ (e = e_1 \cap e_2 \land \left[ (e_1 = N_D^+(v_1) \land e_2 = N_D^-(v_2) \lor (e_1 = N_D^+(v_1) \land e_2 = N_D^+(v_2)) \lor (e_1 = N_D^-(v_1) \land e_2 = N_D^+(v_2)) \right) \lor e = N_D^-(v_1) = N_D^+(v_2) \right] \right] \]

\[ \iff \exists v_1, v_2 \in V \exists e_1, e_2 \subseteq V : e = e_1 \cap e_2 \land \left[ (e_1 = N_D^-(v_1) \neq N_D^-(v_2) = e_2 \lor e_1 = N_D^+(v_1) \neq N_D^+(v_2) = e_2 \lor (e_1 = N_D^-(v_1) \land e_2 = N_D^+(v_2)) \right] \]

\[ \iff \exists v_1, v_2 \in V \exists e_1, e_2 \subseteq V : e = e_1 \cap e_2 \land \left[ (e_1, e_2 \in E^C \land e_1 \neq e_2) \lor (e_1, e_2 \in E^{CE} \land e_1 \neq e_2) \lor (e_1 \in E^C \land e_2 \in E^{CE}) \right] \]

\[ \iff \exists e_1, e_2 \in E^C : e_1 \neq e_2 \land e = e_1 \cap e_2 \lor (\exists e_1, e_2 \in E^{CE} : e_1 \neq e_2 \land e = e_1 \cap e_2) \lor (\exists v_1, v_2 \in V : e = N_D^-(v_1) \cap N_D^+(v_2)) \]

\[ \iff e \in (E^C)^{EI} \lor e \in (E^{CE})^{EI} \lor e \in E^{DC} \]

\[ \iff e \in (E^C)^{EI} \cup (E^{CE})^{EI} \cup E^{DC} \]

A hypergraph \( H = (V, E) \) has the Helly property if

\[ \forall \mathcal{E}' \subseteq \mathcal{E} : (\forall e_1, e_2 \in \mathcal{E}' : e_1 \cap e_2 \neq \emptyset) \rightarrow \bigcap_{e' \in \mathcal{E}'} e' \neq \emptyset \]

(see Berge [1]); next we show that the Helly property is hereditary for edge intersection hypergraphs.

**Theorem 3.** If \( H = (V, E) \) has the Helly property then \( EI(H) = (V, E^{EI}) \) has this property, too.

**Proof.** Let \( E^{EI}_S = \{ e_1, ..., e_t \} \subseteq E^{EI} \) with \( t \geq 1 \) and \( e_i \cap e_j \neq \emptyset \) for \( i, j \in \{1, ..., t\} \). Clearly

\[ \forall i \in \{1, ..., t\} \exists e_i', e_i'' \in E : e_i = e_i' \cap e_i'' \land e_i' \neq e_i''. \]

Let \( E_S := \{ e_1', ..., e_t', e_1'', ..., e_t'' \} \). By \( e_i \cap e_j \neq \emptyset \), for \( i, j \in \{1, ..., t\} \), we have \( \bar{e} \cap \bar{e} \neq \emptyset \) for arbitrary \( \bar{e}, \bar{e} \in E_S \) and the Helly property of \( H \) yields

\[ \emptyset \neq \bigcap_{\bar{e} \in E_S} \bar{e} = e_1' \cap ... \cap e_t' \cap e_1'' \cap ... \cap e_t'' = (e_1' \cap e_1'') \cap ... \cap (e_t' \cap e_t'') = e_1 \cap ... \cap e_t = \bigcap_{e \in E^{EI}_S} e, \]

i.e. \( EI(H) \) has the Helly property.

\[ \square \]

From the definition of edge intersection hypergraphs it follows immediately that for \( k \geq 2 \)

\[ \max\{ |e| \mid e \in E^{EI^k}(H) \} < \max\{ |e| \mid e \in E^{EI^{k-1}}(H) \}. \]

Hence the EI-number \( k^{EI}(H) \) is well defined. In the following we determine the edge intersection hypergraph and the EI-number \( k^{EI} \) for some special classes of hypergraphs. The strong \( d \)-uniform
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The result is trivial for $k=n$. Theorem 4. Let $\hat{C}_n^d$ and $\hat{P}_n^d$ be a strong $d$-uniform hypercycle and a strong $d$-uniform hyperpath, respectively.

(i) $EI^k(\hat{C}_n^d) = \hat{C}_n^{d-k} \cup \hat{C}_n^{d-k-1} \cup \ldots \cup \hat{C}_n^2$ for $d \geq 3$, $n \geq 2d-1$ and $k = 1, \ldots, d-2$.

(ii) $k^{EI}(\hat{C}_n^d) = d-1$ for $d \geq 2$ and $n \geq 2d-1$.

(iii) $k^{EI}(\hat{P}_n^d) = \begin{cases} d-1 & \text{for } d \geq 2 \text{ and } n \geq 2d-1, \\ n-d+1 & \text{for } d \geq 2 \text{ and } n < 2d-1. \end{cases}$

Proof. (i) In strong $d$-uniform hypercycles $\hat{C}_n^d$ with $n \geq 2d-1$ there are intersections of cardinalities at least two between the edges $e_i, e_{i+1}, \ldots, e_{i+d-2}$; $i = 1, \ldots, n$ (indices taken modulo $n$). Hence $EI(\hat{C}_n^d)$ contains the following edges (see Figure 1):

$$e_{j,i} := e_i \cap e_{i+j} = \{v_{i+j}, \ldots, v_{i+d-1}\},$$

with $i = 1, \ldots, n$ and $j = 1, \ldots, d-2$ (indices taken modulo $n$). This yields $EI(\hat{C}_n^d) = \hat{C}_n^{d-1} \cup \hat{C}_n^{d-2} \cup \ldots \cup \hat{C}_n^2$, i.e. by using the EI-operator the maximum edge cardinality decreases by one. For the $k$-th iteration we obtain

$$EI^k(\hat{C}_n^d) = \hat{C}_n^{d-k} \cup \ldots \cup \hat{C}_n^2, \quad k = 1, \ldots, d-2.$$

(ii) The case $d = 2$ is trivial; for $d \geq 3$ it follows with (i) that $EI^{d-2}(\hat{C}_n^d) = \hat{C}_n^2 = C_n$ and hence $k^{EI}(\hat{C}_n^d) = d-1$.

(iii) The result is trivial for $d = 2$ in both cases; in the following we assume $d \geq 3$.

For $n \geq 2d-1$ we have $|e_1 \cap e_{n-d+1}| \leq 1$, i.e. the intersection of the first edge and of the last edge of $\hat{P}_n^d$ do not generate an edge in $EI(\hat{P}_n^d)$. The edges of $EI(\hat{P}_n^d)$ are generated by the following intersections (see Figure 2):

$$e_{1,i} := e_i \cap e_{i+1} = \{v_i, v_{i+1}, \ldots, v_{i+d-1}\} \cap \{v_{i+1}, v_{i+2}, \ldots, v_{i+d}\} = \{v_{i+1}, \ldots, v_{i+d-1}\} \quad \text{for} \quad i = 1, \ldots, n-d,$$

$$e_{2,i} := e_i \cap e_{i+2} = \{v_i, v_{i+1}, \ldots, v_{i+d-1}\} \cap \{v_{i+2}, v_{i+3}, \ldots, v_{i+d+1}\} = \{v_{i+2}, \ldots, v_{i+d-1}\} \quad \text{for} \quad i = 1, \ldots, n-d-1,$$

$$\vdots$$

$$e_{d-2,i} := e_i \cap e_{i+d-2} = \{v_i, v_{i+1}, \ldots, v_{i+d-1}\} \cap \{v_{i+d-2}, v_{i+d-1}, \ldots, v_{i+d+1}\} = \{v_{i+d-2}, v_{i+d-1}\} \quad \text{for} \quad i = 1, \ldots, n-2d+3.$$
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Fig. 1. The strong 4-uniform hypercycle $\hat{C}_{10}^4$ and the corresponding edge intersection hypergraphs.

Hence $EI(\hat{C}_d^n)$ has edges of cardinalities $d - 1, d - 2, \ldots, 2$ and the edge set

$$E(EI(\hat{C}_d^n)) = \{e_1, \ldots, e_{n-d-1}, e_{d-2,1}, \ldots, e_{n-d-2, n-2d+3}\}$$

$$= E(\hat{P}^d_n) \setminus \{\{v_1, \ldots, v_{d-1}\}, \{v_{n-d+2}, \ldots, v_n\} \} \cup \ldots \cup E(\hat{P}^d_n) \setminus \{\{v_1, v_2\}, \ldots, \{v_{d-2}, v_{d-1}\}, \{v_{n-d+2}, v_{n-d+3}\}, \ldots, \{v_{n-1}, v_n\}\}.$$ 

Fig. 2. The strong 4-uniform hyperpath $\hat{P}_{10}^4$ and the corresponding edge intersection hypergraphs.

The reapplication of the EI-operator yields a hypergraph without the edges of maximum cardinality $(d - 1)$, while all other edges of $EI(\hat{P}_d^n)$ remain (because they are contained in the
edges of cardinality \((d - 1)\). After \((d - 2)\) iterations we obtain \(EI^{d-2}(\hat{P}_n^d) = P_{n-2d+4} \cup I_{2d-4}\), where \(I_t\) denotes a set of \(t\) isolated vertices; hence \(k^{n,t}(\hat{P}_n^d) = d - 1\).

For \(n < 2d - 1\) we have \(|e_1 \cap e_{n-d+1}| \geq 2\), i.e. the intersection of the first and the last edge of \(\hat{P}_n^d\) generates in \(EI(\hat{P}_n^d)\) the edge of minimum cardinality \((2d - n)\). All edges of \(EI(\hat{P}_n^d)\) are generated by the following intersections (see Figure 3):

\[
e_{1,i} := e_i \cap e_{i+1} = \{v_1, v_2, \ldots, v_{i+1}, v_{i+2}, \ldots, v_d\}
\]

\[
e_{2,i} := e_i \cap e_{i+2} = \{v_1, v_2, \ldots, v_{i+1}, v_{i+2}, \ldots, v_{i+d+1}\}
\]

\[
\vdots
\]

\[
e_{n-d+1,i} := e_i \cap e_{i+n-d-1} = \{v_1, v_2, \ldots, v_{i+n-d-1}, \ldots, v_d\}
\]

\[
e_{n-d,1} := e_1 \cap e_{n-d+1} = \{v_1, v_2, \ldots, v_d\} \cap \{v_{n-d+1}, \ldots, v_n\}
\]

Hence \(EI(\hat{P}_n^d)\) has edges of cardinalities \(d - 1, d - 2, \ldots, 2d - n\) and the edge set

\[
E(EI(\hat{P}_n^d)) = \{e_{1,1}, \ldots, e_{1,n-d}, e_{2,1}, \ldots, e_{2,n-d-1}, e_{n-d-1,1}, e_{n-d-1,2}, e_{n-d,1}\}
\]

\[
= E(\hat{P}_n^{d-1}) \setminus \{\{v_1, \ldots, v_{d-1}\}, \{v_{n-d+2}, \ldots, v_n\}\} \cup
E(\hat{P}_n^{d-2}) \setminus \{\{v_1, \ldots, v_{d-2}\}, \{v_{2, \ldots, v_{d-1}}\}, \{v_{n-d+2}, \ldots, v_n\}\}
\]

\[
\cup \ldots \cup
E(\hat{P}_n^{d-n}) \setminus \{\{v_1, \ldots, v_{2d-n}\}, \ldots, \{v_{n-d}, \ldots, v_{d-1}\}\},
\{v_{n-d+2}, \ldots, v_{d+1}\}, \ldots, \{v_{2(n-d)+1}, \ldots, v_n\}\}.
\]

![Fig. 3. The strong 5-uniform hyperpath \(\hat{P}_5^5\) and the corresponding edge intersection hypergraphs.](image)

Again, the reapplication of the EI-operator yields a hypergraph without the edges of maximum cardinality \((d - 1)\), while all the other edges remain. After \((n - d)\) iterations we obtain
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From (i) we know that $EI^{n-d}(\hat{P}^d_n) = \hat{e} \cup I_{2(n-d)}$, where $\hat{e} = \{v_{n-d+1}, \ldots, v_d\}$ with cardinality $|\hat{e}| = 2d - n$, hence $k^{EI}(\hat{P}^d_n) = n - d + 1$. 

Berge [1] generalized the complete graph $K_n$ by the definition of the complete $d$-uniform hypergraph $K^d_n$ as follows:

$$V(K^d_n) = \{v_1, \ldots, v_n\}, \quad E(K^d_n) = \{T \subseteq V(K^d_n) \mid |T| = d\}.$$ 

**Theorem 5.** Let $K^d_n$ be a complete $d$-uniform hypergraph with $n - 1 \geq d \geq 3$.

(i) $EI^{k}(K^d_n) = K^{d-k}_n \cup K^{d-k-1}_n \cup \ldots \cup K^0_n$ for $1 \leq k \leq d - 2$, where $t_k := \max\{2, 2^k(d - n) + n\}$ for $1 \leq k \leq d - 2$.

(ii) $k^{EI}(K^d_n) = d - 1$ for $d \geq 2$.

**Proof.** (i) The intersections with cardinality of a least two of edges in $K^d_n$ are all subsets $T \subseteq V(K^d_n)$ with cardinalities in the range $d - 1 \geq |T| \geq t_1 = \max\{2, 2^k(d - n)\}$; hence $EI(K^d_n) = K^{d-1}_n \cup K^{d-2}_n \cup \ldots \cup K^0_n$.

Using induction, the reapplication of the EI-operator to $EI^k(K^d_n)$ yields all subsets $T \subseteq V(K^d_n)$ of cardinalities in the range $d - (k + 1) \geq |T| \geq \max\{2, 2^{k+1}(d - n) - n\} = \max\{2, 2^{k+1}(d - n) + n\}$.

(ii) From (i) we know that $EI^{d-2}(K^d_n) = K^2_n = K_n$, hence $k^{EI}(K^d_n) = d - 1$.

3. Generating $C_n$ as an edge intersection hypergraph

3.1. A lower bound for the number of hyperedges

At first we will give some notations. For this end, let $n \geq 24$ be even and $H = (V, E)$ a 6-uniform hypergraph with $EI(H) = (V, E^{EI}) = C_n$. In detail, let $C_n = (V, E), V = \{1, 2, \ldots, n\}$ and $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}\}$. In general, the vertices in $V$ will be always taken modulo $n$.

For $i \in \{1, 2, \ldots, n\}$ and $e \in E$, a sequence $(i, i+1, \ldots, i+k-1)$ with $\{i, i+1, \ldots, i+k-1\} \subseteq e$, such that $i - 1 \notin e$ and $i + k \notin e$, is referred to as a $k$-section of $e$ on $C_n$.

Any subset $\{v_1, v_2, \ldots, v_k\} \subseteq V$ of $k \geq 2$ vertices containing two vertices $v, v'$ with $|v - v'| \geq 2$ is called a chord of $C_n$. Since $E(C_n) = E^{EI}$ cannot contain any chord, for any two distinct hyperedges $e, e' \in E$ it holds $|e \cap e'| \leq 2$. For the same reason, in case of $|e \cap e'| = 2$ there exists a vertex $i \in V$ with $|e \cap e'| = \{i, i + 1\}$.

In our first theorem, we will prove that $\frac{n}{2}$ is a lower bound for the cardinality of the edge set $E$ of a 6-uniform hypergraph $H = (V, E)$ with $EI(H) = C_n$, for even $n \geq 24$.

The motivation for $24$ as a lower bound for $n$ results from the fact that our main theorem in Subsection 3.2 provides the construction of such hypergraphs $H$ for all even $n = |V| \geq 24$.

**Theorem 6.** Let $n \geq 24$ be even and $H = (V, E)$ a 6-uniform hypergraph with $EI(H) = C_n$. Then $|E| \geq \frac{n}{2}$.

**Proof.** Let $H$ fulfill the assumptions of the Theorem and $e, e' \in E(H)$ with $e \cap e' = \{i, i + 1\} \in E(C_n)$, where $i \in V$. We say that the hyperedge $e$ half-generates the edge $\{i, i + 1\}$ of $C_n$. The term "half-generate" comes from the fact that we always need at least two hyperedges $e \neq e'$ to generate an edge of $C_n$.

First, we discuss the number $k_e := |\{i, i + 1\} | i \in V \setminus \{i, i + 1\} \subseteq e|$ of the edges of $C_n$ being half-generated by the hyperedge $e$. The following values for $k_e$ may occur:
1. \( k_e = 5 \) – then \( e \) has to consist of a 6-section.

2. \( k_e = 4 \) – then we have the following three possibilities:
   (a) \( e \) contains a 5-section and a 1-section;
   (b) \( e \) consists of a 4-section and a 2-section;
   (c) \( e \) has two 3-sections.

3. \( k_e = 3 \) – again three variants are possible:
   (a) \( e \) includes a 4-section and two 1-sections;
   (b) \( e \) contains a 3-section, a 2-section and a 1-section;
   (c) \( e \) is composed of three 2-sections.

4. \( k_e = 2 \) – the hyperedge \( e \) consists of
   (a) a 3-section and three 1-sections or
   (b) two 2-sections and two 1-sections.

5. \( k_e = 1 \) – now \( e \) is the union of a 2-section and four 1-sections.

6. \( k_e = 0 \) implies that in \( e \) we have six 1-sections, i.e., six vertices being non-adjacent in \( C_n \).

Assume \( |\mathcal{E}| < \frac{n}{2} \) and \( k_e \leq 4 \), for all hyperedges \( e \in \mathcal{E} \). This leads to the contradiction

\[
|E(C_n)| = n \leq \frac{1}{2} \sum_{e \in \mathcal{E}} k_e \leq \frac{1}{2} \cdot |\mathcal{E}| \cdot 4 < n.
\]

Therefore, in case of \( |\mathcal{E}| < \frac{n}{2} \) there has to exist at least one hyperedge \( e' \in \mathcal{E} \) with \( k_{e'} = 5 \).

Let \( e' = \{i, i + 1, \ldots, i + 5\} \), where \( i \in V \).

Moreover, for simplicity we label \( V \) so that \( e' = \{1, 2, \ldots, 6\} \) holds.

The ”inner \( C_n \)-edges” of \( e' \), i.e. \( \{2, 3\} \), \( \{3, 4\} \) and \( \{4, 5\} \) arise as intersections \( e_1 \cap e', e_2 \cap e' \) and \( e_3 \cap e' \) of \( e' \) with pairwise distinct hyperedges \( e_1, e_2, e_3 \in \mathcal{E} \setminus \{e'\} \). For this reason, each of the three hyperedges \( e_1, e_2 \) and \( e_3 \) has to possess at least one 2-section.

Hence, for \( i \in \{1, 2, 3\} \) we have \( k_{e_i} \leq 4 \) and we obtain \( k_{e'} + \sum_{i=1}^{3} k_{e_i} \leq 5 + 12 = 17 \). In order to get \( k_{e'} + \sum_{i=1}^{3} k_{e_i} = 17 > 16 = 4 \cdot |\{e', e_1, e_2, e_3\}| \), necessarily each of the hyperedges \( e_1, e_2 \) and \( e_3 \) has to include a 4-section.

Considering the three ”middle” pairs of the vertices in the 4-sections \( (i, i + 1, i + 2, i + 3), (j, j + 1, j + 2, j + 3) \) and \( (k, k + 1, k + 2, k + 3) \) of \( e_1, e_2 \) and \( e_3 \), respectively, an analog argumentation is true: again we need three hyperedges \( e_1', e_2' \) and \( e_3' \), each of them consisting of a 2-section \((i + 1, i + 2), (j + 1, j + 2) \) and \((k + 1, k + 2) \), respectively) and 4-section, to half-generate now \( 29 > 28 = 4 \cdot |\{e', e_1, e_2, e_3, e_1', e_2', e_3'\}| \) edges of \( C_n \). Because \( \mathcal{H} \) is finite, this leads inductively to the contradiction that there is a hyperedge \( e^* \in \mathcal{E} \) containing at least one 2-section but no 4-section. Then \( e^* \) half-generates at most 3 edges of \( C_n \). Let \( e', e_1, e_2, e_3, e_1', e_2', e_3', \ldots, e^* \) be the set of all hyperedges used up to this point and \( t \) be the number of these hyperedges. We easily see

\[
k_{e'} + k_{e_1} + k_{e_2} + k_{e_3} + k_{e_1'} + k_{e_2'} + k_{e_3'} + \ldots + k_{e^*} = 5 + k_{e_1} + k_{e_2} + k_{e_3} + k_{e_1'} + k_{e_2'} + k_{e_3'} + \ldots + 3 \leq 4t.
\]

This argument is valid for each hyperedge \( e' \in \mathcal{E} \) with \( k_{e'} = 5 \). Moreover, for all other hyperedges \( e \in \mathcal{E} \) we know \( k_e \leq 4 \) and this leads to \( \sum_{e \in \mathcal{E}} k_e \leq 4 \cdot |\mathcal{E}| \). This yields the same contradiction as above, namely

\[
n = |E(C_n)| \leq \frac{1}{2} \sum_{e \in \mathcal{E}} k_e \leq 2 \cdot \frac{1}{2} \cdot |\mathcal{E}| < 2 \cdot \frac{n}{2}
\]

and the proof is complete.
3.2. The construction of hypergraphs $\mathcal{H}$ with $EI(\mathcal{H}) = C_n$

The main result of Section 3 is the following.

**Theorem 7.** Let $n \geq 24$. Then there exists a hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $EI(\mathcal{H}) = C_n$ such that the following holds.

(i) If $n$ is even, then $\mathcal{H}$ is 3-regular, 6-uniform and $|\mathcal{E}| = \frac{n}{2}$.

(ii) If $n$ is odd, then $\mathcal{H}$ is 3-regular, $|\mathcal{E}| = \frac{n+1}{2}$, $\mathcal{H}$ contains one hyperedge of cardinality 3 and all other hyperedges in $\mathcal{H}$ have cardinality 6.

Note that the lower bound $\frac{n}{2}$ given in Theorem 6 for the cardinality $|\mathcal{E}|$ is sharp due to Theorem 7(i).

The proof of Theorem 7 will be done in three steps by constructing hypergraphs with the required properties. Depending on $n$, we have to distinguish the following cases, where each case will be considered in a separate lemma.

Case 1. $n = 8k + l$, where $k \in \mathbb{N}$, $k \geq 3$, $k \neq 4$ and $l \in \{0, 2, 4, 6\}$.

Case 2. $n = 32 + l$, where $l \in \{0, 2, 4, 6\}$.

Case 3. $n$ odd.

For the first two cases we give only the construction of the hyperedges of $\mathcal{H}$ (see Lemma 1 and Lemma 2). The lengthy verification of $EI(\mathcal{H}) = C_n$ (as well as an argumentation for the 3-regularity of $\mathcal{H}$) can be found in the Appendix. In each case, this verification consists of two distinct parts:

(a) the hyperedges of $\mathcal{H}$ generate all edges of $C_n$ and

(b) the hyperedges of $\mathcal{H}$ do not generate any chord in $C_n$.

The case $n = 8k + l$ with $k = 4$ requires a special modification of the construction of the first case. For odd $n$, the combination of the constructions in Cases 1 and 2 with a relatively simple additional hyperedge consisting of only three vertices proves Theorem 7(ii) (see Lemma 3).

**Lemma 1.** Let $k, l, n \in \mathbb{N}$ with $k \geq 3$, $k \neq 4$, $l \in \{0, 2, 4, 6\}$ and $n = 8k + l$.

Then there exists a 3-regular, 6-uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $EI(\mathcal{H}) = C_n$ and $|\mathcal{E}| = \frac{n}{2}$.

**Proof.** At first we give a rough description of the construction principle for the hyperedges of $\mathcal{H}$.

In the basic construction (this corresponds to $l = 0$) for $j \in \{1, 5, 9, \ldots, \frac{n}{2} - 3\}$ and $\tilde{j} = \frac{j - 1}{2} \in \{0, 1, \ldots, \frac{n}{8} - 1\}$ we form so-called 4-groups $G_j = \{e_j, e_{j+1}, e_{j+2}, e_{j+3}\}$ of hyperedges. Each of the constructed hyperedges consists of two 3-sections on $C_n$, in detail the hyperedges will have the structure $e = \{p', p' + 1, p' + 2, q', q' + 1, q' + 2\}$ with $p', q' \in V$ and $|q' - p'| \geq 6$. Note that we take the vertices of $V = \{1, 2, \ldots, n\}$ modulo $n$, therefore $|q' - p'| \geq 6$ is meant in the sense that the distance between $p'$ and $q'$ on the cycle $C_n$ is at least 6.

For every $j \in \{1, 5, 9, \ldots, \frac{n}{2} - 3\}$ the so-called first 3-sections of the hyperedges $e_j, e_{j+1}, e_{j+2}, e_{j+3}$ in such a 4-group overlap each other in the following way:

- $e_j = \{p, p + 1, p + 2, \ldots\}$,
- $e_{j+1} = \{p + 1, p + 2, p + 3, \ldots\}$,
- $e_{j+2} = \{p + 2, p + 3, p + 4, \ldots\}$,
- $e_{j+3} = \{p + 3, p + 4, p + 5, \ldots\}$, for certain $p \in V$.

This property of overlapping (by two vertices, if we consider $e_k$ and $e_{k+1}$ $(k \in \{j, j+1, j+2\})$) determines the first 3-sections of the hyperedges uniquely, since the other 3-sections do not overlap. These other 3-sections are referred to as the second 3-sections of the hyperedges.
Considering those 3-sections, we will see that no hyperedge \( e \in G_j \) has a non-empty intersection with a second 3-section of any of the other hyperedges \( e' \in G_j \setminus \{e\} \).

If \( l \in \{2, 4, 6\} \), then \( \frac{l}{2} \) of the 4-groups \( G_j^l \) will be replaced by 5-groups \( G_j^l = \{e_j, e_{j+1}, e_{j+2}, e_{j+3}, e_{j+4}\} \) of hyperedges. In comparison with a 4-group \( G_j^l = \{e_j, e_{j+1}, e_{j+2}, e_{j+3}\} \) in the basic construction (see above) we add yet another hyperedge \( e_{j+4} = \{p''+1, p''+2, q'', q''+1, q''+2\} \) in order to obtain the needed 5-group. Looking at the detailed definitions of the hyperedges, later we will see that most of the properties described above for the hyperedges in the 4-groups will be preserved for the hyperedges in the new 5-groups, only little modifications will appear.

So \( e_{j+4} \) will continue the overlapping of the first 3-sections; using the number \( |q'' - p''| \) will be preserved for the hyperedges in the new 5-groups, only little modifications will appear.

Now we give the detailed definitions of the hyperedges. We begin with the \( \frac{l}{2} \) 5-groups of hyperedges, where \( l \in \{0, 2, 4, 6\} \).

(A): \( \tilde{j} \in \{0, 1, \ldots, \frac{l}{2} - 2\} \) or \( n = 30 \land \tilde{j} = \frac{l}{2} - 1 \).

Let \( j = 5\tilde{j} + 1 \), i.e. \( j \in \{1, 6, 11\} \) (note that \( j = 11 \) is possible only for \( n = 30 \)).
\[
e_j = \{10\tilde{j} + 1, 10\tilde{j} + 2, 10\tilde{j} + 3, 10\tilde{j} + 8, 10\tilde{j} + 9, 10\tilde{j} + 10\},
\]
\[
e_{j+1} = \{10\tilde{j} + 2, 10\tilde{j} + 3, 10\tilde{j} + 4, 10\tilde{j} + 19, 10\tilde{j} + 20, 10\tilde{j} + 21\},
\]
\[
e_{j+2} = \{10\tilde{j} + 3, 10\tilde{j} + 4, 10\tilde{j} + 5, 10\tilde{j} + 16, 10\tilde{j} + 17, 10\tilde{j} + 18\},
\]
\[
e_{j+3} = \{10\tilde{j} + 4, 10\tilde{j} + 5, 10\tilde{j} + 6, 10\tilde{j} + 3, 10\tilde{j} + 2, 10\tilde{j} + 1\},
\]
\[
e_{j+4} = \{10\tilde{j} + 5, 10\tilde{j} + 6, 10\tilde{j} + 7, 10\tilde{j} + 10, 10\tilde{j} + 1 + 10\tilde{j} + 2\}.
\]

Whereas in (A) we have 0, 1 or 2 such 5-groups \( G_j^l \) (depending on \( l \in \{0, 2, 4\} \)) – with the exception \( n = 30 \), where three 5-groups can occur – the following case (B) describes only one 5-group, namely the largest 5-group \( G_{\frac{l}{2} - 1} \), which is the 5-group with the largest index \( \tilde{j} = \frac{l}{2} - 1 \).

(B): \( n \neq 30 \land l \geq 2 \land \tilde{j} = \frac{l}{2} - 1 \).

\( j = 5\tilde{j} + 1, e_j, e_{j+3} \) and \( e_{j+4} \) are the same as in (A). Since the next group \( G_{\frac{l}{2} - 1} \) after \( G_{\frac{l}{2} - 1} \) is a 4-group, we have to modify \( e_{j+1} \) and \( e_{j+2} \) as follows.
\[
e_{j+1} = \{10\tilde{j} + 2, 10\tilde{j} + 3, 10\tilde{j} + 4, 10\tilde{j} + 17, 10\tilde{j} + 18, 10\tilde{j} + 19\},
\]
\[
e_{j+2} = \{10\tilde{j} + 3, 10\tilde{j} + 4, 10\tilde{j} + 5, 10\tilde{j} + 14, 10\tilde{j} + 15, 10\tilde{j} + 16\}.
\]

Because of \( |E| = \frac{n}{l} \) and since we have defined \( \frac{l}{2} \) of such 5-groups, namely
\[
G_0 = \{e_1, e_2, e_3, e_4, e_5\},
\]
\[
G_1 = \{e_6, e_7, e_8, e_9, e_{10}\},\ldots ,
\]
\[
G_{\frac{l}{2} - 1} = \{e_{\frac{l}{2} - 4}, e_{\frac{l}{2} - 3}, e_{\frac{l}{2} - 2}, e_{\frac{l}{2} - 1}, e_{\frac{l}{2}}\},
\]
we need \( \frac{n}{l} (\frac{l}{2} - \frac{1}{2}) = \frac{1}{l} (n - 5l) \) 4-groups, in detail
\[
G_{\frac{l}{2}} = \{e_{\frac{l}{2} + 1}, e_{\frac{l}{2} + 2}, e_{\frac{l}{2} + 3}, e_{\frac{l}{2} + 4}\},
\]
\[
G_{\frac{l}{2} + 1} = \{e_{\frac{l}{2} + 5}, e_{\frac{l}{2} + 6}, e_{\frac{l}{2} + 7}, e_{\frac{l}{2} + 8}\},\ldots ,
\]
\[
G_{\frac{l}{2} - 1} = \{e_{\frac{l}{2} - 3}, e_{\frac{l}{2} - 2}, e_{\frac{l}{2} - 1}, e_{\frac{l}{2}}\}.\]
Here are the 4-groups.

(C): \( \tilde{j} \in \{ \frac{1}{2}, \frac{1}{2} + 1, \ldots, \frac{n-1}{8} - 2 \} \) or \( l = 0 \land \tilde{j} = \frac{n-1}{8} - 1 \).

Let \( j = 4 \tilde{j} + \frac{1}{2} + 1 \), i.e. \( j \in \{ \frac{1}{2}l + 1, \frac{5}{2}l + 5, \ldots, \frac{n}{2} - 7, \frac{n}{2} - 3 \} \) (note that \( j = \frac{n}{2} - 3 \) is possible only for \( l = 0 \)).

With

\[
x = \begin{cases} 8\tilde{j} & , \ l = 0 \\ 8\tilde{j} + l - 1 & , \ l > 0 \end{cases}
\]

we define the hyperedges of \( G_{\tilde{j}} \).

\( e_j = \{ x + 1, x + 2, x + 3, x + 7, x + 8, x + 9 \} \),

\( e_{j+1} = \{ x + 2, x + 3, x + 4, x + 16, x + 17, x + 18 \} \),

\( e_{j+2} = \{ x + 3, x + 4, x + 5, x + 13, x + 14, x + 15 \} \),

\( e_{j+3} = \{ x + 4, x + 5, x + 6, x - 2, x - 1, x \} \).

(D): \( l \geq 2 \land \tilde{j} = \frac{n-1}{8} - 1 \).

In comparison with (C) we have to modify only the second and the third hyperedge; we set

\( e_{j+1} = \{ x + 2, x + 3, x + 4, x + 18, x + 19, x + 20 \} \) and

\( e_{j+2} = \{ x + 3, x + 4, x + 5, x + 15, x + 16, x + 17 \} \).

Owing to \( j = \frac{n}{2} - 3 \) and \( x = 8\tilde{j} + l - 1 = n - 9 \) we obtain finally

\( e_{n-3} = \{ n - 8, n - 7, n - 6, n - 2, n - 1, n \} \),

\( e_{n-2} = \{ n - 7, n - 6, n - 5, 9, 10, 11 \} \),

\( e_{n-1} = \{ n - 6, n - 5, n - 4, 6, 7, 8 \} \),

\( e_n = \{ n - 5, n - 4, n - 3, n - 11, n - 10, n - 9 \} \).

Now we come to Case 2, where \( k = 4 \) and therefore \( n = 8k + l = 32 + l \) is valid. At first we want to explain why the construction in the proof of Lemma 1 cannot be applied in the present case.

To see the problem, it suffices to consider \( n = 32 \) and have a look at the construction in the proof of Lemma 1 (cf. the above case (C) for \( l = 0 \)). Then in one case the construction of \( e_{j+1} \) and \( e_{j'+1} \) for certain \( j \neq j' \) causes the conflict that \( e_{j+1} \cap e_{j'+1} \) will be a chord in \( C_n \). The reason is that if \( \tilde{j} = \tilde{j'} + 2 \) holds, then because of

\[
\{8\tilde{j} + 2, 8\tilde{j} + 3, 8\tilde{j} + 4\} \cap \{8\tilde{j'} + 16, 8\tilde{j'} + 17, 8\tilde{j'} + 18\} = \{8\tilde{j} + 2\}
\]

and

\[
\{8\tilde{j'} + 2, 8\tilde{j'} + 3, 8\tilde{j'} + 4\} \cap \{8\tilde{j} + 16, 8\tilde{j} + 17, 8\tilde{j} + 18\} = \{8\tilde{j'} + 2\}
\]

we would obtain the chord \( \{8\tilde{j} + 2, 8\tilde{j'} + 2\} = \{8\tilde{j'} + 18, 8\tilde{j'} + 2\} \).

Hence we have to modify slightly the construction of the hyperedges.

**Lemma 2.** Let \( l, n \in \mathbb{N} \) with \( l \in \{0, 2, 4, 6\} \) and \( n = 32 + l \).

Then there exists a 3-regular, 6-uniform hypergraph \( \mathcal{H} = (V,E) \) with \( EI(\mathcal{H}) = C_n \) and \( |E| = \frac{n}{2} \).

**Proof.** Take the construction of the hyperedges described in the proof of Lemma 1. Let \( G_{\tilde{j}} = \{e_j, e_{j+1}, e_{j+2}, \ldots\} \) be an arbitrarily chosen 4-group or 5-group of hyperedges. Then we swap the second 3-sections of the second hyperedge \( e_{j+1} \) and the third hyperedge \( e_{j+2} \). In detail, for each case we give the modified hyperedges. We use the same distinction of cases as above, i.e. we consider at first the 5-groups \( G_{\tilde{j}} \).
(A): $\tilde{j} \in \{0, 1, \ldots, \frac{l}{2} - 2\}$.

$e_{j+1} = \{10\tilde{j} + 2, 10\tilde{j} + 3, 10\tilde{j} + 4, 10\tilde{j} + 16, 10\tilde{j} + 17, 10\tilde{j} + 18\}$,
$e_{j+2} = \{10\tilde{j} + 3, 10\tilde{j} + 4, 10\tilde{j} + 5, 10\tilde{j} + 19, 10\tilde{j} + 20, 10\tilde{j} + 21\}$.

(B): $l \geq 2 \land \tilde{j} = \frac{l}{2} - 1$.

$e_{j+1} = \{10\tilde{j} + 2, 10\tilde{j} + 3, 10\tilde{j} + 4, 10\tilde{j} + 14, 10\tilde{j} + 15, 10\tilde{j} + 16\}$,
$e_{j+2} = \{10\tilde{j} + 3, 10\tilde{j} + 4, 10\tilde{j} + 5, 10\tilde{j} + 17, 10\tilde{j} + 18, 10\tilde{j} + 19\}$.

Now we come to the 4-groups.

(C): $\tilde{j} \in \{\frac{l}{2}, \frac{l}{2} + 1, \ldots, 2\}$ or $l = 0 \land \tilde{j} = 3$.

$e_{j+1} = \{x + 2, x + 3, x + 4, x + 13, x + 14, x + 15\}$,
$e_{j+2} = \{x + 3, x + 4, x + 5, x + 16, x + 17, x + 18\}$.

(D): $l \geq 2 \land \tilde{j} = 3$.

$e_{\frac{l}{2} - 2} = \{n - 7, n - 6, n - 5, 6, 7, 8\}$,
$e_{\frac{l}{2} - 1} = \{n - 6, n - 5, n - 4, 9, 10, 11\}$.

The following remark justifies the need for the different constructions for $k = 3$ and $k = 4$.

**Remark 1.** The construction from the proof of Lemma 2 does not work for all even $n \geq 24$.

**Proof.** To give a simple counterexample, take $n = 24$ and assume that the construction of the proof of Lemma 2 does work. It follows $l = 0$, therefore we only have 4-groups of hyperedges. We look at $e_2$ and $e_7$, which are from $G_0$ and $G_1$. Then we have $\tilde{j} = 0, \tilde{j'} = 1, j = 1, j' = 5, e_2 = e_{j+1}$ and $e_7 = e_{j'+2}$. Hence, we would have to use the construction from (C) of the proof and with $x = 8\tilde{j}$ we obtain for the hyperedges $e_2 = \{2, 3, 4, 13, 14, 15\}$ and $e_7 = \{11, 12, 13, 24, 25, 26\} = \{11, 12, 13, 24, 1, 2\}$ (modulo 24). This would result in the chord $e_2 \cap e_7 = \{2, 13\}$, which is a contradiction.

**Remark 2.** For $n = 8k$, the constructions from the proofs of Lemma 1 and Lemma 2 do not work if $k \leq 2$.

**Proof.** If $k \in \{0, 1\}$, the remark is trivially true; so let $k = 2$. Obviously, we have $l = 0$. Since 16 is an integral multiple of 8, we have only two 4-groups of hyperedges in $\mathcal{H}$. It suffices to consider the hyperedges constructed in the part (C).

First, we use the proof of Lemma 1 and consider the hyperedge $e_2$. Then $j = 1, \tilde{j} = 0$ and $e_2 = e_{j+1} = \{2, 3, 4, 16, 17, 18\} = \{2, 3, 4, 16, 1, 2\}$ (modulo 16). This leads to $e_1 \cap e_2 = \{1, 2, 3\}$ which results in a chord contradicting $EI(\mathcal{H}) = C_n$.

Secondly, concerning the construction in the proof of Lemma 2, we look at the hyperedge $e_3$. Again, we get $j = 1$ and $\tilde{j} = 0$. Thus $e_3 = e_{j+2} = \{3, 4, 5, 16, 17, 18\} = \{3, 4, 5, 16, 1, 2\}$ (modulo 16). We obtain the same chord as in the previous case, namely $e_1 \cap e_3 = \{1, 2, 3\}$, a contradiction.
Finally, to complete the proof of Theorem 7, we investigate Case 3, i.e. the odd cardinalities $n = |V|$.

**Lemma 3.** Let $n \in \mathbb{N}$ be odd.

Then there exists a 3-regular hypergraph $H$ with $EI(H) = C_n$, $|E| = \frac{n+1}{2}$, $H$ contains one hyperedge of cardinality 3 and all other hyperedges in $H$ have cardinality 6.

**Proof.** Let $n = 8k + l + 1$, where $k \geq 3$, $l \in \{0, 2, 4, 6\}$; $H' = (V', E')$ with $EI(H') = C_{n-1}$, $V' = \{1, 2, \ldots, n - 1\}$ and $E' = \{e'_1, e'_2, \ldots, e'_{n-1}\}$. Since $n - 1$ is even, we can assume that $H'$ (including the numbering of the vertices and the hyperedges) is constructed according to the proofs of Lemma 1 and Lemma 2 (of course with the (only) modification that now we have $n - 1$ vertices instead of $n$ in the lemmas).

Owing to the construction, the vertex 3 and the vertex 4 is contained exclusively in the hyperedges $e_1, e_2, e_3$ and $e_2, e_3, e_4$, respectively; namely in the first 3-sections of these hyperedges. We obtain $H = (V, E)$ with $EI(H) = C_n$ and $V = \{1, 2, \ldots, n\}$ from $H' = (V', E')$ as follows.

Looking at the cycle $C_{n-1}$, we add a new vertex $n$ ”between” the vertices 3 and 4 in this cycle, i.e. in $C_{n-1} = EI(H')$, and get $C_n = EI(H)$ by the following construction of $H = (V, E)$.

$V := V' \cup \{n\}$,

$E := \{e_1, e_2, \ldots, e_{\frac{n+1}{2}}\}$, where

$$e_i = \begin{cases} e'_i, & i = 1, 4, 5, 6, \ldots, \frac{n-1}{2} \\ (e'_2 \setminus \{4\}) \cup \{n\}, & i = 2 \\ (e'_3 \setminus \{3\}) \cup \{n\}, & i = 3 \\ \{3, n, 4\}, & i = \frac{n+1}{2}. \end{cases}$$

Instead of the edge $e_2 \cap e_3 = \{3, 4\} \in E(EI(H'))$, in $EI(H)$ we have the edges $e_2 \cap e_{\frac{n+1}{2}} = \{3, n\}$ and $e_3 \cap e_{\frac{n+1}{2}} = \{n, 4\}$.

In addition to these non-empty intersections of $e_{\frac{n+1}{2}}$ with $e_2$ and $e_3$, in $H$ we have only two further non-empty intersections with other hyperedges, namely $e_1 \cap e_{\frac{n+1}{2}} = \{3\}$ and $e_4 \cap e_{\frac{n+1}{2}} = \{4\}$. Thus, the hyperedge $e_{\frac{n+1}{2}}$ does not induce any chord in $C_n$. The same holds for the (modified) hyperedges $e_2$ and $e_3$. Hence $EI(H) = C_n$, where $C_n$ is the vertex sequence $(\ldots, n - 2, n - 1, 1, 2, 3, n, 4, 5, 6, \ldots)$.

Restricting Problem 1 to uniform hypergraphs $H$ with $EI(H) = C_n$, we establish the following weakened version of the problem.

**Problem 2.** Let $k \geq 1$, $n_0 \in \mathbb{N}^+$ and $n \geq n_0$. What is the minimum cardinality $|E|$ of the edge set of a $3k$-uniform hypergraph $H = (V, E)$ with $EI(H) = C_n$?

Obviously, for $n \geq 4$, the hypergraphs $H_n^{\star} = (V, E)$ with $V = \{1, 2, \ldots, n\}$ and $E = \{\{i, i + 1, i + 2\} \mid i \in V\}$ (the vertices taken modulo $n$) are edge-minimal 3-uniform hypergraphs with $EI(H_n^{\star}) = C_n$; this answers the question for $k = 1$ and $n_0 = 4$.

In Theorem 7, we solved Problem 2 for $k = 2$ and even $n \geq n_0 = 24$.

In order to motivate the concentration on $3k$-uniform hypergraphs, we consider the proof of Theorem 6. For $k = 2$, we verified that to construct edge-minimal 6-uniform hypergraphs $H$ with $EI(H) = C_n$, the hyperedges of $H$ can be composed from certain 3-sections of $C_n$. The combination of 3-sections results in hyperedges of cardinalities being integral multiples of 3. Following this likely approach also for hyperedges of larger cardinality $r$, this leads to $r = 3k$ ($k \geq 3$).

We conjecture that the solution of the problem is difficult for $k \geq 3$. Some results on edge intersection hypergraphs 14
References


Appendix

For simplicity, we refer to the main part of the preprint as [1].

1. Proof of Lemma 1

This is the first lemma to prove.

**Lemma 1.** Let $k, l, n \in \mathbb{N}$ with $k \geq 3, k \neq 4, l \in \{0, 2, 4, 6\}$ and $n = 8k + l$.
Then there exists a 3-regular, 6-uniform hypergraph $H = (V, E)$ with $EI(H) = C_n$ and $|E| = \binom{n}{2}$.

**Proof.** Deviating from [1], we handle the cases $l = 0$ and $l \in \{2, 4, 6\}$ separately. The reason is that in [1] we only gave the definitions of the hyperedges in a formal and compact form. In the present script we have to verify that $EI(H)$ contains all edges of $C_n$ but no chords. This is much easier by treating not so complex but rather simpler cases what we will do here.

**Part 1 (basic construction): $l = 0$.**

Let $j \in \{1, 5, 9, \ldots, \frac{n}{2} - 3\}$ and $\tilde{j} = \frac{j - 1}{4} \in \{0, 1, \ldots, \frac{n}{8} - 1\}$.

Considering the 4-groups

- $G_0 = \{e_1, e_2, e_3, e_4\}$,
- $G_1 = \{e_5, e_6, e_7, e_8\}, \ldots$,
- $G_{\frac{n}{8} - 1} = \{e_{\frac{n}{2} - 3}, e_{\frac{n}{2} - 2}, e_{\frac{n}{2} - 1}, e_{\frac{n}{2}}\}$,

the 4-group $G_\tilde{j}$ contains the hyperedges

- $e_j = \{8\tilde{j} + 1, 8\tilde{j} + 2, 8\tilde{j} + 3, 8\tilde{j} + 7, 8\tilde{j} + 8, 8\tilde{j} + 9\}$,
- $e_{j+1} = \{8\tilde{j} + 2, 8\tilde{j} + 3, 8\tilde{j} + 4, 8\tilde{j} + 16, 8\tilde{j} + 17, 8\tilde{j} + 18\}$,
- $e_{j+2} = \{8\tilde{j} + 3, 8\tilde{j} + 4, 8\tilde{j} + 5, 8\tilde{j} + 13, 8\tilde{j} + 14, 8\tilde{j} + 15\}$,
- $e_{j+3} = \{8\tilde{j} + 4, 8\tilde{j} + 5, 8\tilde{j} + 6, 8\tilde{j} - 2, 8\tilde{j} - 1, 8\tilde{j}\}$,

where $j = 4\tilde{j} + 1$.

In each hyperedge $e_r$, the vertices printed bold are the vertices of the first 3-section of the hyperedge $e_r$ in the sense of [1].

Remember that we take the numbers of the vertices $1, 2, \ldots, n$ modulo $n$, the indices of the hyperedges $e_1, e_2, \ldots, e_{\frac{n}{2}}$ modulo $\frac{n}{2}$ and the indices of the 4-groups $G_0, G_1, \ldots, G_{\frac{n}{8} - 1}$ modulo $\frac{n}{8}$.

As an instance for the construction of the hyperedges, we choose $n = 24$ and consider the hypergraph $H = (V, \{e_1, e_2, \ldots, e_{12}\})$ with $EI(H) = C_{24}$. This may also ease the understanding of Part 1(a) and Part 1(b) of the proof below.
Example 1. \( \mathcal{H} = (V, \{ e_1, e_2, \ldots, e_{12} \} ) \) with \( EI(\mathcal{H}) = C_{24} \) has the following hyperedges.

\[
\begin{align*}
e_1 &= \{1, 2, 3, 7, 8, 9\}, \\
e_2 &= \{2, 3, 4, 16, 17, 18\}, \\
e_3 &= \{3, 4, 5, 13, 14, 15\}, \\
e_4 &= \{4, 5, 6, 22, 23, 24\}, \\
e_5 &= \{9, 10, 11, 15, 16, 17\}, \\
e_6 &= \{1, 2, 10, 11, 12, 24\}, \\
e_7 &= \{11, 12, 13, 21, 22, 23\}, \\
e_8 &= \{6, 7, 8, 12, 13, 14\}, \\
e_9 &= \{1, 17, 18, 19, 23, 24\}, \\
e_{10} &= \{8, 9, 10, 18, 19, 20\}, \\
e_{11} &= \{5, 6, 7, 19, 20, 21\}, \\
e_{12} &= \{14, 15, 16, 20, 21, 22\},
\end{align*}
\]

and the edges of \( C_{24} \) result from the intersections

\[
\begin{align*}
e_1 \cap e_2 &= \{2, 3\}, \\
e_2 \cap e_3 &= \{3, 4\}, \\
e_3 \cap e_4 &= \{4, 5\}, \\
e_4 \cap e_{11} &= \{5, 6\}, \\
e_5 \cap e_{11} &= \{6, 7\}, \\
e_6 \cap e_8 &= \{7, 8\}, \\
e_7 \cap e_{10} &= \{8, 9\}, \\
e_8 \cap e_{10} &= \{9, 10\}, \\
e_9 \cap e_6 &= \{10, 11\}, \\
e_6 \cap c_7 &= \{11, 12\}, \\
e_7 \cap e_8 &= \{12, 13\}, \\
e_8 \cap e_9 &= \{13, 14\}, \\
e_9 \cap e_{12} &= \{14, 15\}, \\
e_{10} \cap e_{12} &= \{15, 16\}, \\
e_2 \cap e_5 &= \{16, 17\}, \\
e_2 \cap e_9 &= \{17, 18\}, \\
e_9 \cap e_{10} &= \{18, 19\}, \\
e_{10} \cap e_{11} &= \{19, 20\}, \\
e_{11} \cap e_{12} &= \{20, 21\}, \\
e_{12} \cap e_7 &= \{21, 22\}, \\
e_4 \cap e_7 &= \{22, 23\}, \\
e_4 \cap e_9 &= \{23, 24\} \text{ and } e_6 \cap e_9 &= \{24, 1\}.
\end{align*}
\]

For \( i \neq j \), it can be shown easily that each of the remaining intersections \( e_i \cap e_j \) of pairs \( e_i, e_j \in \mathcal{E}(\mathcal{H}) \) contains less than two vertices and, therefore, the above edges of \( C_{24} \) are the only (hyper-)edges in \( EI(\mathcal{H}) \), i.e. there are no chords in \( EI(\mathcal{H}) \) or - by other words - we have \( EI(\mathcal{H}) = C_{24} \).

Now we show

Part 1(a): The hyperedges of \( \mathcal{H} \) generate all edges of \( C_n \).

We distinguish several subcases. In each subcase, the given equations are valid for all \( \tilde{j} \in \{0, 1, \ldots, \frac{n}{2} - 1\} \) and \( j = 4\tilde{j} + 1 \in \{1, 5, 9, \ldots, \frac{n}{2} - 3\} \).

(a): Edges of \( C_n \) generated only by the first 3-sections of hyperedges.

\[
\begin{align*}
\{8\tilde{j} + 2, 8\tilde{j} + 3\} &= e_{\tilde{j}} \cap e_{\tilde{j}+1} = e_{4\tilde{j}+1} \cap e_{4\tilde{j}+2}, \\
\{8\tilde{j} + 3, 8\tilde{j} + 4\} &= e_{\tilde{j}+1} \cap e_{\tilde{j}+2} = e_{4\tilde{j}+2} \cap e_{4\tilde{j}+3}, \text{ and} \\
\{8\tilde{j} + 4, 8\tilde{j} + 5\} &= e_{\tilde{j}+2} \cap e_{\tilde{j}+3} = e_{4\tilde{j}+3} \cap e_{4\tilde{j}+4}.
\end{align*}
\]

(\(\beta\)): Edges of \( C_n \) generated by a first 3-section and a second 3-section of two hyperedges.

The edge \( \{8\tilde{j} + 1, 8\tilde{j} + 2\} \) and \( \{8\tilde{j} + 5, 8\tilde{j} + 6\} \) of \( C_n \) is contained only in one first 3-section, namely in the hyperedge \( e_{\tilde{j}} = e_{4\tilde{j}+1} \) and \( e_{\tilde{j}+3} = e_{4\tilde{j}+4} \), respectively.

With \( \tilde{j}' = \tilde{j} - 2 \) and \( j' = 4\tilde{j}' + 1 \) we obtain

\[
\begin{align*}
\{8\tilde{j} + 1, 8\tilde{j} + 2\} &= \{8\tilde{j}' + 17, 8\tilde{j}' + 18\} = e_{\tilde{j}'} \cap e_{\tilde{j}'+1} = e_{4\tilde{j}'+1} \cap e_{4\tilde{j}'+2} = e_{4\tilde{j}+1} \cap e_{4\tilde{j}+6}.
\end{align*}
\]

Analogously, with \( j' = \tilde{j} - 1 \) and \( j' = 4\tilde{j}' + 1 \) it follows

\[
\begin{align*}
\{8\tilde{j} + 5, 8\tilde{j} + 6\} &= \{8\tilde{j}' + 13, 8\tilde{j}' + 14\} = e_{\tilde{j}'+3} \cap e_{\tilde{j}'+2} = e_{4\tilde{j}'+4} \cap e_{4\tilde{j}'+3} = e_{4\tilde{j}+4} \cap e_{4\tilde{j}+1}.
\end{align*}
\]
(γ): Edges of $C_n$ generated only by the second 3-sections of two hyperedges.

The edges $\{8j+1, 8j+2\}, \{8j+2, 8j+3\}, \{8j+3, 8j+4\}, \{8j+4, 8j+5\}, \{8j+5, 8j+6\}$ of $C_n$ have been generated in the subcases (α) and (β), for all $j \in \{0, 1, \ldots, \frac{n}{8} - 1\}$.

To complete the proof of Part1(a), now we consider the edges $\{8j+6, 8j+7\}, \{8j+7, 8j+8\}, \{8j+8, 8j+9\}$.

With $j' = j - 1$ and $j'' = j + 1$ we get
$$\{8j + 6, 8j + 7\} = \{8j' + 14, 8j' + 15\} = \{8j'' - 2, 8j'' - 1\}$$
$$= e_{j' + 2} \cap e_{j'' + 3} = e_{4j' + 3} \cap e_{4j'' + 4} = e_{4j - 1} \cap e_{4j + 8}.$$

Setting $j' = j$ and $j'' = j + 1$ we have
$$\{8j + 7, 8j + 8\} = \{8j + 7, 8j + 8\} = \{8j'' - 1, 8j''\}$$
$$= e_j \cap e_{j'' + 3} = e_{4j'' + 1} \cap e_{4j'' + 4} = e_{4j + 1} \cap e_{4j + 8}.$$

Finally, we choose $j' = j$ and $j'' = j - 1$. This leads to
$$\{8j + 8, 8j + 9\} = \{8j + 8, 8j + 9\} = \{8j'' + 16, 8j'' + 17\}$$
$$= e_j \cap e_{j'' + 1} = e_{4j'' + 1} \cap e_{4j'' + 2} = e_{4j + 1} \cap e_{4j - 2}.$$

With this, we have shown that $C_n$ is a subhypergraph of $EI(\mathcal{H})$.

Part 1(b): $EI(\mathcal{H})$ does not contain any chord of $C_n$.

(i) Using the construction of the hyperedges of $\mathcal{H}$, for all hyperedges $e_f \neq e_g$ we verify that $e_f$ and $e_g$ do never have a 3-section in common. Note that we use the numbering of the hyperedges from the beginning of Part 1.

Let's have a look at the remainders modulo 8 of the vertices contained in a 3-section of an arbitrary hyperedge. We find the following sets of remainders for the first 3-sections: $\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5, 6\}$. For the second 3-sections we have the sets $\{7, 0, 1\}, \{0, 1, 2\}, \{5, 6, 7\}, \{6, 7, 0\}$. Therefore, all these sets of remainders are pairwise distinct.

Assume that $e_f$ and $e_g$ have a 3-section in common. Because of the pairwise distinctness of the sets of the remainders mentioned above, it follows $f \equiv g \mod 4$ and the common 3-section has to be the first or the second 3-section of both, $e_f$ as well as $e_g$. Looking at the definition of the hyperedges, this leads to $f = g$ in contradiction to $e_f \neq e_g$.

Consequently, a chord in $EI(\mathcal{H})$ cannot be obtained as an intersection of a 3-section $\{x, x+1, x+2\} \subset e$ and a 3-section $\{y, y+1, y+2\} \subset e'$ of any two hyperedges $e$ and $e'$ of $\mathcal{H}$.

(ii) If $e = \{p, p+1, p+2, q, q+1, q+2\} \in \mathcal{E}$ is an arbitrary hyperedge, then the construction of $\mathcal{H}$ provides that the distance $d_{C_n}(p+2, q)$ of the vertices $p+2$ and $q$ along the cycle $C_n$ is at least 4; analogously $d_{C_n}(q+2, p) \geq 4$. Hence the intersection of $e$ with one 3-section $\{y, y+1, y+2\} \subset e'$ of another hyperedge $e'$ can only result in the (welcome) edges $\{p, p+1\}, \{p+1, p+2\}, \{q, q+1\}$ or $\{q+1, q+2\}$, which are edges of the cycle $C_n$.

Because of (i) and (ii), for any chord $\{k, l\}$ ($|k - l| > 1 \land \{k, l\} \neq \{u, 1\}$) in $EI(\mathcal{H})$ resulting from the hyperedge $e = \{p, p+1, p+2, q, q+1, q+2\}$ and a second hyperedge $e' = \{r, r+1, r+2, s, s+1, s+2\}$ one of the following situations must occur. (For our investigations let $\{p, p+1, p+2\}$ and $\{r, r+1, r+2\}$ be the first 3-section of the hyperedge
In the case \( n = e \) a contradiction.

(B).

A contradiction.

Consequently, situation (B) has to occur and we assume

\[ e = \{p, p + 1, p + 2, q, q + 1, q + 2\}, \quad e' = \{r, r + 1, r + 2, s, s + 1, s + 2\}, \]

\[ k \in \{p, p + 1, p + 2\} \cap \{s, s + 1, s + 2\} \quad \text{and} \quad l \in \{q, q + 1, q + 2\} \cap \{r, r + 1, r + 2\}. \]

Obviously, \( e \in G_{j_{\rightarrow}} = \{e_j, e_{j+1}, e_{j+2}, e_{j+3}\} \) and \( e' \in G_{j_{\leftarrow}} = \{e'_j, e'_{j+1}, e'_{j+2}, e'_{j+3}\} \) with \( j, j' \in \{1, 5, 9, \ldots, \frac{n}{2} - 3\} \) and \( \tilde{j} = \frac{j - 1}{4}, \tilde{j'} = \frac{j' - 1}{4} \in \{0, 1, \ldots, \frac{n}{2} - 1\} \) as well as \( j \neq j' \), \( \tilde{j} \neq \tilde{j'} \).

We discuss all possible choices of \( e \) and \( e' \) in the sets \( \{e_j, e_{j+1}, e_{j+2}, e_{j+3}\} \) and \( \{e'_j, e'_{j+1}, e'_{j+2}, e'_{j+3}\} \), respectively. In order to find the wanted \( k \) and \( l \) in the intersections of the corresponding 3-sections, we are searching for numbers, i.e. vertices, \( 8\tilde{j} + x \) and \( 8j'+y \) in the 3-sections being under investigation, which have one and the same remainder modulo 8.

**B1:** \( e = e_j \land e' = e'_j \).

- From \( k \in \{p, p + 1, p + 2\} \cap \{s, s + 1, s + 2\} \)
  - \( \{8j + 1, 8j + 2, 8j + 3\} \cap \{8j' + 7, 8j' + 8, 8j' + 9\} \)

it follows \( k = 8\tilde{j} + 1 = 8\tilde{j'} + 9 \) and, therefore, \( \tilde{j} = \tilde{j'} + 1 \).

- Thereby, \( l \in \{q, q + 1, q + 2\} \cap \{r, r + 1, r + 2\} \)
  - \( \{8j' + 7, 8j' + 8, 8j' + 9\} \cap \{8\tilde{j}' + 1, 8\tilde{j}' + 2, 8\tilde{j}' + 3\} \)
  - \( \{8\tilde{j}' + 15, 8\tilde{j}' + 16, 8\tilde{j}' + 17\} \cap \{8\tilde{j}' + 1, 8\tilde{j}' + 2, 8\tilde{j}' + 3\} = \emptyset \),

a contradiction.

**B2:** \( e = e_j \land e' = e'_{j+1} \).

- From \( k \in \{p, p + 1, p + 2\} \cap \{s, s + 1, s + 2\} \)
  - \( \{8j + 1, 8j + 2, 8j + 3\} \cap \{8\tilde{j}' + 16, 8\tilde{j}' + 17, 8\tilde{j}' + 18\} \)

it follows \( k = 8\tilde{j} + 1 = 8\tilde{j'} + 17 \) or \( k = 8\tilde{j} + 2 = 8\tilde{j'} + 18 \) and, therefore, \( \tilde{j} = \tilde{j'} + 2 \).

- Thereby, \( l \in \{q, q + 1, q + 2\} \cap \{r, r + 1, r + 2\} \)
  - \( \{8j + 7, 8j + 8, 8j + 9\} \cap \{8\tilde{j}' + 2, 8\tilde{j}' + 3, 8\tilde{j}' + 4\} \)
  - \( \{8\tilde{j}' + 23, 8\tilde{j}' + 24, 8\tilde{j}' + 25\} \cap \{8\tilde{j}' + 2, 8\tilde{j}' + 3, 8\tilde{j}' + 4\} \).

In the case \( n = 24 \) this intersection is equal to \( \{8\tilde{j}' - 1, 8\tilde{j}', 8\tilde{j}' + 1\} \cap \{8\tilde{j}' + 2, 8\tilde{j}' + 3, 8\tilde{j}' + 4\} = \emptyset \), and in the case \( n \geq 40 \) the intersection is trivially equal to \( \emptyset \), incompatible to (B).
B3: \(e = e_j \land e' = e_{j'+2}\).

For all \(j, j' \in \{1, 5, 9, \ldots, \frac{n}{2} - 3\}\) it follows \(\{p, p + 1, p + 2\} \cap \{s, s + 1, s + 2\}\)
\[\{8j' + 1, 8j' + 2, 8j' + 3\} \cap \{8j' + 13, 8j' + 14, 8j' + 15\} = \emptyset.\]

B4: \(e = e_j \land e' = e_{j'+3}\).

For all \(j, j' \in \{1, 5, 9, \ldots, \frac{n}{2} - 3\}\) it follows \(\{p, p + 1, p + 2\} \cap \{s, s + 1, s + 2\}\)
\[\{8j' + 1, 8j' + 2, 8j' + 3\} \cap \{8j' - 2, 8j' - 1, 8j'\} = \emptyset.\]

B5: \(e = e_{j+1} \land e' = e_{j'+1}\).

From \(k \in \{p, p + 1, p + 2\} \cap \{s, s + 1, s + 2\}\)
\[\{8j + 2, 8j + 3, 8j + 4\} \cap \{8j' + 16, 8j' + 17, 8j' + 18\} \text{ it follows } k = 8j + 2 = 8j' + 18 \text{ and, therefore, } j = j' + 2.\]

Thereby, \(l \in \{q, q + 1, q + 2\} \cap \{r, r + 1, r + 2\}\)
\[\{8j + 16, 8j + 17, 8j + 18\} \cap \{8j' + 2, 8j' + 3, 8j' + 4\} \text{ it follows } k = 8j + 2 = 8j' + 18 \text{ and, therefore, } j = j' + 2.\]

Note that in the case \(n = 32\) we would have \(l = 8j' + 34 = 8j' + 2.\) This is the reason why for \(n = 32\) a modified construction of the hyperedges of \(\mathcal{H}\) has to be used (cf. Lemma 2).

B6: \(e = e_{j+1} \land e' = e_{j'+2}\).

For all \(j, j' \in \{1, 5, 9, \ldots, \frac{n}{2} - 3\}\) it follows \(\{p, p + 1, p + 2\} \cap \{s, s + 1, s + 2\}\)
\[\{8j + 2, 8j + 3, 8j + 4\} \cap \{8j' + 13, 8j' + 14, 8j' + 15\} = \emptyset.\]

B7: \(e = e_{j+1} \land e' = e_{j'+3}\).

For all \(j, j' \in \{1, 5, 9, \ldots, \frac{n}{2} - 3\}\) we obtain \(\{p, p + 1, p + 2\} \cap \{s, s + 1, s + 2\}\)
\[\{8j + 2, 8j + 3, 8j + 4\} \cap \{8j' - 2, 8j' - 1, 8j'\} = \emptyset.\]

B8: \(e = e_{j+2} \land e' = e_{j'+2}\).

From \(k \in \{p, p + 1, p + 2\} \cap \{s, s + 1, s + 2\}\)
\[\{8j + 3, 8j + 4, 8j + 5\} \cap \{8j' + 13, 8j' + 14, 8j' + 15\} \text{ we get } k = 8j + 5 = 8j' + 13 \text{ and, therefore, } j = j' + 1.\]

Thereby, \(l \in \{q, q + 1, q + 2\} \cap \{r, r + 1, r + 2\}\)
\[\{8j + 13, 8j + 14, 8j + 15\} \cap \{8j' + 3, 8j' + 4, 8j' + 5\} \text{ we get } k = 8j + 5 = 8j' + 13 \text{ and, therefore, } j = j' + 1.\]

B9: \(e = e_{j+2} \land e' = e_{j'+3}\).

For all \(j, j' \in \{1, 5, 9, \ldots, \frac{n}{2} - 3\}\) we have \(\{p, p + 1, p + 2\} \cap \{s, s + 1, s + 2\}\)
\[\{8j + 3, 8j + 4, 8j + 5\} \cap \{8j' - 2, 8j' - 1, 8j'\} = \emptyset.\]

B10: \(e = e_{j+3} \land e' = e_{j'+3}\).

From \(k \in \{p, p + 1, p + 2\} \cap \{s, s + 1, s + 2\}\)
\[\{8j + 4, 8j + 5, 8j + 6\} \cap \{8j' - 2, 8j' - 1, 8j'\} \text{ we get } k = 8j + 6 = 8j' - 2 \text{ and, therefore, } j = j' + 1.\]

Thereby, \(l \in \{q, q + 1, q + 2\} \cap \{r, r + 1, r + 2\}\)
\[\{8j + 4, 8j + 5, 8j + 6\} \cap \{8j' - 2, 8j' - 1, 8j'\} \text{ we get } k = 8j + 6 = 8j' - 2 \text{ and, therefore, } j = j' + 1.\]
Claim. The hypergraph $H$.

Part 1. Then there is no hyperedge $H$ (given in [1]) from the hypergraph $H$.

Proof. $H$ with $n = 8$ and one hyperedge in each step. As the initial step, Part 1 includes the verification for above proof of Part 1, our present proof will be done inductively by adding two vertices

by $\{\{n\} \cap \{j\}, \{j\} \cap \{n, 1\}, \{n, 1, 2\}\}$. Therefore each vertex $v \in V$ is contained in exactly three hyperedges.

Part 2 (Supplemental construction): $l \in \{2, 4, 6\}$.

First of all, we sketch the idea of the proof.

Note that in [1] (cf. the proof of Lemma 1 in [1]) we gave only the definitions of the hyperedges of the wanted hypergraph with $n$ vertices; now let us denote this hypergraph by $H'$. To avoid the lengthy verification of $EI(H') = C_n = C_{sk+l}$ analogously to the above proof of Part 1, our present proof will be done inductively by adding two vertices and one hyperedge in each step. As the initial step, Part 1 includes the verification for $n = 8k$. For the induction hypothesis, let $H$ be the hypergraph given in Lemma 1 of [1] with $n = 8k + l - 2$ vertices, which has the property $EI(H) = C_{n-2} = C_{8k+l-2}$. We show how to construct the hypergraph $H'$ with $n = 8k + l$ vertices and $EI(H') = C_n$ (given in [1]) from the hypergraph $H$.

For this end, we will add two new vertices $x$ and $y$ as well as one new hyperedge $e_\ast$ to $H$ and obtain $H' = (V', E') = (V(H) \cup \{x, y\}, E(H) \cup \{e_\ast\})$.

It will be easy to see that - after relabelling the vertices and the hyperedges - the hyperedges of $H'$ are exactly the hyperedges given in [1].

Finally, we will verify that $EI(H') = C_n$ is valid. The 6-regularity and 3-uniformity of the hypergraph $H$ follows analogously to Part 1.

As a preliminary consideration, we prove the following.

Claim. Let $n = 8k$, and $j, j, G_j = \{e_j, e_{j+1}, e_{j+2}, e_{j+3}\}$ be defined as at the beginning of Part 1. Then there is no hyperedge $e_z \in E$ with $\{\tilde{8}j + 1, \tilde{8}j + 2\} \cap e_z \neq \emptyset$ and $\{\tilde{8}j + 5, \tilde{8}j + 6\} \cap e_z \neq \emptyset$.

Proof. The definitions of $e_j, e_{j+1}, e_{j+2}$ and $e_{j+3}$ provide $e_z \in E \setminus G_j$.

Assume $p \in \{\tilde{8}j + 1, \tilde{8}j + 2\} \cap e_z$ and $q \in \{\tilde{8}j + 5, \tilde{8}j + 6\} \cap e_z$. Then there is a $j' \in \{1, 5, 9, \ldots, \frac{3}{2} - 3\} \setminus \{j\}$ such that $z \in \{j', j' + 1, j' + 2, j' + 3\}$ and $j' = \frac{8k - 1}{4} \neq j$.

Obviously, the remainder of $p$ and $q$ modulo 8 is from the set \{1, 2\} and \{5, 6\}, respectively. Since the sets of the remainders modulo 8 of the vertices in the two 3-sections of $e_z$ are
Now we add two vertices and one hyperedge to the hypergraph $H$

(ii) We relabel the vertices of

(i) We preliminarily remark that, owing to the above Claim, adding a new hyperedge $e \in \text{Subcase 2.1}$ will make use of an analog construction.

Subcase 2.1: $l = 2$

Note that also in this Subcase we will give an example (Example 2, see at the end of Subcase 2.1) for the constructed hypergraph, namely for $n = 26$. But before we will describe the idea of our construction.

For this end, let $n = 8k + 2$, $H = (V, E)$ with $V = \{1, 2, \ldots, n - 2\}$ and $E = \{e_1, e_2, \ldots, e_{2^{n/2}}\}$, where the hyperedges in $E$ should be constructed as described at the beginning of Part 1 (now with $n - 2$ instead of $n$).

Consequently, $EI(H) = C_{n-2}$.

To give a sketchy idea, consider the 4-group $G_0 = \{e_1, e_2, e_3, e_4\}$ consisting of the hyperedges

$e_1 = \{1, 2, 3, 7, 8, 9\}$,
$e_2 = \{2, 3, 4, 16, 17, 18\}$,
$e_3 = \{3, 4, 5, 13, 14, 15\}$,
$e_4 = \{4, 5, 6, n - 4, n - 3, n - 2\} = \{4, 5, 6, 8k - 2, 8k - 1, 8k\}$.

We proceed as follows: we insert the new vertex $x$ "between" the (old) vertices 6 and 7 (remember that the vertices are numbered by 1, 2, \ldots, $n - 2$ along the cycle $C_{n-2}$). Then we insert a second new vertex $y$ "between" the (old) vertices $n - 2$ and 1. Simultaneously, we add a new hyperedge $e_\ast$ to the 4-group $G_0$ such that the new $G_0$ becomes a 5-group, i.e. $G_0 = \{e_1, e_2, e_3, e_4, e_\ast\}$. This procedure results in a new hypergraph $H'$ with - as we will prove - $EI(H') = C_n$.

To this end, a few modifications of several (old) hyperedges of the original hypergraph $H$ and some relabelling of vertices and hyperedges have to be carried out.

Now we come to the detailed steps of our construction.

(i) We preliminarily remark that, owing to the above Claim, adding a new hyperedge $e' = \{1, 2, 5, 6\}$ to $H$ would not induce a chord in $EI(H \cup \{e'\}) = EI(H) = C_{n-2}$.

(ii) We relabel the vertices of $V$ (including the vertices inside the hyperedges of $H$) according to

$$v := \begin{cases} v & \text{if } v \in \{1, 2, \ldots, 6\} \\ v + 1 & \text{if } v \in \{7, 8, \ldots, n - 2\} \end{cases}$$

Then we have $V = \{1, 2, \ldots, 6, 8, 9, \ldots, n - 1\}$ and (i) remains valid (with this modified numbering of the vertices).

(iii) Now we add two vertices and one hyperedge to the hypergraph $H = (V, E)$ and obtain $H' = (V', E')$:

$V' := V \cup \{x, y\}$, where $x = 7$ and $y = n$.
Some 3-sections of four special hyperedges

Because of (i) and the fact that \( e_\gamma \) is the only hyperedge containing the new vertices 7 and 8, (i) remains valid in the sense that there are no vertices \( k, l \in V' \) with \( |k - l| > 1 \) and \( \{k, l\} \subset e \), for all hyperedges \( e \in \mathcal{E}(EI(\mathcal{H}')) \). The only exception is the (unoffending) case \( k = 1 \) and \( l = n \). In other words: the new hyperedge \( e_\gamma \) will not generate a chord in the cycle \( C_n \), which will arise as \( EI(\mathcal{H}') \) in the next step (iv).

In Part 1(b)(i) we saw that no two hyperedges have a 3-section in common. Obviously, this remains valid up to now, also if we include the new hyperedge \( e_\gamma \). Therefore, each hyperedge \( e \in \mathcal{E}(\mathcal{H}') \) is uniquely determined by giving its first or its second 3-section. This property will simplify the considerations in step (iv).

Finally, note that the relabelling of the vertices in (ii) and the insertion of the new vertices 7 and 8 in (iii) induce that in some 3-sections of several hyperedges there is a gap between the vertices 6, 8 and \( n - 1, 1 \), respectively. Originally, each 3-section of a hyperedge consists of three vertices \( p, p + 1, p + 2 \) being consecutive on the cycle \( C_n \). The next step of our construction retrieves this previous state for all hyperedges.

(iv) Some 3-sections of four special hyperedges \( e_\alpha, e_\beta, e_\gamma, e_\delta \) have to be modified and the modifications exclusively bear on the second 3-sections of these hyperedges. For clearness, in each case we give the modifications in the form

\[
e\kappa : \{x, y, z\} \Rightarrow \{x', y', z'\} \Rightarrow \{x'', y'', z''\}, \quad \kappa \in \{\alpha, \beta, \gamma, \delta\} \quad \text{and} \quad \{x, y, z\} \text{ is the original 3-section in the hypergraph } \mathcal{H} \text{ (before step (i))},
\]

\[
\{x', y', z'\} \text{ is the 3-section after step (ii) (this is the same as after step (iii)) and } \{x'', y'', z''\} \text{ is the final form of the 3-section after step (iv)}.
\]

As mentioned above, the other (these are the first) 3-sections of the modified hyperedges remain unchanged. They are not needed for our argumentation, so in the following we present only the (modified) second 3-sections.

\[
e_\alpha : \{5, 6, 7\} \Rightarrow \{5, 6, 8\} \Rightarrow \{6, 7, 8\},
\]

\[
e_\beta : \{6, 7, 8\} \Rightarrow \{6, 8, 9\} \Rightarrow \{7, 8, 9\},
\]

\[
e_\gamma : \{n - 3, n - 2, 1\} \Rightarrow \{n - 2, n - 1, 1\} \Rightarrow \{n - 2, n - 1, n\} \quad \text{and}
\]

\[
e_\delta : \{n - 2, 1, 2\} \Rightarrow \{n - 1, 1, 2\} \Rightarrow \{n - 1, n, 1\}.
\]

Please note that \( e_\alpha \neq e_\gamma \), since \( e_\gamma \) comes into play not before step (iii). Clearly, considering only the hyperedges in \( \mathcal{E} \setminus \{e_\alpha, e_\beta, e_\gamma, e_\delta, e_\gamma\} \), these hyperedges induce a subgraph of \( C_n \) in \( EI(\mathcal{H}') \), therefore they do not cause any chords. Owing to (iii), no chord emerges if we add \( e_\gamma \) to this set of hyperedges.

Hence, we only have to show that the new hyperedges \( e_\alpha, e_\beta, e_\gamma, e_\delta \) do not generate a chord:

(a) Obviously, the relabelling of the vertices in (ii) does not result in a chord, since the relabelling is carried out simultaneously in all hyperedges, i.e. apart from the modified vertex numbers the hyperedges remain unchanged.

(b) It is clear that the addition of the new vertices 7 and 8 in step (iii) cannot lead to a chord, because none of the "old" hyperedges (i.e. the hyperedges in \( \mathcal{E} \)) contains one of these vertices.
(c) Above we demonstrated that, in step (iii), $e_\ast$ does not generate any chord.

(d) It suffices to show that the modifications in step (iv) do not produce any chord. Since these modifications involve only the hyperedges $e_\alpha, e_\beta, e_\gamma, e_\delta$, we have to consider solely the intersections of these hyperedges with other hyperedges of $\mathcal{H}'$.

i. It can be easily seen that (analogously as in Part 1(b)(i)), for all hyperedges $e_f \neq e_g$ in $\mathcal{E}' = \mathcal{E} \cup \{e_\ast\}$, the hyperedges $e_f$ and $e_g$ do never have a 3-section in common. Therefore, a chord $\{k, l\}$ could result only from the intersection of two hyperedges $e_f$ and $e_g$, where $k$ is contained in one 3-section of $e_g$ as well as in one 3-section of $e_f$ and $l$ is contained in the other 3-section of $e_g$ and the other 3-section of $e_f$.

ii. We consider $e_\alpha = \{\ldots, 6, 7, 8\}$.

In step (iv), the replacement of the vertex 5 by the vertex 7 does not result in a chord, since at that moment 7 was only in the hyperedge $e_\ast$ and, moreover, already before step (iv) we had a nonempty intersection $e_\alpha \cap e_\alpha \supseteq \{5, 6\}$. Therefore - owing to the non-existence of chords at this time - the other 3-sections of $e_\alpha$ and $e_\alpha$ have to be disjoint.

iii. Let's look at $e_\beta = \{\ldots, 7, 8, 9\}$.

In (iv), superseding 6 by 7 does not lead to a chord, because till then 7 had been only in the hyperedges $e_\ast$ and $e_\alpha$. Additionally, before (iv) we had also nonempty intersections $e_\alpha \cap e_\beta \supseteq \{6\}$ and $e_\alpha \cap e_\beta \supseteq \{6, 8\}$. Hence, for the same reason as in ii, the other 3-sections of $e_\alpha$ and $e_\beta$ as well as of $e_\alpha$ and $e_\beta$ must be disjoint.

iv. We come to $e_\gamma = \{\ldots, n - 2, n - 1, n\}$.

The substitution of the vertex 1 by $n$ in step (iv) does not generate a chord. The reason is that before this substitution the vertex $n$ had been contained only in $e_\ast$. What is more, before (iv) obviously $e_\ast \cap e_\gamma \supseteq \{1\} \neq \emptyset$ and, analogously to i and ii, the remaining 3-sections of both hyperedges cannot include any vertex in common.

v. Finally, we investigate $e_\delta = \{\ldots, n - 1, n, 1\}$.

Now we look at the replacement of the vertex 2 by $n$ in (iv) and see that until then $n$ had been only an element of the hyperedges $e_\ast$ and $e_\gamma$. Again, before step(iv) we had nonempty intersections $e_\ast \cap e_\delta \supseteq \{1, 2\}$ and $e_\gamma \cap e_\delta \supseteq \{n - 1, 1\}$, consequently the intersections of the other 3-sections of these hyperedges have to be empty.

So we see that the steps (ii) – (iv) of our construction do not generate any chord in $EI(\mathcal{H}')$.

It remains to verify, that $E(C_n) \subseteq \mathcal{E}(\mathcal{H}')$. We do not want to discuss here the - in a certain sense "trivial" - edges $\{i,i+1\}$ of $C_n$ which result from "simply incremented vertices" (see (ii)). Before the relabelling step (ii), such edges had been edges of the cycle $C_{n-2} = EI(\mathcal{H})$. Therefore, it suffices to consider the remaining edges of $C_n$, i.e. the edges in the vertex ranges $\{5,6,7,8,9\}$ and $\{n-2,n-1,n,1,2\}$.

These edges result from the following intersections:

\[
\begin{align*}
\{5,6\} &= e_4 \cap e_\ast, \{6,7\} = e_\ast \cap e_\alpha, \{7,8\} = e_\alpha \cap e_\beta, \{8,9\} = e_\beta \cap e_1, \\
\{n-2,n-1\} &= e_4 \cap e_\gamma, \{n-1,n\} = e_\gamma \cap e_\delta, \{n,1\} = e_\delta \cap e_\ast \text{ and } \{1,2\} = e_\ast \cap e_1.
\end{align*}
\]
Now we come to Example 2. Note that for the hyperedges the labelling from [1] is used, i.e. the hyperedge $e_*$ becomes $e_5$ and the indices of the former $e_5, e_6, \ldots$ have to be increased by 1.

Example 2. $\mathcal{H} = (V, \{e_1, e_2, \ldots, e_{13}\})$ with $EI(\mathcal{H}) = C_{26}$ has the following hyperedges.

\[
\begin{align*}
& e_1 = \{ 1 , 2 , 3 , 8 , 9 , 10 \}, \\
& e_2 = \{ 2 , 3 , 4 , 17 , 18 , 19 \}, \\
& e_3 = \{ 3 , 4 , 5 , 14 , 15 , 16 \}, \\
& e_4 = \{ 4 , 5 , 6 , 23 , 24 , 25 \}, \\
& e_5 = \{ 1 , 2 , 5 , 6 , 7 , 26 \}, \\
& e_6 = \{ 10 , 11 , 12 , 16 , 17 , 18 \}, \\
& e_7 = \{ 1 , 11 , 12 , 13 , 25 , 26 \}, \\
& e_8 = \{ 12 , 13 , 14 , 22 , 23 , 24 \}, \\
& e_9 = \{ 7 , 8 , 9 , 13 , 14 , 15 \}, \\
& e_{10} = \{ 18 , 19 , 20 , 24 , 25 , 26 \}, \\
& e_{11} = \{ 9 , 10 , 11 , 19 , 20 , 21 \}, \\
& e_{12} = \{ 6 , 7 , 8 , 20 , 21 , 22 \}, \\
& e_{13} = \{ 15 , 16 , 17 , 21 , 22 , 23 \},
\end{align*}
\]

where in $EI(\mathcal{H})$ the edges of $C_{26}$ are generated analogously as shown in Example 1.

Note that now $G_0 = \{e_1, e_2, e_3, e_4, e_5\}$ is a 5-group and the other groups $G_1$ and $G_2$ are 4-groups of hyperedges.

Subcase 2.2: $l = 4$.

Again, we make use of our construction of the hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $n - 2 = (8k + 4) - 2 = 8k + 2$ vertices (cf. Subcase 2.1). Looking at Example 2, we remember that $G_0 = \{e_1, e_2, e_3, e_4, e_5\}$ is a 5-group and the other groups $G_1, G_2, \ldots$ are 4-groups.

In a first step, we relabel

- the vertices $v \in V = \{1, 2, \ldots, n - 2\}$ by $v := v - 9$ (modulo $(n - 2)$),
- the hyperedges $e_i \in \mathcal{H} = \{e_1, e_2, \ldots, e_{n-2}\}$ by $e_i := e_{i-4}$ (the indices taken modulo $\frac{n-2}{2}$) and
- the edge groups $G_j \in \{G_0, G_1, \ldots, G_{\frac{n-4}{8}-1}\}$ by $G_j := G_{j-1}$ (the indices taken modulo $\frac{n-4}{8}$).

That way, the former 4-group $G_{\frac{n-4}{8}-1}$ becomes the 4-group $G_0$ with the new label 0 and the former 5-group $G_0$ is now the new 5-group $G_1$ (with the corresponding relabeled vertices and hyperedges). Consequently, in our present hypergraph we have locally (i.e., in the vertex range, where in Subcase 2.1 the essential modifications in $\mathcal{H}$ in connection with the insertion of the new vertices $x$ and $y$ and the new hyperedge $e_*$ take place) the same structure as we had in Subcase 2.1. Therefore, also in the present situation (i.e., where $n = 8k + 4$ holds) the same procedure as in Subcase 2.1 can be used in order to obtain a new hypergraph $\mathcal{H}'$ with $n$ vertices from the hypergraph $\mathcal{H}$ with $n - 2$ vertices.

It can be verified analogously to Subcase 2.1 that $EI(\mathcal{H}') = C_n$ is valid.

Choosing $n = 28$ we give the last example in Part 2. Again $e_*$ becomes $e_5$ and the indices of the former $e_5, e_6, \ldots$ have to be increased by 1.
Example 3. \( \mathcal{H} = (V, \{e_1, e_2, \ldots, e_{14}\}) \) with \( EI(\mathcal{H}) = C_{28} \) has the following hyperedges.

\[
\begin{align*}
e_1 &= \{1, 2, 3, 8, 9, 10\}, \\
e_2 &= \{2, 3, 4, 19, 20, 21\}, \\
e_3 &= \{3, 4, 5, 16, 17, 18\}, \\
e_4 &= \{4, 5, 6, 25, 26, 27\}, \\
e_5 &= \{1, 2, 5, 6, 7, 28\}, \\
e_6 &= \{11, 12, 13, 18, 19, 20\}, \\
e_7 &= \{1, 12, 13, 14, 27, 28\}, \\
e_8 &= \{13, 14, 15, 24, 25, 26\}, \\
e_9 &= \{7, 8, 9, 14, 15, 16\}, \\
e_{10} &= \{10, 11, 12, 15, 16, 17\}, \\
e_{11} &= \{20, 21, 22, 26, 27, 28\}, \\
e_{12} &= \{9, 10, 11, 21, 22, 23\}, \\
e_{13} &= \{6, 7, 8, 22, 23, 24\}, \\
e_{14} &= \{17, 18, 19, 23, 24, 25\},
\end{align*}
\]

now \( G_0 = \{e_1, e_2, e_3, e_4, e_5\} \) and \( G_1 = \{e_6, e_7, e_8, e_9, e_{10}\} \) are 5-groups and \( G_2 = \{e_{11}, e_{12}, e_{13}, e_{14}\} \) is a 4-group of hyperedges.

Subcase 2.3: \( l = 6 \).

We use an analog verification as in Subcase 2.2.

Concerning the 6-uniformity and the 3-regularity the argumentation given at the end of Part 1 remains valid also for Part 2.

\[ \square \]

2. Proof of Lemma 2

Lemma 2. Let \( l, n \in \mathbb{N} \) with \( l \in \{0, 2, 4, 6\} \) and \( n = 32 + l \). Then there exists a 3-regular, 6-uniform hypergraph \( \mathcal{H} = (V, \mathcal{E}) \) with \( EI(\mathcal{H}) = C_n \) and \( |\mathcal{E}| = \frac{n}{2} \).

Proof. Looking at the construction of the hyperedges in [1] it is obvious that there are only little modifications in comparison with the construction in the proof of Lemma 1 (cf. [1]), i.e. with the case \( n = 8k + l \), where \( k \neq 4 \). In principle, for any 4-group or 5-group \( G_j = \{e_j, e_{j+1}, e_{j+2}, \ldots\} \) of hyperedges (for \( n = 8k + l, k \neq 4 \)) we have only to swap the second 3-sections of the second hyperedge \( e_{j+1} \) and the third hyperedge \( e_{j+2} \) in order to obtain the corresponding hyperedges \( e_{j+1} \) and \( e_{j+2} \) for \( n = 8k + l, k = 4 \). For details, see the Cases (A)–(D) in the proof of Lemma 2 in [1].

Hence we could argue that the verification of the present case can be done analogously to that of Lemma 1.

But since Lemma 2 includes only the four possible values 32, 34, 36, 38 for the number \( n \) of the vertices, alternatively the hyperedges can be written down easily and their intersections can be computed. At first, we give explicitly the edge set of the wanted hypergraph \( \mathcal{H} \) with \( EI(\mathcal{H}) = C_{32} \).
Example 4. \( \mathcal{H} = (V, \{e_1, e_2, \ldots, e_{16}\}) \) with \( EI(\mathcal{H}) = C_{32} \) has the following hyperedges.

\[
\begin{align*}
e_1 &= \{ 1, 2, 3, 7, 8, 9 \}, \\
e_2 &= \{ 2, 3, 4, 13, 14, 15 \}, \\
e_3 &= \{ 3, 4, 5, 16, 17, 18 \}, \\
e_4 &= \{ 4, 5, 6, 30, 31, 32 \}, \\
e_5 &= \{ 9, 10, 11, 15, 16, 17 \}, \\
e_6 &= \{ 10, 11, 12, 21, 22, 23 \}, \\
e_7 &= \{ 11, 12, 13, 24, 25, 26 \}, \\
e_8 &= \{ 6, 7, 8, 12, 13, 14 \}, \\
e_9 &= \{ 17, 18, 19, 23, 24, 25 \}, \\
e_{10} &= \{ 18, 19, 20, 29, 30, 31 \}, \\
e_{11} &= \{ 1, 2, 19, 20, 21, 32 \}, \\
e_{12} &= \{ 14, 15, 16, 20, 21, 22 \}, \\
e_{13} &= \{ 1, 25, 26, 27, 31, 32 \}, \\
e_{14} &= \{ 5, 6, 7, 26, 27, 28 \}, \\
e_{15} &= \{ 8, 9, 10, 27, 28, 29 \}, \\
e_{16} &= \{ 22, 23, 24, 28, 29, 30 \},
\end{align*}
\]

and, secondly, the hyperedges of \( \mathcal{H} = (V, \{e_1, e_2, \ldots, e_{17}\}) \) with \( EI(\mathcal{H}) = C_{34} \) follow.

Example 5. Now we have one 5-group \( G_0 = \{e_1, e_2, e_3, e_4, e_5\} \) and three 4-groups \( G_1 = \{e_6, e_7, e_8, e_9\}, G_2 = \{e_{10}, e_{11}, e_{12}, e_{13}\}, G_3 = \{e_{14}, e_{15}, e_{16}, e_{17}\} \).

\[
\begin{align*}
e_1 &= \{ 1, 2, 3, 8, 9, 10 \}, \\
e_2 &= \{ 2, 3, 4, 14, 15, 16 \}, \\
e_3 &= \{ 3, 4, 5, 17, 18, 19 \}, \\
e_4 &= \{ 4, 5, 6, 31, 32, 33 \}, \\
e_5 &= \{ 1, 2, 5, 6, 7, 34 \}, \\
e_6 &= \{ 10, 11, 12, 16, 17, 18 \}, \\
e_7 &= \{ 11, 12, 13, 22, 23, 24 \}, \\
e_8 &= \{ 12, 13, 14, 25, 26, 27 \}, \\
e_9 &= \{ 7, 8, 9, 13, 14, 15 \}, \\
e_{10} &= \{ 18, 19, 20, 24, 25, 26 \}, \\
e_{11} &= \{ 19, 20, 21, 30, 31, 32 \}, \\
e_{12} &= \{ 1, 20, 21, 22, 33, 34 \}, \\
e_{13} &= \{ 15, 16, 17, 21, 22, 23 \}, \\
e_{14} &= \{ 26, 27, 28, 32, 33, 34 \}, \\
e_{15} &= \{ 6, 7, 8, 27, 28, 29 \}, \\
e_{16} &= \{ 9, 10, 11, 28, 29, 30 \}, \\
e_{17} &= \{ 23, 24, 25, 29, 30, 31 \},
\end{align*}
\]

the remaining cases \( n = 36 \) and \( n = 38 \) can be handled analogously.
References