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FETI-DP AND BDDC METHODS WITH A GENERALIZED TRANSFORMATION OF BASIS FOR HETEROGENEOUS PROBLEMS: CONNECTIONS TO DEFLATION

AXEL KLAWONN, MARTIN KÜHN, AND OLIVER RHEINBACH

Abstract. In FETI-DP (Finite Element Tearing and Interconnecting) and BDDC (Balancing Domain Decomposition by Constraints) domain decomposition methods, the convergence behavior of the iterative scheme can be improved by implementing a coarse space using a transformation of basis and local assembly. This is an alternative to coarse spaces implemented by deflation or balancing. In this paper, we show a correspondence of FETI-DP or BDDC methods using a transformation of basis and of FETI-DP methods using deflation or balancing, where the deflation vectors are obtained from the transformation of basis. The methods then have essentially the same eigenvalues. As opposed to existing theory, this result also applies to general scalings and heterogeneous problems including coefficient jumps inside subdomains.

The new methods, however, slightly differ from the classic FETI-DP and BDDC methods using a transformation of basis, and the classic theory has to be replaced. An important application for the theory presented in this paper are FETI-DP and BDDC methods with adaptive coarse spaces, i.e., where deflation vectors are obtained from approximating local eigenvectors. These methods have gained considerable interest, recently.

1. Introduction

1.1. **Main Result.** In this section, we briefly describe our main result without dwelling on the details of notation for which we refer to the later sections of this article. For quite general problems, including heterogeneous elliptic finite element problems and general scalings, we show the equality of the spectra of the deflated preconditioned FETI-DP operator $(I-P)M_D^{-1}(I-P)^TF$ and that of a new preconditioned FETI-DP operator $\widehat{M}_D^{-1}\widehat{F}$ using a new basis constructed, by transformation of basis, from the deflation vectors, i.e.,

$$\sigma((I-P)M_D^{-1}(I-P)^TF) = \sigma(\widehat{M}_D^{-1}\widehat{F}) \tag{1.1}$$

where σ denotes the spectrum. The new algorithm, denoted generalized transformation-of-basis approach, is different from known methods since in the scaling on the subdomain interface an interaction of primal and dual variables can occur. This is possible since we keep Lagrange multipliers corresponding to some primal variables in the FETI-DP system. However, as a result, standard theory does not apply anymore and has to be replaced by (1.1).

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We also show that a corresponding, transformed BDDC preconditioned system also essentially has the same eigenvalues, i.e.,

$$\sigma(\widehat{M}_{\mathrm{BDDC}}^{-1}\mathcal{S}) \setminus \{0,1\} \subset \sigma(\widehat{M}_D^{-1}\widehat{F}) = \sigma(M_{PP}^{-1}F). \tag{1.2}$$

Our results thus generalize the findings from [29] to general heterogeneous problems. The new theory is also of interest for recent adaptive FETI-DP and BDDC methods [36, 37, 7, 25, 26, 21, 22, 2, 40, 4, 20, 41]. The transformation-of-basis approach allows a more flexible treatment of the coarse problem in domain decomposition including inexact solvers which is crucial to obtain scalability to the order of million cores [24] and beyond.

1.2. **State of the Art.** The numerical solution of partial differential equations by finite elements often requires a fine discretization of the given domain in order to obtain a good approximate solution of the original problem. This leads to large sparse linear systems of equations, which often cannot be solved by sparse direct solvers anymore. Instead, iterative methods are used. Domain decomposition methods [46] are widely-used iterative algorithms for the parallel solution of implicit finite element problems. In these methods, the finite element problem is decomposed into independent, parallel local problems. Additionally, to obtain scalability in the number of iterations, a coarse space ensures global transport of information.

Originally, the coarse space of FETI-DP and BDDC domain decomposition methods is formed from coupling the subdomains in a few primal variables, i.e., partial finite element assembly is used to enforce continuity across subdomain boundaries in vertices of subdomains. In three dimensions, this is not sufficient to obtain a good condition number bound. A possible remedy is, in each iteration, to enforce edge averages to be continuous across the interface. A transformation of basis, to explicitly introduce the averages as new variables in combination with subsequent partial finite element assembly in these new variables, is a technique to implement such average constraints in FETI-DP and BDDC methods; see, e.g., [32, 34, 31, 27, 44]. An alternative is the use of the edge characteristic functions as deflation vectors in a deflation or balancing approach; see, e.g., [29, 17].

Many different FETI-DP and BDDC coarse spaces have been developed over time for different problems. For highly heterogeneous problems, e.g., in almost incompressible elasticity with coefficient jumps, automatic or adaptive coarse spaces have been introduced. Here, instead of using coarse spaces defined (and analyzed) a priori, the adaptive (i.e., problem-specific) coarse spaces are constructed automatically, during the computation [36, 37, 7, 25, 26, 21, 22, 2, 40, 4, 20, 41].

It has been shown in [29, 17] for homogeneous problems (and multiplicity scaling) that for any FETI-DP method with a transformation of basis there exists a FETI-DP method using deflation (also known as projector preconditioning) or balancing with essentially the same spectrum [29, Theorem 6.7]. Note that the reverse is not always true, notably, the FETI-DP method using deflation in [15] for almost incompressible elasticity (which uses sums of edge averages as constraints) cannot be implemented using the transformation-of-basis approach.

In the present paper, a corresponding relation is proven also for heterogeneous problems with jumps along and across subdomain boundaries. It is therefore of interest for domain decomposition methods with adaptive coarse spaces. Here, the combination of a generalized transformation-of-basis approach with general scalings is required. We will revisit the standard theory and then discuss, how the transformed P_D operator is defined (see (5.11)), which is central to the theory. We then show that the corresponding methods using a transformation of basis, deflation, or balancing have essentially the same spectrum; see Theorem 6.7. Potential advantages of approaches using a transformation of basis are high robustness when machine precision is approached (see [28]) and the possibility to solve the coarse space inexactly.

The transformation-of-basis approaches presented here slightly differ from the classic approach. The reason is that, in deflation, whenever a nondiagonal scaling (e.g., deluxe) is used, an interaction between deflated and non-deflated variables can result. Such an interaction is not present in the standard transformation-of-basis approach as in [32, 34, 31, 27]. This is relevant even for diagonal scalings, since a transformation of basis for heterogeneous problems will, in general, result in a nondiagonal scaling; see (7.8). An example will briefly be discussed in the next section. In some cases, e.g., for multiplicity scaling or for scalings which are constant on edges or faces, the new transformed scaling is identical to the initial scaling (cf. [29]) but this is not true in general; see also Definition 7.1. Even if the original scaling is diagonal (as is the case with ρ-scaling) the transformed scaling is, in general, not diagonal. For FETI-DP, we consider a transformation of basis in the space of Lagrange multipliers; see Lemma 5.1. We also need an important identity (Lemma 6.3) to obtain the desired equality of the essential eigenvalues (see Theorem 6.7) for heterogeneous coefficients.

The focus of the present paper is thus to construct, for heterogeneous problems, a (generalized) transformation-of-basis approach for FETI-DP and BDDC with full theoretical justification and to show the equivalence to a corresponding FETI-DP method with a deflation or balancing approach. It is natural to combine the approaches presented here with adaptive coarse spaces, e.g., our approach from [22] or the many other adaptive coarse space approaches developed, recently. This, however, is out of the scope of this paper.

The remainder of the paper is structured as follows. In the next section, we will give a short motivation for the need of the generalized transformation-of-basis approach. In Section 3, we will introduce our elliptic model problem and introduce the underlying finite element geometry used in the domain decomposition approach. In Section 4, we will present the standard FETI-DP and BDDC algorithm with a transformation of basis and deflation or balancing. In Section 5, we introduce a generalized variant of the transformation-of-basis approach for arbitrary scalings, coefficient distributions, and constraint vectors. In Section 6, we will finally prove that FETI-DP with the generalized transformation of basis has the same nontrivial spectrum as FETI-DP with a deflation or balancing approach. The corresponding theory and a short discussion on how to apply the generalized transformation of basis for BDDC follows in Section 7. In Section 8, we will present some results to visualize our theoretical findings. Eventually, we will draw a conclusion in Section 9.

2. Motivation of a Generalized Transformation-of-Basis Approach for General Scalings and Arbitrary Constraints

2.1. **Context and Nomenclature.** When FETI-DP and BDDC methods are combined with deflation or balancing, typically, an initial coarse space is defined, which introduces sufficient coupling to obtain invertibility of the subdomain problems. A simple vertex coarse space can suffice as an initial coarse space. At the same time, an initial scaling is chosen; see, e.g., Section 5.1. For heterogeneous problems, the scaling used in the preconditioner (i.e., ρ-scaling, deluxe scaling, etc.) is an important ingredient to obtain a robust iterative method. Then, a second coarse space is implemented by deflation or balancing to obtain fast convergence; see, e.g., [29, 17]. In this paper, we will denote the first coarse space also as "a priori coarse space" and the second coarse space, defined after the scaling, as "a posteriori coarse space".

In adaptive FETI-DP and BDDC methods [36, 37, 7, 25, 26, 21, 22, 2, 40, 4, 20, 41], the a posteriori coarse space is highly dependent on the a priori scaling, i.e., the computation of the approximate eigenvectors for the a posteriori coarse space makes use of the a priori scaling. Indeed, the choice of an inappropriate a priori scaling (e.g., use of multiplicity scaling for heterogeneous problems) will lead to an (unnecessary) large a posteriori coarse space. This can be observed, e.g., in [26].

Let us denote our a priori coarse space by the index set Π' and the corresponding index set of dual variables by Δ'. We will also denote the index set Π' as "a priori set of primal variables". After a transformation of basis, we will use additional partial finite element assembly in the index set Π (our a posteriori coarse space), i.e., the final set of primal variables is $\Pi' \cup \Pi$. The index set of the final (or remaining) dual variables is therefore $\Delta = \Delta' \setminus \Pi$. We will denote the index set Π as "a posteriori set of primal variables".

Let us briefly illustrate the difference between the a priori and a posteriori coarse space. Let us assume that after an initial coarse space and a scaling D have been defined, based on the use of this scaling, a deflation vector $c_D := c(D)$ has been chosen to further accelerate convergence. As an example, c_D could be defined by the solution of local eigenvalue problems, where the eigenvalue problems are defined using the scaling, as it is done in domain decomposition methods with adaptive coarse spaces. The deflated method using the constraint $c_D^T Bw = 0$ may allow the construction of a bound

$$|P_D w|^2_{\widetilde{S}} \leq C|w|^2_{\widetilde{S}} \quad \forall w \in \{w \in \widetilde{W} \,:\, c_D^T Bw = 0\}; \tag{2.1}$$

cf. Section 4. However, the estimate (2.1) depends on the use of the scaling D in combination with the constraint $c_D^T Bw = 0$ and may not be valid anymore if a different scaling $\widetilde{D}$ is used.

Therefore, for the theory in this paper, we will start with the original operator P_D; cf. (5.11); see Section 2. Since the estimate (2.1) can be considered as an estimate for the deflated FETI-DP method, we will argue in this paper, that a generalized transformation of basis has to be used in order to guarantee a corresponding estimate as for the transformed and assembled operators.

We show how the scaling has to be transformed for heterogeneous problems for BDDC; see Definition 7.1. For FETI-DP, this is done implicitly; see (5.13). This is different from the homogeneous context in [29], where it was shown that the scaling in the new basis is identical to that of the old basis. It is an important consequence

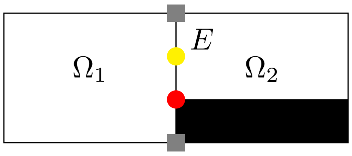

Figure 1. Decomposition of $\Omega = [0,1] \times [0, \frac{1}{2}]$ into two subdomains Ω_1, Ω_2 with homogeneous Dirichlet boundary conditions on $\partial\Omega$ and given coefficient distribution (Top left and bottom left). A non-homogeneous coefficient distribution with $\rho_1 = 1$ (white) and $\rho_2 = 1e6$ (black) is considered. Initial primal variables (Π') are indicated by gray squares. Initial dual variables are indicated by circles. The red circle represents an a posteriori primal variable (Π'), i.e., here, we enforce a scaling-dependent constraint by a transformation of basis. The yellow circle represents the remaining dual variable.

that, after transformation, a diagonal scaling may not be diagonal anymore. This occurs for non-nodal degrees of freedom, e.g., edge averages. For nodal degrees of freedom an interaction between dual and primal variables results when a non-diagonal scaling, e.g., deluxe scaling, is used. The interaction between dual and primal variables is not the present in classical theory, and a standard argument used in the classical theory, i.e., that iterates are zero in the primal variables, cannot be used, anymore. In our theory, the use of Lemma 6.3 and Lemma 6.4 replaces this standard argument. Constructing the scaling for the transformed displacements only in the remaining dual variables will, in general, not give the desired result.

This discussion is relevant for FETI-DP and BDDC methods with adaptive coarse spaces where first a scaling is chosen (e.g., ρ- or deluxe-scaling) and then a coarse space is constructed based on this scaling. We therefore believe that this is also of interest for the analysis of other adaptive approaches [2, 40, 4, 20]. We will discuss the implications for adaptive FETI-DP and BDDC methods in detail elsewhere.

Let us now revisit the classical theory considering an example where the assumption of diagonal and constant scaling for the equivalence class elements (as assumed in [29]) is not fulfilled anymore.

2.2. **Example.** We now show that the transformation approach can lead to nonzero values in non-nodal primal variables, even for a diagonal scaling.

Consider the edge E with nodes Π (red circle) and Δ (yellow circle) between the two subdomains Ω_1 and Ω_2 as depicted in Figure 1.

The ρ-scaling for the degrees of freedom for the nodal basis is given by

$$D_u^{(1)} = \operatorname{diag}\left(\frac{1}{1+1e6}, \frac{1}{2}\right), \text{ and } D_u^{(2)} = I - D_u^{(1)}.$$

Then, assuming a (nondiagonal) transformation T of the form

$$T = \begin{pmatrix} t_{rr} & t_{ry} \\ t_{yr} & t_{yy} \end{pmatrix} \tag{2.2}$$

with $t_{yr}, t_{ry} \neq 0$, $T^T T = I$, and where the first column is given by a (scaling dependent) constraint vector $c_D^T = [t_{rr}, t_{yr}]^T$ enabling an estimate such as (2.1). The constraint is scaling-dependent in the way that the estimate can only be obtained when the constraint is used in combination with the given scaling. This is, e.g., typically the case in methods with adaptive coarse spaces. If the scaling in the adaptive methods is changed, e.g., by restriction (see below), the condition number bound of the adaptive method may not be valid anymore.

The indices y and r in (2.2) denote the relation to the nodes colored yellow (y) and red (r) in Figure 1.

In the new basis, the transformed ρ-scaling (see Definition 7.1) is

$$\widehat{D}_u^{(1)} = \begin{pmatrix} \frac{1}{1+1e6} t_{rr}^2 + \frac{1}{2} t_{ry}^2 & \frac{1}{1+1e6} t_{rr} t_{yr} + \frac{1}{2} t_{ry} t_{yy} \\ \frac{1}{1+1e6} t_{rr} t_{yr} + \frac{1}{2} t_{ry} t_{yy} & \frac{1}{1+1e6} t_{yr}^2 + \frac{1}{2} t_{yy}^2 \end{pmatrix} =: \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$\text{and } \widehat{D}_u^{(2)} = I - \widehat{D}_u^{(1)} = \begin{pmatrix} 1-\alpha & -\beta \\ -\gamma & 1-\delta \end{pmatrix}.$$

After enforcing continuity in the a posteriori primal variable, we have with $w_\Pi = w_{1,\Pi} = w_{2,\Pi}$,

$$\begin{aligned} (P_{\widehat{D}} w)_\Pi^{(1)} = ((I - E_{\widehat{D}_u}) w)_\Pi^{(1)} &= w_\Pi - (\alpha w_\Pi + \beta w_{1,\Delta} + (1-\alpha) w_\Pi - \beta w_{2,\Delta}) \\ &= \beta (w_{1,\Delta} - w_{2,\Delta}). \end{aligned}$$

Since in general $\beta \neq 0$ and $w_{1,\Delta} \neq w_{2,\Delta}$, we obtain, in general, a nonzero value in the a posteriori primal variables after $P_{\widehat{D}} = I - \widehat{E}_{D_u}$ is applied; this is contrary to the assumptions of the standard theory. The interaction of a posteriori primal and dual variables was also observed in [20].

Neither the use of the standard scaling $(D_u^{(1)})_\Delta = \frac{1}{2}$ nor the transformed and restricted scaling $(\widehat{D}_u^{(1)})_\Delta = \frac{1}{1+1e6} t_{yr}^2 + \frac{1}{2} t_{yy}^2$ are adequate, here.

3. Model Problem and Geometry

Given a bounded polyhedral domain $\Omega \subset \mathbb{R}^3$ where the Dirichlet boundary $\partial\Omega_D \subset \partial\Omega$ is a closed subset of positive surface measure and $\partial\Omega_N := \partial\Omega \setminus \partial\Omega_D$, we will consider the weak formulation of an elliptic problem

$$a(u, v) = F(v) \quad \forall v \in H_0^1(\Omega, \partial\Omega_D).$$

The domain Ω will be decomposed into N nonoverlapping open subdomains Ω_i, $i = 1, \ldots, N$. The interface Γ is defined as the union of the interior subdomain boundaries, i.e., $\Gamma := \{x \in \overline{\Omega}_i \cap \overline{\Omega}_j;\ i \neq j\}$. Then, the subdomains are each triangulated and then discretized by the finite element method with matching nodes on the interface. For simplicity, we use piecewise linear conforming finite elements.

In three dimensions, the interface can be decomposed into vertices, edges, and faces; for a detailed definition of these sets; cf. [31]. Edges and faces are defined as open sets. A face shared by two arbitrary subdomains Ω_i and Ω_j will be denoted by $\mathcal{F}^{ij}$ while we denote edges shared by Ω_i, Ω_j, Ω_l and possibly more subdomains by $\mathcal{E}^{il}$. Vertices of Ω_i that belong to multiple subdomains are denoted by $\mathcal{V}^{ik}$.

By $W^h(\Omega_i)$, $i = 1, \ldots, N$, we denote the local finite element space on Ω_i. The local trace space $W_i := W^h(\Gamma_i)$ is defined on $\Gamma_i := \overline{\Omega_i} \cap \Gamma$. We also introduce the global product space $W := \Pi_{i=1}^N W_i$, and denote the space of functions that are continuous on the interface by $\widehat{W} \subset W$.

4. FETI-DP and BDDC with a Transformation of Basis and FETI-DP with Deflation and Balancing

To make this paper self-contained, in this section, we recall standard FETI-DP and BDDC methods with a transformation of basis and FETI-DP with a deflation or balancing approach. Note that the deflation or balancing approach for BDDC is different from the BDDC method with a transformation of basis and is therefore not described; see [29].

Transformation of basis, deflation or balancing are approaches to add additional constraints to the coarse space of the underlying FETI-DP or BDDC domain decomposition method. Another possibility is the use of optional Lagrange multipliers; see [12, 16, 35, 31, 36]. For more details on FETI-DP and BDDC methods; see, e.g., [46, 12, 13, 6, 8, 14] and, e.g., [32, 34, 31, 27] for the use of a transformation of basis in FETI-DP and BDDC as well as, e.g., [39, 10, 11, 38, 29, 17] for FETI-DP with deflation or balancing.

4.1. **Standard FETI-DP and BDDC Methods.** We partition the set of displacement degrees of freedom into interior, dual, and primal degrees of freedom, denoted by an index I, Δ', and Π', respectively. Interior displacement degrees of freedom belong to nodes in the interior of subdomains and on the Neumann boundary $\partial\Omega_N$. Dual and primal displacement degrees of freedom belong to nodes on the interface Γ, and every node on Γ can be classified as either one of them.

For every subdomain $\Omega_i, i = 1, \ldots, N$, we assemble the local stiffness matrix $K^{(i)}$ and the local load vector $f^{(i)}$. The displacements u are partitioned into interior degrees of freedom u_I, dual degrees of freedom $u_{\Delta'}$, and primal degrees of freedom $u_{\Pi'}$. Assuming an appropriate ordering of the degrees of freedom, we obtain the following partitioning of the local stiffness matrices, displacement and load vectors

$$K^{(i)} = \begin{bmatrix} K_{II}^{(i)} & K_{\Delta' I}^{(i)T} & K_{\Pi' I}^{(i)T} \\ K_{\Delta' I}^{(i)} & K_{\Delta'\Delta'}^{(i)} & K_{\Pi'\Delta'}^{(i)T} \\ K_{\Pi' I}^{(i)} & K_{\Pi'\Delta'}^{(i)} & K_{\Pi'\Pi'}^{(i)} \end{bmatrix}, u^{(i)} = \begin{bmatrix} u_I^{(i)} \\ u_{\Delta'}^{(i)} \\ u_{\Pi'}^{(i)} \end{bmatrix}, \text{ and } f^{(i)} = \begin{bmatrix} f_I^{(i)} \\ f_{\Delta'}^{(i)} \\ f_{\Pi'}^{(i)} \end{bmatrix}.$$

Further, we introduce the following matrices and vectors

$$K_{BB}^{(i)} := \begin{bmatrix} K_{II}^{(i)} & K_{\Delta' I}^{(i)T} \\ K_{\Delta' I}^{(i)} & K_{\Delta'\Delta'}^{(i)} \end{bmatrix}, K_{\Pi' B}^{(i)} := \begin{bmatrix} K_{\Pi' I}^{(i)} \, K_{\Pi'\Delta'}^{(i)} \end{bmatrix}, \text{ and } f_B^{(i)} := \begin{bmatrix} f_I^{(i)T} \, f_{\Delta'}^{(i)T} \end{bmatrix}^T$$

$$\text{as well as} \quad K_{\Gamma\Gamma}^{(i)} := \begin{bmatrix} K_{\Delta'\Delta'}^{(i)} & K_{\Pi'\Delta'}^{(i)T} \\ K_{\Pi'\Delta'}^{(i)} & K_{\Pi'\Pi'}^{(i)} \end{bmatrix} \quad \text{and} \quad K_{\Gamma I}^{(i)T} := \begin{bmatrix} K_{\Delta' I}^{(i)T} \, K_{\Pi' I}^{(i)T} \end{bmatrix}.$$

Next, we define the global block matrices

$$K_{II} := \mathrm{diag}_{i=1}^N K_{II}^{(i)}, \; K_{\Delta'\Delta'} := \mathrm{diag}_{i=1}^N K_{\Delta'\Delta'}^{(i)}, \; \text{ and } \; K_{\Pi'\Pi'} := \mathrm{diag}_{i=1}^N K_{\Pi'\Pi'}^{(i)},$$
$$\text{as well as} \quad K_{BB} := \mathrm{diag}_{i=1}^N K_{BB}^{(i)}, \; K_{\Gamma\Gamma} := \mathrm{diag}_{i=1}^N K_{\Gamma\Gamma}^{(i)},$$

and $K_{\Gamma I}$ the corresponding global off-diagonal block.

We also need assembly operators. The first, $R_{\Pi'}^T := \left[R_{\Pi'}^{(1)T}, \ldots, R_{\Pi'}^{(N)T}\right]$, performs the assembly in the primal variables $u_{\Pi'}^{(i)}$ and is needed for FETI-DP and BDDC. The second, $R_{\Delta'}^T := \left[R_{\Delta'}^{(1)T}, \ldots, R_{\Delta'}^{(N)T}\right]$, performs assembly in the dual variables $u_{\Delta'}^{(i)}$ and is only needed for BDDC. The transposed operators $R_{\Pi'}$ and $R_{\Delta'}$ then are extension operators that distribute global information to the local subdomains.

For FETI-DP, in place of $R_{\Delta'}$, we need a jump operator $B_\Gamma = \left[B_\Gamma^{(1)}, \ldots, B_\Gamma^{(N)}\right]$ which is built from values 0 and ± 1 such that $B_\Gamma u = 0$ holds for $u \in \widehat{W}$.

By assembly in the primal variables, we obtain

$$\widetilde{K}_{\Pi'\Pi'} = \sum_{i=1}^{N} R_{\Pi'}^{(i)T} K_{\Pi'\Pi'}^{(i)} R_{\Pi'}^{(i)}, \quad \widetilde{K}_{\Pi' B} = \left[R_{\Pi'}^{(1)T} K_{\Pi' B}^{(1)}, \ldots, R_{\Pi'}^{(N)T} K_{\Pi' B}^{(N)} \right],$$

$$\widetilde{f} = \left[f_B^T, (\sum_{i=1}^{N} R_{\Pi'}^{(i)T} f_{\Pi'}^{(i)})^T \right]^T, \text{ and } \widetilde{S}_{\Pi'\Pi'} = \widetilde{K}_{\Pi'\Pi'} - \widetilde{K}_{\Pi' B} K_{BB}^{-1} \widetilde{K}_{\Pi' B}^T.$$

Then, the FETI-DP master system is given by

$$\begin{bmatrix} K_{BB} & \widetilde{K}_{\Pi' B}^T & B_B^T \\ \widetilde{K}_{\Pi' B} & \widetilde{K}_{\Pi'\Pi'} & 0 \\ B_B & 0 & 0 \end{bmatrix} \begin{bmatrix} u_B \\ \widetilde{u}_{\Pi'} \\ \lambda \end{bmatrix} = \begin{bmatrix} \widetilde{f} \\ 0 \\ 0 \end{bmatrix}.$$

Here, B_B is the nontrivial part of B, and is has the form $B_B = \left[B_B^{(1)}, \ldots, B_B^{(N)} \right]$. The (unpreconditioned) FETI-DP system

$$F\lambda = d$$

is obtained after elimination of $\widetilde{u}^T = [u_B^T,\ \widetilde{u}_{\Pi'}^T]^T$, where

$$\begin{aligned} F &= \begin{bmatrix} B_B & 0 \end{bmatrix} \begin{bmatrix} K_{BB} & \widetilde{K}_{\Pi' B}^T \\ \widetilde{K}_{\Pi' B} & \widetilde{K}_{\Pi'\Pi'} \end{bmatrix}^{-1} \begin{bmatrix} B_B^T \\ 0^T \end{bmatrix} \\ &= B_B K_{BB}^{-1} B_B^T + B_B K_{BB}^{-1} \widetilde{K}_{\Pi' B}^T \widetilde{S}_{\Pi'\Pi'}^{-1} \widetilde{K}_{\Pi' B} K_{BB}^{-1} B_B^T = B_\Gamma \widetilde{S}^{-1} B_\Gamma^T, \end{aligned} \tag{4.1}$$

$$d = B_B K_{BB}^{-1} f_B + B_B K_{BB}^{-1} \widetilde{K}_{\Pi' B}^T \widetilde{S}_{\Pi'\Pi'}^{-1} \Big((\sum_{i=1}^{N} R_{\Pi}^{\prime(i)T} f_{\Pi}^{\prime(i)}) - \widetilde{K}_{\Pi' B} K_{BB}^{-1} f_B \Big). \tag{4.2}$$

For heterogeneous problems, the scaling is an important ingredient of FETI-DP and BDDC methods. It is used in the corresponding preconditioners. We will now introduce two kind of commonly used scalings.

First, we introduce the standard ρ-scaling; see, e.g., [45, 30, 46, 28, 42]. For $x \in \Gamma_i$ we introduce $\mathcal{N}_x$ as the set of indices of subdomains that have x on their boundaries. We define the coefficient evaluation by $\widehat{\rho}_i(x) := \sup_{x \in \text{supp}(\varphi_x) \cap \Omega_i} \rho(x)$. There, φ_x is the nodal finite element function at x and $\text{supp}(\varphi_x)$ its support. Let Ω_j and Ω_i share either a face or an edge and let $x \in \partial\Omega_i \cap \partial\Omega_j$. The corresponding nontrivial row of $B^{(j)}$ is then multiplied by the scaling $\delta_i^\dagger(x) := \widehat{\rho}_i(x) / \sum_{k \in \mathcal{N}_x} \widehat{\rho}_k(x)$ and vice versa. By this, we obtain the local scaling $D^{(j)}$. In BDDC the degrees of freedom on $\partial\Omega_i$ are scaled by $\delta_i^\dagger(x)$. This defines the scaling $D_u^{(i)}$; see (4.5).

Secondly, we introduce deluxe scaling; see, e.g, [9, 7, 1, 5, 3]. Therefore, the Schur complement of the stiffness matrix on the interface Γ, i.e., $S_{\Gamma\Gamma} := \text{diag}_{i=1}^N S^{(i)} := \text{diag}_{i=1}^N \big(K_{\Gamma\Gamma}^{(i)} - K_{\Gamma I}^{(i)} \big(K_{II}^{(i)} \big)^{-1} K_{\Gamma I}^{(i)T} \big)$, has to be defined. Let us either consider a face $\mathcal{F}^{ij}$ shared by the two subdomains Ω_i and Ω_j or an edge $\mathcal{E}^{ik}$ shared by the subdomains Ω_i, Ω_j, Ω_k. Multiplicities greater than three can be handled analogously.

Define for $l \in \{i, j\}$ the matrix $S_{F_{ij},0}^{(l)}$ as the restriction of $S^{(l)}$ to the (open) face $\mathcal{F}^{ij}$ and for $l \in \{i, j, k\}$. Further, define $S_{E_{ij},0}^{(l)}$ as the restriction of $S^{(l)}$ to the (open) edge $\mathcal{E}^{ik}$.

For a face $\mathcal{F}^{ij}$, in deluxe scaling the nontrivial rows of $B^{(j)}$ corresponding to the Lagrange multipliers on this face are multiplied by $D_{F_{ij}}^{(i)} = (S_{F_{ij},0}^{(i)} + S_{F_{ij},0}^{(j)})^{-1} S_{F_{ij},0}^{(i)}$

if the orientation of the constraints in B are chosen consistently. Otherwise, entries of $D^{(i)}_{F_{ij}}$ have to be scaled by -1.

For an edge $\mathcal{E}^{ik}$, in deluxe scaling the nontrivial rows of $B^{(j)}_{\Gamma}$ corresponding to the Lagrange multipliers coupling Ω_i and Ω_j on this edge are multiplied by $D^{(i)}_{E_{ik}} = (S^{(i)}_{E_{ik},0} + S^{(j)}_{E_{ik},0} + S^{(k)}_{E_{ik},0})^{-1} S^{(i)}_{E_{ik},0}$. Again, a consistent orientation of the Lagrange multipliers is assumed. The rows of $B^{(j)}_{\Gamma}$ corresponding to Lagrange multipliers coupling Ω_j and Ω_k on $\mathcal{E}^{ik}$ and the rows of $B^{(i)}_{\Gamma}$ and $B^{(k)}_{\Gamma}$ are scaled anologously. The final scaling matrix $D^{(j)}$ is obtained from the local scaling matrices on the equivalence class elements of Ω_j. In BDDC, the degrees of freedom on $\mathcal{F}^{ij}$ and $\mathcal{E}^{ik}$ of $\partial\Omega_i$ are scaled by $D^{(i)}_{F_{ij}}$ and $D^{(i)}_{E_{ik}}$, respectively. This defines the scaling $D^{(i)}_u$ for the BDDC preconditioner; see (4.5).

Let us note that for the case of ρ-scaling, we have $D^{(i)T} = D^{(i)}$ for $i = 1, \dots, N$ while deluxe scaling is not symmetric.

The standard preconditioner for the FETI-DP system is the Dirichlet preconditioner; see [46]. We introduce the restriction operator R_Γ, which restricts a vector $\tilde{u}$ to the interface by removing values on interior nodes and its transpose R_Γ^T, which extends a vector u_Γ by zero.

For both kinds of scalings, depending on the definition of the local scalings, we obtain the scaled jump operator

$$B_{D,\Gamma} = \left[B^{(1)}_{D,\Gamma}, \dots, B^{(N)}_{D,\Gamma}\right] = \left[D^{(1)T} B^{(1)}_{\Gamma}, \dots, D^{(N)T} B^{(N)}_{\Gamma}\right]. \tag{4.3}$$

The standard Dirichlet preconditioner then writes (see [46])

$$M_D^{-1} := B_{D,\Gamma} R_\Gamma^T S_{\Gamma\Gamma} R_\Gamma B_{D,\Gamma}^T = B_{D,\Gamma} \widetilde{S} B_{D,\Gamma}^T. \tag{4.4}$$

For BDDC, we will use an ordering according to $\left[u^{(1)}_{\Delta'}, \dots, u^{(N)}_{\Delta'}, u^{(1)}_{\Pi'}, \dots, u^{(N)}_{\Pi'}\right]$ instead of $\left[u^{(1)}_{\Delta'}, u^{(1)}_{\Pi'}, \dots, u^{(N)}_{\Delta'}, u^{(N)}_{\Pi'}\right]$. First, we need $\mathcal{K}_{\Gamma\Gamma}$ and $\mathcal{K}_{\Gamma I}$ to define the Schur complement on the interface Γ,

$$\mathcal{S}_{\Gamma\Gamma} := \mathcal{K}_{\Gamma\Gamma} - \mathcal{K}_{\Gamma I} K_{II}^{-1} \mathcal{K}_{\Gamma I}^T.$$

The right hand side is then given by $[g_{\Delta'}^T,\ g_{\Pi'}^T]^T$, which is obtained by the corresponding elimination of the interior degrees of freedom from f. The BDDC system matrix is the fully assembled global Schur complement, which can also be written

$$\mathcal{S} := \begin{bmatrix} R_{\Delta'}^T & 0 \\ 0 & I_{\Pi'} \end{bmatrix} \begin{bmatrix} I_{\Delta'} & 0 \\ 0 & R_{\Pi'}^T \end{bmatrix} \mathcal{S}_{\Gamma\Gamma} \begin{bmatrix} I_{\Delta'} & 0 \\ 0 & R_{\Pi'} \end{bmatrix} \begin{bmatrix} R_{\Delta'} & 0 \\ 0 & I_{\Pi'} \end{bmatrix}.$$

The BDDC system for $u_\Gamma^T = [u_{\Delta'}^T,\ u_{\Pi'}^T]^T$ with the corresponding right hand side $g^T = [R_{\Delta'} g_{\Delta'}^T,\ R_{\Pi'} g_{\Pi'}^T]^T$ writes

$$\mathcal{S} u_\Gamma = g.$$

Using the scaling D_u, we can define the scaled assembly operator R^T_{Δ',D_u} acting on the dual degrees of freedom Δ'. The standard BDDC preconditioner then reads, in our notation,

$$M_{\mathrm{BDDC}}^{-1} := \begin{bmatrix} R^T_{\Delta',D_u} R_\Gamma & 0 \\ 0 & I_{\Pi'} \end{bmatrix} \begin{bmatrix} K_{BB} & \widetilde{K}_{\Pi B}^T \\ \widetilde{K}_{\Pi B} & \widetilde{K}_{\Pi\Pi} \end{bmatrix}^{-1} \begin{bmatrix} R_\Gamma^T R_{\Delta',D_u} & 0 \\ 0 & I_{\Pi'} \end{bmatrix}. \tag{4.5}$$

4.2. **Transformation of Basis in FETI-DP and BDDC methods.** Let us now briefly recall the standard approach of the transformation of basis for FETI-DP and BDDC and constant scaling on all equivalence classes' elements, e.g., multiplicity scaling or ρ-scaling for certain coefficient distributions; see, e.g., [32, 34, 31, 27, 29]. Using partial finite element assembly, continuity across the subdomain boundary of certain displacement degrees of freedom can be enforced for the finite element function. However, using a transformation of basis from a nodal to a different basis, also general constraints can be enforced using the same technique.

Consider an edge E shared by Ω_i, Ω_j and other subdomains. Suppose that, during the Krylov iteration, the iterates u should fulfill a constraint given by the normalized vector c defined on $\partial\Omega_i \cap E$ (and equal to that on $\partial\Omega_j \cap E$), i.e., it holds

$$c^T \left(u_E^{(i)} - u_E^{(j)} \right) = 0 \quad \Leftrightarrow \quad c^T u_E^{(i)} = c^T u_E^{(j)} \tag{4.6}$$

for $u_E^{(l)} = u_{|\partial\Omega_l \cap E}$, $l \in \{i, j\}$. For instance, $c = \frac{1}{n_E}(1, \ldots, 1)^T$ means a continuous edge average shared by Ω_i and Ω_j, where n_E is the length of c. This corresponds to the use of a non-nodal basis function.

We refer to equations such as (4.6) as *constraint*, while we denote c a *constraint vector*. We then define a (square) transformation matrix

$$T_E^{(l)} = \left[c,\, C^{(l)\perp} \right], \quad l \in \{i, j\}$$

where $C^{(l)\perp}$ is computed such that $T_E^{(l)}$ is orthogonal. We then define the transformation matrix $T^{(l)}$, which acts on the complete subdomain, and which is identical to $T_E^{(l)}$ on the edge E and identity elsewhere. We obtain the transformed variables $\overline{u}_E^{(l)}$, stiffness matrices $\overline{K}^{(l)}$, load vectors $\overline{f}^{(l)}$ on Ω_l as

$$\overline{K}^{(l)} = T^{(l)T} K^{(l)} T^{(l)}, \quad \overline{u}^{(l)} = T^{(l)T} u^{(l)}, \quad \overline{f}^{(l)} = T^{(l)} f^{(l)}, \quad l \in \{i, j\}.$$

After the transformation of basis has been performed, assembly in the new (a posteriori) primal variables is used to enforce the given constraint.

This procedure indeed enforces our original constraint corresponding to c as follows. We have

$$c^T u_E^{(l)} = c^T \left[c,\, C^{(l)\perp} \right] \overline{u}_E^{(l)} = \overline{u}_{E,1}^{(l)}, \quad l \in \{i, j\}$$

where $\overline{u}_{E,1}^{(l)}$ is the displacement at the first degree of freedom on the edge E. Let us now identify the variables $\overline{u}_{E,1}^{(i)}$ and $\overline{u}_{E,1}^{(j)}$ by using partial finite element assembly in this degree of freedom. We denote this degree of freedom as $\widehat{u}_E^{(l)}$ with $\widehat{u}_{E,1} := \widehat{u}_{E,1}^{(i)} := \widehat{u}_{E,1}^{(j)} := \frac{1}{2}\overline{u}_{E,1}^{(i)} + \frac{1}{2}\overline{u}_{E,1}^{(j)}$ on the first degree of freedom and $\widehat{u}_{E,k}^{(l)} = \overline{u}_{E,k}^{(l)}$ for $k > 1$. For the values transformed back to the initial basis, we now see by

$$c^T T_E^{(i)} \widehat{u}_E^{(i)} = \frac{1}{2}\overline{u}_{E,1}^{(i)} + \frac{1}{2}\overline{u}_{E,1}^{(j)} = c^T T_E^{(j)} \widehat{u}_E^{(j)}$$

that the constraint is enforced.

4.3. **Deflation and Balancing in FETI-DP methods.** In this section, we briefly explain the deflation and the balancing approach for FETI-DP. Our presentation is based on [29, Section 2 and Section 5] and [22, Section 4]. Note that the deflation or balancing approach for BDDC is different from the BDDC method with a transformation of basis and is therefore not described; see [29].

Deflation (see [39]) is also known as projector preconditioning; see [10]. Deflation (Projector Preconditioning) and Balancing approaches are used in order to add additional constraints to the FETI-DP coarse space.

A short introduction to deflation and balancing in the context of FETI-DP and domain decomposition methods can be found in [39, 10, 11, 38, 29, 17] and the references therein.

In the following, for a matrix A, by A^+ we denote an arbitrary pseudoinverse satisfying $AA^+A = A$ and $A^+AA^+ = A^+$.

The following description is the extension of [29] for the case of a semidefinite matrix F; it can be found in detail in [22]. Let $U = (u_1, \dots, u_s)$ be given as the matrix where the constraint vectors u_i, $i = 1, \dots, s$, for the Lagrange multipliers are stored as columns. Then, we define

$$P := U(U^T FU)^+ U^T F \tag{4.7}$$

and multiply the FETI-DP system by $(I - P)^T$. This yields the deflated system

$$(I - P)^T F\lambda = (I - P)^T d \tag{4.8}$$

which is still consistent. Since range $U \subset \ker((I - P)^T F)$, we have range $(F(I - P)) \subset \ker U^T$ also for a semidefinite matrix F. Since $(I - P)^T$ is also a projection, we have

$$(I - P)^T F = F(I - P) = (I - P)^T F(I - P) \tag{4.9}$$

such that only components of the dual variable in range $(I - P)$ are relevant to the construction of the Krylov spaces. Let λ^* denote the solution of the original system $F\lambda = d$, which is unique only up to an element in $\ker B^T$. Let $\hat{\lambda} \in$ range $(I - P)$ be a solution of (4.8). Then, $\hat{\lambda}$ is identical to $(I - P)\lambda^*$ up to an element in $\ker B^T$. We have the decomposition

$$\lambda^* = P\lambda^* + (I - P)\lambda^* =: \bar{\lambda} + (I - P)\lambda^*,$$

where $\bar{\lambda}$ can be expressed by $\bar{\lambda} = P\lambda^* = U(U^T FU)^+ U^T FF^+F\lambda^* = PF^+d$. As already argued in [22], the solution in terms of the displacements does not change if $(I - P)\lambda^*$ is replaced by $\hat{\lambda}$, i.e.,

$$u_\Delta = \widetilde{S}^{-1}\left(\widetilde{f}_\Delta - B^T\lambda^*\right) = \widetilde{S}^{-1}\left(\widetilde{f}_\Delta - B^T(\bar{\lambda} + \hat{\lambda})\right).$$

Preconditioning the deflated system of equations by the Dirichlet preconditioner M_D^{-1} defined in (4.4) gives

$$M_D^{-1}(I - P)^T F\lambda = M_D^{-1}(I - P)^T d.$$

Another multiplication with $(I-P)$ from the left gives the (symmetric) deflation or projector preconditioner

$$M_{PP}^{-1} := (I - P)M_D^{-1}(I - P)^T. \tag{4.10}$$

As shown in [29, Theorem 6.1], the nonzero eigenvalues are not changed. The deflated and preconditioned problem then writes: Find $\lambda \in$ range $(I - P)$, such that

$$M_{PP}^{-1}F\lambda = M_{PP}^{-1}d.$$

Instead of computing $\bar{\lambda}$ a posteriori, the computation can be executed iteratively in the (iterative) solver. This gives the balancing preconditioner $M_{BP}^{-1} := M_{PP}^{-1} + PF^+$.

As alredy stated in [22] the balancing preconditioner for a semidefinite matrix F is of the form $M_{BP}^{-1} = M_{PP}^{-1} + U(U^T FU)^+ U^T FF^+$ but we can equivalently use

$$M_{BP}^{-1} = M_{PP}^{-1} + U(U^T FU)^+ U^T$$

since it will be applied to $F\lambda = d$.

Using [29] and [22], we obtain that the eigenvalues of $M_{BP}^{-1}F$ and $M_{PP}^{-1}F$ are essentially the same.

5. Correspondence of FETI-DP with a Generalized Transformation of Basis to FETI-DP using Deflation or Balancing

In this section, we consider the FETI-DP method in the transformed variables. For convenience, we order the primal variables first.

We will show that for every FETI-DP or BDDC method with a generalized transformation a corresponding FETI-DP method exists using deflation or balancing with essentially the same eigenvalues. The reverse is true under certain conditions. First, a constraint vector should not span several equivalence classes (which is not true, e.g., in [15]). Second, if for an edge a constraint is enforced between two neighboring subdomain then this constraint should be enforced between all neighboring subdomains. If the second condition is not valid then the transformation-of-basis approach will enforce more constraints and the corresponding condition number will be smaller than that of the deflation of balancing approach.

For simplicity, we consider a single edge Z_{l_1} common to the three subdomains Ω_i, Ω_j, and Ω_k. Without loss of generality, we can assume that the transformations $T_{Z_{l_1}}^{(i)}, T_{Z_{l_1}}^{(j)}, T_{Z_{l_1}}^{(k)}$ on the subdomains Ω_i, Ω_j, and Ω_k are identical. This implies that the numbering of the edge nodes is consistent for all three subdomains.

We implicitly use the assumption that a constraint vector does not span several equivalence classes, which is not valid, e.g., for the FETI-DP method in [15] for almost incompressible elasticity using only one deflation vector for each almost incompressible subdomain.

We will describe all steps in detail and for general scalings. Our results are therefore also of interest for the adaptive BDDC methods in [2, 40, 4, 20] which combine deluxe scaling with a transformation of basis.

For simplicity, we always assume an initial (a priori) coarse space with all a priori constraints enforced by partial assembly as in [32, 34, 31, 27]. Then, our a posteriori coarse space consisting of a posteriori constraints is implemented using a generalized transformation of basis and partial assembly.

In the following, we will work with the space $\widetilde{W}_{T,a}$ (see (5.10)) where all displacements are transformed to the new basis and are continuous in all primal variables. We also need

$$\widetilde{W}_Q := \{w \in \widetilde{W} : Q^T w = 0\} \tag{5.1}$$

where the constraint vectors are stored in the columns of Q. Another notation for this space in the context of deflation and balancing will be

$$\widetilde{W}_U := \{w \in \widetilde{W} : U^T Bw = 0\}, \tag{5.2}$$

i.e., we have $Q = B^T U$.

In the following, we will just work with B_Γ and $B_{D,\Gamma}$ and in order to simplify the notation, we will use $B = B_\Gamma$ and $B_D = B_{D,\Gamma}$.

For our theoretical considerations, we would like to work with $\widetilde{W}_Q$, but in the implementation, $\widetilde{W}_Q$ is obtained via partial subassembly and scattering of the corresponding values, i.e., we iterate in $\widetilde{W}_{T,a}$. The two spaces enforce the same constraints but correspond to different methods. $\widetilde{W}_{T,a}$ corresponds to a transformation-of-basis approach and $\widetilde{W}_Q$ corresponds to a deflation or balancing approach.

As motivated in the introduction and shown in the following sections, the construction of a transformation-of-basis approach with a posteriori constraints that yields the same condition number as the deflation approach, requires some modifications of the theory compared to standard FETI-DP approaches, i.e., where only a priori constraints are used. This results from the fact that the primal components of $P_D w$ do, in general, not vanish – opposed to standard theory; cf. the motivation in Section 2.

Let us assume that a posteriori constraints are only associated with the edge Z_{l_1}. Consider the orthonormalized set of constraint vectors $[q^1_{Z_{l_1}}, \dots, q^r_{Z_{l_1}}]$. Then, introduce $T_{Z_{l_1},\Pi_{Z_{l_1}}} := [q^1_{Z_{l_1}}, \dots, q^r_{Z_{l_1}}]$. Using a modified Gram-Schmidt algorithm, we compute $T_{Z_{l_1},\Delta_{Z_{l_1}}}$ so that $T_{Z_{l_1}} := [T_{Z_{l_1},\Pi_{Z_{l_1}}},\ T_{Z_{l_1},\Delta_{Z_{l_1}}}]$ is a square matrix and $T^T_{Z_{l_1}} T_{Z_{l_1}} = I$, i.e., T_{Z,Δ_Z} will be orthogonal to the constraint space span $[q^1_{Z_{l_1}}, \dots, q^r_{Z_{l_1}}]$.

For each subdomain Ω_l, we denote the faces and/or edges by $Z_{l_1}, \dots, Z_{l_s}$. For $n > 1$ the matrix $T_{Z_{l_n},\Pi_{Z_{l_n}}}$ is void and $T_{Z_{l_n}} = [T_{Z_{l_n},\Delta_{Z_{l_n}}}] = I$. We assume that the degrees of freedom of all the corresponding equivalence classes of Ω_l are ordered such that the degrees of freedom on Z_{l_1} are ordered first, those of Z_{l_2} are ordered second, etc. Then,

$$T^{(l)}_{\Delta'_l} := [T^{(l)}_{\Pi_l},\ T^{(l)}_{\Delta_l}] := \left(\begin{array}{cc|cccc} T_{Z_{l_1},\Pi_{Z_{l_1}}} & T_{Z_{l_1},\Delta_{Z_{l_1}}} & 0 & \dots & \dots & 0 \\ 0 & 0 & I & 0 & \dots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 & I \end{array}\right) \tag{5.3}$$

represents the transformation from the new basis to the old basis, still missing an assembly operation, i.e.,

$$w_{l,\Delta'_l} = T^{(l)}_{\Delta'_l} \overline{w}_{l,\Delta'_l}. \tag{5.4}$$

As mentioned before, the transformations are chosen consistently, i.e., for the three subdomains, we have

$$T_{Z_{i_1},\Pi_{Z_{i_1}}} = T_{Z_{j_1},\Pi_{Z_{j_1}}} = T_{Z_{k_1},\Pi_{Z_{k_1}}},\ T_{Z_{i_1},\Delta_{Z_{i_1}}} = T_{Z_{j_1},\Delta_{Z_{j_1}}} = T_{Z_{k_1},\Delta_{Z_{k_1}}}, \tag{5.5}$$

$T^{(l)T}_{\Delta'_l} T^{(l)}_{\Delta'_l} = I$, $l \in \{i,j,k\}$ and that the columns of $T^{(l)}_{\Pi_l}$ span the range of all constraint vectors associated with Ω_l, $l \in \{i,j,k\}$. Therefore, using (5.3) and (5.4), we have

$$T^{(l)T}_{\Pi_l} w_{l,\Delta'_l} = \begin{pmatrix} I & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \overline{w}_{l,\Pi_{Z_{l_1}}} \\ \overline{w}_{l,\Delta_{Z_{l_1}}} \\ \vdots \\ \overline{w}_{l,\Delta_{Z_{l_s}}} \end{pmatrix} = \overline{w}_{l,\Pi_{Z_{l_1}}} \text{ for } l \in \{i,j,k\}.$$

Next, we define the global transformation matrix

$$T = \begin{pmatrix} I_{\Pi'} & 0 \\ 0 & \text{blockdiag}_{l=1,\ldots,N}(T^{(l)}_{\Delta'_l}) \end{pmatrix}.$$

We will often use T^T. Note that T^T has to be replaced by T^{-1} if the columns of T are not orthonormalized.

The transformed variables then still lack an assembly operation. In the following, we will also use the index Z_1 instead of Z_{l_1} for $l \in \{i, j, k\}$ since this edge is shared by these three subdomains and since (5.5) holds. In order to enforce

$$\overline{w}_{i,\Pi_{Z_1}} = \overline{w}_{j,\Pi_{Z_1}} = \overline{w}_{k,\Pi_{Z_1}}, \tag{5.6}$$

we introduce the global restriction operator R, which replicates the a posteriori primal degrees of freedom (given by the index set Π), and its transpose R^T, which sums a posteriori primal degrees of freedom.

The restriction operator R is of the form

$$R = \begin{pmatrix} I_{\Pi'} & 0 & 0 & \ldots & \ldots & 0 \\ 0 & (*)_{\Pi_1} & (*)_{\Delta_1} & 0 & \ldots & 0 \\ \vdots & \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ 0 & (*)_{\Pi_N} & 0 & \ldots & 0 & (*)_{\Delta_N} \end{pmatrix}, \tag{5.7}$$

where the matrix $((*)_{\Pi_i}, (*)_{\Delta_i})$, $i = 1, \ldots, N$, is a permutation of the columns of the identity matrix. The operator R replicates the a posteriori degrees of freedom to the different subdomains but does not change the a priori set of primal variables.

The assembly operator for the a posteriori primal variables is defined as

$$R_\mu^T := (R^T R)^{-1} R^T \tag{5.8}$$

and therefore $R_\mu^T R = I$. Here, the index μ stands for multiplicity.

The local versions of R_μ^T and R, restricted to the considered edge Z_1, are given by

$$R^T_{Z_1,\mu} := \begin{pmatrix} R^T_{\Pi_{Z_1,\mu}} \\ R^T_{\Delta_{Z_1}} \end{pmatrix} := \begin{pmatrix} \frac{1}{3} I_{i,\Pi_{Z_1}} & 0 & \frac{1}{3} I_{j,\Pi_{Z_1}} & 0 & \frac{1}{3} I_{k,\Pi_{Z_1}} & 0 \\ 0 & I_{i,\Delta_{Z_1}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{j,\Delta_{Z_1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{k,\Delta_{Z_1}} \end{pmatrix},$$

and the version which ignores the multiplicity is

$$R^T_{Z_1} := \begin{pmatrix} 3R^T_{\Pi_{Z_1,\mu}} \\ R^T_{\Delta_{Z_1}} \end{pmatrix}.$$

Then, $\widehat{w} := R_\mu^T T^T w$ is also continuous in the a posteriori set of primal variables given by the index set Π' as described by (5.6).

5.1. **Transformed Variables and Transformed Operators.** Our orthogonal transformation T will perform the change of basis from the standard nodal finite element basis to a non-nodal basis, e.g., with explicit averages. The inverse T^T then changes back to the nodel basis.

In the new basis, the new (assembled) variables are given by

$$\widehat{w} := R_\mu^T T^T w = R_\mu^T \overline{w}. \tag{5.9}$$

By construction, $\widehat{w}$ is continuous in the a posteriori set of primal variables given by Π' and the a priori primal variables given by Π. We also introduce the corresponding space

$$\widetilde{W}_{T,a} := \{\widehat{w} = R_\mu^T T^T w : w \in \widetilde{W}\}. \tag{5.10}$$

For these new (assembled) variables, we also define the transformed operators $\widehat{P}_D$ and $\widehat{\widetilde{S}}$ by

$$\widehat{P}_D := R_\mu^T T^T P_D TR \quad \text{and} \quad \widehat{\widetilde{S}} := R^T T^T \widetilde{S} TR, \tag{5.11}$$

where $P_D = B_D^T B$, i.e., the operator P_D corresponds to the jump operator B and the a priori scaling D used with the a priori coarse space corresponding to the index set Π'. For the theory, we will also use

$$\widehat{B} := BTR \text{ and } \widehat{B}_D := B_D TR_\mu. \tag{5.12}$$

In practice, we will not implement a transformed version of B or B_D.

The transformed, preconditioned FETI-DP system matrix, using the transformed Dirichlet preconditioner $\widehat{M}_D^{-1}$, is thus given by

(5.13)

$$\widehat{M}_D^{-1}\widehat{F} := (\widehat{B}_D \widehat{\widetilde{S}} \widehat{B}_D^T)\, (\widehat{B}\widehat{\widetilde{S}}^{-1} \widehat{B}^T) = \Big(B_D TR_\mu \widehat{\widetilde{S}} R_\mu^T T^T B_D^T\Big)\Big(BTR\widehat{\widetilde{S}}^{-1} R^T T^T B^T\Big).$$

where $\widehat{\widetilde{S}} = R^T T^T \widetilde{S} TR$ is the transformed Schur complement assembled also in the a posteriori primal variables; see (5.11)

For the case of nonredundant Lagrange multipliers, by [29, Theorem 6.5], we know that there exists a matrix T_λ so that $BT = T_\lambda B$. In the next section, we provide a lemma which extends [29, Theorem 6.5] to the case of redundant Lagrange multipliers.

The preconditioned system (5.13) is different from the standard FETI-DP method using a transformation of basis, as, e.g., in [28, 29, 34, 31, 27] which can be written, iterating in the transformed basis,

$$M_D^{-1} F = (B_{D,\Delta} \widehat{\widetilde{S}} B_{D,\Delta}^T)\, (B_\Delta \widehat{\widetilde{S}}^{-1} B_\Delta^T), \tag{5.14}$$

where the operator B_Δ only enforces continuity on the a posteriori dual variables and $B_{D,\Delta}$ is its scaled variant; cf., e.g., [32, 34].

5.2. **Transformation in the Space of Lagrange Multipliers.**

A transformed scaling for FETI-DP can be defined by constructing a transformation of basis in the space of Lagrange multipliers, as in [29]. This will, however, typically not be used in implementations. Instead the formulation given by (5.13) is implemented.

Remark 5.1. (Transformation in the Space of Lagrange Multipliers) Consider FETI-DP using redundant Lagrange multipliers and transformation matrices $T^{(i)}$, $i = 1, \ldots, N$, such that $T^{(j)}_{|Z} = T^{(k)}_{|Z}$ for any equivalence class element Z shared by at least two subdomains Ω_j and Ω_k. Then, there exists a transformation of basis T_λ in the space of Lagrange multipliers such that

$$BT = T_\lambda B.$$

For constant coefficients on each face and edge, we also have $B_D T = T_\lambda B_D$. In general, however, we only have

$$B_D T = (D^{(1)T} T_\lambda B^{(1)}, \ldots, D^{(N)T} T_\lambda B^{(N)}). \tag{5.15}$$

A corresponding relationship was formulated for nonredundant Lagrange multipliers in [29, Theorem 6.5].

Now the transformed scaling can be defined.

Remark 5.2. (Transformed Lagrange Multiplier Scaling) For a scaling matrix $D^{(i)}$ the transformed scaling $\widehat{D}^{(i)}$ matrix is defined by

$$\widehat{D}^{(i)} := T_\lambda^T D^{(i)} T_\lambda \quad \text{for} \quad i = 1, \ldots, N. \tag{5.16}$$

For problems with constant coefficients on edges or faces the transformed scaling in Definition 7.1 remains diagonal if the original scaling was diagonal. For heterogeneous problems this is not generally the case.

6. Eigenvalues of FETI-DP with a Generalized Transformation of Basis and Deflation or Balancing

In this section, we will show that FETI-DP using our generalized transformation-of-basis approach results in essentially the same eigenvalues as FETI-DP using the corresponding deflation or balancing approach. The generalized transformation-of-basis approach is different from the standard FETI-DP and BDDC methods using a transformation of basis [32, 34, 31, 27] in that it allows an interaction of dual and primal variables in the scaling.

Remark 6.1. The generalized transformation-of-basis approach also results in the same number of zero eigenvalues as FETI-DP using deflation; cf. Figure 4, and analogously to $(I-P)FU = 0$, we also have $\widehat{F}U = 0$

To establish the equality of eigenvalues of FETI-DP (and BDDC) using a generalized transformation of basis and of FETI-DP using deflation, we will show

(6.1)
$$\langle \widehat{M}_D^{-1} \widehat{F} \widehat{\lambda}, \widehat{F} \widehat{\lambda} \rangle = \langle \widehat{P}_D \widehat{u}, \widehat{P}_D \widehat{u} \rangle_{\widehat{\widetilde{S}}} \stackrel{!}{=} \langle P_D u_0, P_D u_0 \rangle_{\widetilde{S}} = \langle M_{PP}^{-1} F(I-P)\widehat{\lambda}, F(I-P)\widehat{\lambda} \rangle,$$

where $\widehat{u} \in \widetilde{W}_{T,a}$ and $u_0 \in \widetilde{W}_Q$; see Theorem 6.7.

For this, we will show that for any assembled and transformed displacement $\widehat{w} \in \widetilde{W}_{T,a}$ we have an $w_0 = TR\widehat{w} \in \widetilde{W}_Q$ such that

$$|\widehat{P}_D \widehat{w}|^2_{\widehat{\widetilde{S}}} = |P_D w_0|^2_{\widetilde{S}}. \tag{6.2}$$

Vice versa, we will show that for any $w_0 \in \widetilde{W}_Q$ an $\widehat{w} = R_\mu^T T^T w_0 \in \widetilde{W}_{T,a}$ exists such that (6.2) holds, too.

We therefore have, for arbitrary scalings and coefficients, the same eigenvalues for the deflation approach and the corresponding generalized transformation-of-basis approach.

Remark 6.2. In standard FETI-DP and BDDC theory, bounds of the form $|P_D w|^2_{\widetilde{S}} \leq C|w|^2_{\widetilde{S}}$ are established. For the adaptive coarse space approach in [22], we have $C = 4\max\{N_{\mathcal{F}}, N_{\mathcal{E}}M_{\mathcal{E}}\}^2\mathrm{TOL}$, where $N_{\mathcal{F}}$ denotes the maximum number of faces of a subdomain, $N_{\mathcal{E}}$ the maximum number of edges of a subdomain, $M_{\mathcal{E}}$ the maximum multiplicity of an edge and TOL a given tolerance. For more details, we will refer the reader to [23].

Using the definitions (5.9), (5.11), and (5.12), we have

(6.3)
$$\begin{aligned} |\widehat{P}_D\widehat{w}|^2_{\widehat{\widetilde{S}}} &= \widehat{w}^T\,(\widehat{B}^T\widehat{B}_D)\;\widehat{\widetilde{S}}\;(\widehat{B}_D^T\widehat{B})\;\widehat{w} \\ &= w^T T R_\mu\,(R^T T^T B^T B_D T R_\mu)\;R^T T^T \widetilde{S} T R\;(R_\mu^T T^T B_D^T B T R)\,R_\mu^T T^T w. \end{aligned}$$

Let us now define $w_0 := TRR_\mu^T T^T w$. We would like to show $TRR_\mu^T T^T B_D^T B w_0 = B_D^T B w_0$. This, however, is not directly clear and is the subject of Lemma 6.3.

Classically, it is argued (see, e.g., [33, 34, 31, 46]) that the operator $TRR_\mu^T T^T$ reduces to identity on the dual variables and that $B_D^T B w_0$ is zero in the primal variables. This latter argument, however, is not valid here since $B_D^T B w_0$ is not zero in the a posteriori set of primal variables if the transformation of basis is established according to the corresponding deflation approach. Lemma 6.3 essentially states that $TRR_\mu^T T^T$ can be seen as a projection onto span $\{B_D^T B w_0\}$ with w_0 given as before.

In the following lemma, we also show the identity $TRR_\mu^T T^T w_0 = w_0$, which will be of use in Lemma 6.4.

Lemma 6.3. *Given* $w_0 \in \widetilde{W}_Q$*, we have*

$$TRR_\mu^T T^T\; w_0 = w_0 \quad \textit{and} \quad TRR_\mu^T T^T\; B_D^T B w_0 = B_D^T B w_0.$$

Proof. In the following, we also will use u, where $u = w_0$ or $u = B_D^T B w_0$. It will be replaced by the corresponding function in the following.

First, consider $TRR_\mu^T T^T u$. From $u = (u_l)_{l=1,\ldots,N}$, we obtain the local functions $u_l \in W_l$ and for $l \in \{i,j,k\}$, we define u_{l,Z_1} as the values at the degrees of freedom on the edge Z_1. For $l = \{i,j,k\}$, the values of the local function u_l on all remaining degrees of freedom on $\partial\Omega_{l,h} \cap \Gamma_h$ will be denoted by u_{l,Z_1^C}. For $l \notin \{i,j,k\}$, we have $u_l = u_{l,Z_1^C}$.

We have

$$T^T u = \begin{pmatrix} u_{\Pi'} \\ A_1 \\ \vdots \\ A_N \end{pmatrix} \text{ with } A_l := \begin{cases} {T_{\Delta_l}^{(l)}}^T u_{l,Z_1^C}, & l \notin \{i,j,k\}, \\ \begin{pmatrix} {T_{\Pi_l}^{(l)}}^T \begin{pmatrix} u_{l,Z_1} \\ u_{l,Z_1^C} \end{pmatrix} \\ {T_{\Delta_l}^{(l)}}^T \begin{pmatrix} u_{l,Z_1} \\ u_{l,Z_1^C} \end{pmatrix} \end{pmatrix}, & l \in \{i,j,k\}, \end{cases}$$

since $\Delta_l' = \Delta_l$ for $l \notin \{i,j,k\}$, and thus, we also have

$$RR_\mu^T T^T u = \begin{pmatrix} u_{\Pi'} \\ \widehat{A}_1 \\ \vdots \\ \widehat{A}_N \end{pmatrix}$$

$$\text{where } \widehat{A}_l := \begin{cases} A_l, & l \notin \{i,j,k\}, \\ \begin{pmatrix} \frac{1}{3}\left[{T_{\Pi_i}^{(i)}}^T \begin{pmatrix} u_{i,Z_1} \\ u_{i,Z_1^C} \end{pmatrix} + {T_{\Pi_j}^{(j)}}^T \begin{pmatrix} u_{j,Z_1} \\ u_{j,Z_1^C} \end{pmatrix} + {T_{\Pi_k}^{(k)}}^T \begin{pmatrix} u_{k,Z_1} \\ u_{k,Z_1^C} \end{pmatrix} \right] \\ {T_{\Delta_l}^{(l)}}^T \begin{pmatrix} u_{l,Z_1} \\ u_{l,Z_1^C} \end{pmatrix} \end{pmatrix}, & l \in \{i,j,k\}. \end{cases}$$

Here, we have used (5.3).

From (5.3) in compact form, we have

(6.4)
$$T_{\Delta_l'}^{(l)} = \begin{pmatrix} T_{\Pi_l}^{(l)} & T_{\Delta_l}^{(l)} \end{pmatrix} = \begin{pmatrix} T_{Z_1,\Pi_{Z_1}} & T_{Z_1,\Delta_{Z_1}} & 0 \\ 0 & 0 & T_{Z_1^C}^{(l)} \end{pmatrix} = \begin{pmatrix} T_{Z_1,\Pi_{Z_1}} & T_{Z_1,\Delta_{Z_1}} & 0 \\ 0 & 0 & I \end{pmatrix}$$

for $l \in \{i,j,k\}$ and $T_{\Delta_l'}^{(l)} = I$ otherwise.

We now apply T to $RR_\mu^T T^T u$ or, locally, $T_{\Delta_l'}^{(l)}$ to $\widehat{A}_l$. We restrict ourselves to the case of $l \in \{i,j,k\}$ since there is nothing to show for $l \notin \{i,j,k\}$, i.e., $T_{\Delta_l'}^{(l)} \widehat{A}_l = u_{l,Z_1^C} = u_l$. Then, for $l \in \{i,j,k\}$, we obtain

$$\begin{aligned} T_{\Delta_l'}^{(l)} \widehat{A}_l &= \frac{1}{3} T_{\Delta_l'}^{(l)} \begin{pmatrix} {T_{\Pi_i}^{(i)}}^T \begin{pmatrix} u_{i,Z_1} \\ u_{i,Z_1^C} \end{pmatrix} \\ {T_{\Delta_l}^{(l)}}^T \begin{pmatrix} u_{l,Z_1} \\ u_{l,Z_1^C} \end{pmatrix} \end{pmatrix} + \frac{1}{3} T_{\Delta_l'}^{(l)} \begin{pmatrix} {T_{\Pi_j}^{(j)}}^T \begin{pmatrix} u_{j,Z_1} \\ u_{j,Z_1^C} \end{pmatrix} \\ {T_{\Delta_l}^{(l)}}^T \begin{pmatrix} u_{l,Z_1} \\ u_{l,Z_1^C} \end{pmatrix} \end{pmatrix} + \frac{1}{3} T_{\Delta_l'}^{(l)} \begin{pmatrix} {T_{\Pi_k}^{(k)}}^T \begin{pmatrix} u_{k,Z_1} \\ u_{k,Z_1^C} \end{pmatrix} \\ {T_{\Delta_l}^{(l)}}^T \begin{pmatrix} u_{l,Z_1} \\ u_{l,Z_1^C} \end{pmatrix} \end{pmatrix} \\ &\overset{(6.4)}{=} \frac{1}{3} T_{\Delta_l'}^{(l)} \begin{pmatrix} \begin{pmatrix} T_{Z_1,\Pi_{Z_1}}^T & 0 \end{pmatrix} \begin{pmatrix} u_{i,Z_1} \\ u_{i,Z_1^C} \end{pmatrix} \\ \begin{pmatrix} T_{Z_1,\Delta_{Z_1}}^T & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} u_{l,Z_1} \\ u_{l,Z_1^C} \end{pmatrix} \end{pmatrix} + \frac{1}{3} T_{\Delta_l'}^{(l)} \begin{pmatrix} \begin{pmatrix} T_{Z_1,\Pi_{Z_1}}^T & 0 \end{pmatrix} \begin{pmatrix} u_{j,Z_1} \\ u_{j,Z_1^C} \end{pmatrix} \\ \begin{pmatrix} T_{Z_1,\Delta_{Z_1}}^T & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} u_{l,Z_1} \\ u_{l,Z_1^C} \end{pmatrix} \end{pmatrix} \\ &\quad + \frac{1}{3} T_{\Delta_l'}^{(l)} \begin{pmatrix} \begin{pmatrix} T_{Z_1,\Pi_{Z_1}}^T & 0 \end{pmatrix} \begin{pmatrix} u_{k,Z_1} \\ u_{k,Z_1^C} \end{pmatrix} \\ \begin{pmatrix} T_{Z_1,\Delta_{Z_1}}^T & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} u_{l,Z_1} \\ u_{l,Z_1^C} \end{pmatrix} \end{pmatrix} \end{aligned}$$

Without loss of generality, we consider $l = i$. Then, the last equation becomes

$$\begin{aligned} T_{\Delta_i'}^{(i)} \widehat{A}_i &= \frac{1}{3} \begin{pmatrix} T_{Z_1,\Pi_{Z_1}} & T_{Z_1,\Delta_{Z_1}} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} T_{Z_1,\Pi_{Z_1}}^T & 0 \\ T_{Z_1,\Delta_{Z_1}}^T & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} u_{i,Z_1} \\ u_{i,Z_1^C} \end{pmatrix} \\ &\quad + \frac{1}{3} \begin{pmatrix} T_{Z_1,\Pi_{Z_1}} & T_{Z_1,\Delta_{Z_1}} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} T_{Z_1,\Pi_{Z_1}}^T & 0 & 0 & 0 \\ 0 & 0 & T_{Z_1,\Delta_{Z_1}}^T & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} u_{j,Z_1} \\ u_{j,Z_1^C} \\ u_{i,Z_1} \\ u_{i,Z_1^C} \end{pmatrix} \\ &\quad + \frac{1}{3} \begin{pmatrix} T_{Z_1,\Pi_{Z_1}} & T_{Z_1,\Delta_{Z_1}} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} T_{Z_1,\Pi_{Z_1}}^T & 0 & 0 & 0 \\ 0 & 0 & T_{Z_1,\Delta_{Z_1}}^T & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} u_{j,Z_1} \\ u_{j,Z_1^C} \\ u_{i,Z_1} \\ u_{i,Z_1^C} \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3}\begin{pmatrix} u_{i,Z_1} \\ u_{i,Z_1^C} \end{pmatrix} + \frac{1}{3}\begin{pmatrix} T_{Z_1,\Pi_{Z_1}} T^T_{Z_1,\Pi_{Z_1}} u_{j,Z_1} + T_{Z_1,\Delta_{Z_1}} T^T_{Z_1,\Delta_{Z_1}} u_{i,Z_1} \\ u_{i,Z_1^C} \end{pmatrix} \\
&\quad + \frac{1}{3}\begin{pmatrix} T_{Z_1,\Pi_{Z_1}} T^T_{Z_1,\Pi_{Z_1}} u_{k,Z_1} + T_{Z_1,\Delta_{Z_1}} T^T_{Z_1,\Delta_{Z_1}} u_{i,Z_1} \\ u_{i,Z_1^C} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{3}\left(\left(I + 2T_{Z_1,\Delta_{Z_1}} T^T_{Z_1,\Delta_{Z_1}}\right) u_{i,Z_1} + \sum_{n\in\{j,k\}} T_{Z_1,\Pi_{Z_1}} T^T_{Z_1,\Pi_{Z_1}} u_{n,Z_1}\right) \\ u_{i,Z_1^C} \end{pmatrix}.
\end{aligned}
$$

This shows that we can focus on the degrees of freedom on the edge Z_1 since $TRR_\mu^T T^T$ reduces to identity on the degrees of freedom on $\Gamma_{l,h} \backslash Z_1$, i.e., $u^T T \widetilde{S} T^T u = u^T T R_\mu \widehat{\widetilde{S}} R_\mu^T T^T u$ for u with $u_{|Z_1} = 0$.

We have

$$
\begin{aligned}
&\left(I + 2T_{Z_1,\Delta_{Z_1}} T^T_{Z_1,\Delta_{Z_1}}\right) u_{i,Z_1} + \sum_{n\in\{j,k\}} T_{Z_1,\Pi_{Z_1}} T^T_{Z_1,\Pi_{Z_1}} u_{n,Z_1} \\
=&\left(I + 2T_{Z_1,\Delta_{Z_1}} T^T_{Z_1,\Delta_{Z_1}} + 2T_{Z_1,\Pi_{Z_1}} T^T_{Z_1,\Pi_{Z_1}}\right) u_{i,Z_1} \\
&\qquad + \sum_{n\in\{j,k\}} T_{Z_1,\Pi_{Z_1}} T^T_{Z_1,\Pi_{Z_1}} u_{n,Z_1} - 2T_{Z_1,\Pi_{Z_1}} T^T_{Z_1,\Pi_{Z_1}} u_{i,Z_1} \\
=&3u_{i,Z_1} + T_{Z_1,\Pi_{Z_1}} T^T_{Z_1,\Pi_{Z_1}} \left(u_{j,Z_1} + u_{k,Z_1} - 2u_{i,Z_1}\right).
\end{aligned}
$$

In two following parts of the proof, we have to distinguish $u = w_0$ and $u = B_D^T B w_0$. We have $w_{0,l,Z_1} = w_{0|\partial\Omega_l \cap Z_1}$ for $l \in \{i,j,k\}$.

Now, consider $u = w_0$ and

$$
\begin{aligned}
u_{j,Z_1} + u_{k,Z_1} - 2u_{i,Z_1} &= w_{0,j,Z_1} + w_{0,k,Z_1} - 2w_{0,i,Z_1} \\
&= (w_{0,j,Z_1} - w_{0,i,Z_1}) + (w_{0,k,Z_1} - w_{0,i,Z_1})
\end{aligned}
$$

Since we know that (w_{0,r_1}, w_{0,r_2}) is orthogonal to the constraint vectors for $r_1, r_2 \in \{i,j,k\}$, $r_1 \neq r_2$ (cf. (6.8)), we obtain

$$
\begin{aligned}
&T_{Z_1,\Pi_{Z_1}} T^T_{Z_1,\Pi_{Z_1}} \left(u_{j,Z_1} + u_{k,Z_1} - 2u_{i,Z_1}\right) \\
=&T_{Z_1,\Pi_{Z_1}} T^T_{Z_1,\Pi_{Z_1}} ((w_{0,j,Z_1} - w_{0,i,Z_1}) + (w_{0,k,Z_1} - w_{0,i,Z_1})) \\
=&T_{Z_1,\Pi_{Z_1}} (T^T_{Z_1,\Pi_{Z_1}} (w_{0,j,Z_1} - w_{0,i,Z_1}) + T^T_{Z_1,\Pi_{Z_1}} (w_{0,k,Z_1} - w_{0,i,Z_1})) = 0.
\end{aligned}
$$

Using $u_{i,Z_1} = B_D^T B w_{0|\partial\Omega_i \cap Z_1}$, we have $u_{i,Z_1} = D^{(j)}(w_{0,i,Z_1} - w_{0,j,Z_1}) + D^{(k)}(w_{0,i,Z_1} - w_{0,k,Z_1})$. With corresponding formulas for j and k, we obtain

$$
\begin{aligned}
u_{j,Z_1} + u_{k,Z_1} - 2u_{i,Z_1} =&D^{(i)}(w_{0,j,Z_1} - w_{0,i,Z_1}) + D^{(k)}(w_{0,j,Z_1} - w_{0,k,Z_1}) \\
&+ D^{(i)}(w_{0,k,Z_1} - w_{0,i,Z_1}) + D^{(j)}(w_{0,k,Z_1} - w_{0,j,Z_1}) \\
&+ 2D^{(j)}(w_{0,j,Z_1} - w_{0,i,Z_1}) + 2D^{(k)}(w_{0,k,Z_1} - w_{0,i,Z_1})) \\
=&(D^{(i)} + D^{(j)})(w_{0,j,Z_1} - w_{0,i,Z_1}) + (D^{(i)} + D^{(k)})(w_{0,k,Z_1} - w_{0,i,Z_1}) \\
&+ D^{(k)}(w_{0,j,Z_1} - w_{0,k,Z_1}) + D^{(k)}(w_{0,k,Z_1} - w_{0,i,Z_1}) \\
&+ D^{(j)}(w_{0,k,Z_1} - w_{0,j,Z_1}) + D^{(j)}(w_{0,j,Z_1} - w_{0,i,Z_1})) \\
=&(\underbrace{D^{(i)} + D^{(j)} + D^{(k)}}_{=I})(w_{0,j,Z_1} - w_{0,i,Z_1}) + (\underbrace{D^{(i)} + D^{(j)} + D^{(k)}}_{=I})(w_{0,k,Z_1} - w_0
\end{aligned}
$$

$$=(w_{0,j,Z_1} - w_{0,i,Z_1}) + (w_{0,k,Z_1} - w_{0,i,Z_1}).$$

Since we know that (w_{0,r_1}, w_{0,r_2}) is orthogonal to the constraint vectors for $r_1, r_2 \in \{i,j,k\}$, $r_1 \neq r_2$ (cf. (6.8)), we obtain as before

$$T_{Z_1,\Pi_{Z_1}} T_{Z_1,\Pi_{Z_1}}^T (u_{j,Z_1} + u_{k,Z_1} - 2u_{i,Z_1}) = 0.$$

Therefore, for $u = w_0$ and $u = B_D^T B w_0$, we have

$$T_{\Delta_l'}^{(l)} \widehat{A}_l = \begin{pmatrix} u_{l,Z_1} \\ u_{l,Z_1^C} \end{pmatrix},$$

which finally yields

$$TRR_\mu^T T^T u = u$$

for any $w_0 \in \widetilde{W}_Q$ and $u = w_0$ or $u = B_D^T B w_0$. □

Let us now have a closer look at $\widehat{B}\widehat{w} = BTRR_\mu^T T^T w$.

Lemma 6.4. *For $\widehat{w} \in \widetilde{W}_{T,a}$ there exists a $w_0 := TR\widehat{w} \in \widetilde{W}_Q$ with*

$$\widehat{B}\widehat{w} = Bw_0. \tag{6.5}$$

Vice versa, for $w_0 \in \widetilde{W}_Q$ there exists a $\widehat{w} := R_\mu^T T^T w_0 \in \widetilde{W}_{T,a}$ with (6.5).

Proof. Let $\widehat{w} \in \widetilde{W}_{T,a}$ be given. We define

$$w_0 := TR\widehat{w} = TRR_\mu^T T^T w = T \begin{pmatrix} * \\ \frac{1}{3}(\overline{w}_{i,\Pi_i} + \overline{w}_{j,\Pi_j} + \overline{w}_{k,\Pi_k}) \\ \overline{w}_{i,\Delta_i} \\ * \\ \frac{1}{3}(\overline{w}_{i,\Pi_i} + \overline{w}_{j,\Pi_j} + \overline{w}_{k,\Pi_k}) \\ \overline{w}_{j,\Delta_j} \\ * \\ \frac{1}{3}(\overline{w}_{i,\Pi_i} + \overline{w}_{j,\Pi_j} + \overline{w}_{k,\Pi_k}) \\ \overline{w}_{k,\Delta_k} \\ * \end{pmatrix}. \tag{6.6}$$

By construction, we have $\widehat{B}\widehat{w} = Bw_0$. Then, with $\overline{w}_{0,\Pi} := \frac{1}{3}(\overline{w}_{i,\Pi_i} + \overline{w}_{j,\Pi_j} + \overline{w}_{k,\Pi_k})$ and

$$w_{0,l,\Delta_l'} := [T_{\Pi_l}^{(l)}\ T_{\Delta_l}^{(l)}] \begin{pmatrix} \overline{w}_{0,\Pi} \\ \overline{w}_{l,\Delta_l} \end{pmatrix}, \quad l \in \{i,j,k\}, \tag{6.7}$$

we also have

$$\overline{w}_{0,\Pi} = T_{\Pi_{r_1}}^{(r_1)^T} w_{0,r_1,\Delta_{r_1}'} = T_{\Pi_{r_2}}^{(r_2)^T} w_{0,r_2,\Delta_{r_2}'} \tag{6.8}$$

for $r_1, r_2 \in \{i,j,k\}$, $r_1 \neq r_2$. From the construction of $T_{\Pi_l}^{(l)}$, $l \in \{i,j,k\}$, it follows that (w_{0,r_1}, w_{0,r_2}) is orthogonal to the constraint vectors; cf. the beginning of Section 5. Since the constraints are local, and we assumed that a posteriori constraints are only associated with the edge common to Ω_i, Ω_j, and Ω_k, also all other local combinations (w_{0,r_1}, w_{0,r_2}), $r_1 \neq r_2$, $r_1, r_2 \in \{1,\dots,N\}$, satisfy the constraints. Thus, w_0 fulfills all constraints introduced before, i.e., $w_0 \in \widetilde{W}_Q$.

More general cases can be treated analogously.

Let $w_0 \in \widetilde{W}_Q$ be given. By the first identity of Lemma 6.3, we have $w_0 = TRR_\mu^T T^T w_0$. Define $\widehat{w} := R_\mu^T T^T w_0$. Then, it yields

$$Bw_0 = BTRR_\mu^T T^T w_0 = \widehat{B}\widehat{w}.$$

Again, more general cases can be treated analogously. □

We now prove the main relation for deflation or balancing and the generalized transformation of basis, i.e., the first part of equation (6.2).

Lemma 6.5. *For $\widehat{w} \in \widetilde{W}_{T,a}$ there exists a $w_0 := TR\widehat{w} \in \widetilde{W}_Q$ such that*

$$|\widehat{P}_D\widehat{w}|^2_{\widehat{\widetilde{S}}} = |P_D w_0|^2_{\widetilde{S}} \tag{6.9}$$

holds. Vice versa, for $w_0 \in \widetilde{W}_Q$ there exists a $\widehat{w} := R_\mu^T T^T w_0 \in \widetilde{W}_{T,a}$ such that (6.9) *holds.*

Proof. Let $\widehat{w} \in \widetilde{W}_{T,a}$ be given. Then, by using the first part of Lemma 6.4 and the second part of Lemma 6.3, we have

$$\begin{aligned}|\widehat{P}_D\widehat{w}|^2_{\widehat{\widetilde{S}}} &= \widehat{w}^T\widehat{B}^T\widehat{B}_D\widehat{\widetilde{S}}\widehat{B}_D^T\widehat{B}\widehat{w} = w_0 B^T\widehat{B}_D\widehat{\widetilde{S}}\widehat{B}_D^T B w_0\\ &= w_0 B^T\widetilde{S}B_D^T B w_0 = |P_D w_0|^2_{\widetilde{S}}\end{aligned}$$

with $w_0 := TR\widehat{w} \in \widetilde{W}_Q$.

Let $w_0 \in \widetilde{W}_Q$ be given. Then, by using the second part of Lemma 6.3 and the second part of Lemma 6.4, we have

$$|P_D w_0|^2_{\widetilde{S}} = w_0^T P_D^T \widetilde{S} P_D w_0 = \widehat{w}^T\widehat{B}^T\widehat{B}_D\widehat{\widetilde{S}}\widehat{B}_D^T\widehat{B}\widehat{w} = |\widehat{P}_D\widehat{w}|^2_{\widehat{\widetilde{S}}}$$

with $\widehat{w} := R_\mu^T T^T w_0 \in \widetilde{W}_{T,a}$. □

We will now present a lemma essentially based on Lemma 6.4, Lemma 6.3, and [31, equation (8.1)].

Lemma 6.6. *For $\widehat{w} \in \widetilde{W}_{T,a}$, we have $\widehat{B}\widehat{P}_D\widehat{w} = \widehat{B}\widehat{w}$.*

Proof. By the arguments from Lemma 6.4, Lemma 6.3, and the identity $BP_Dw = Bw$ for $w \in \widetilde{W}$ from [31, equation (8.1)] and $\widetilde{W}_Q \subset \widetilde{W}$, with $w_0 := TR\widehat{w}$, we have,

$$\widehat{B}\widehat{P}_D\widehat{w} = \widehat{B}\widehat{B}_D^T\widehat{B}\widehat{w} = \widehat{B}\widehat{B}_D^T B w_0 = BTRR_\mu^T T^T B_D^T B w_0 = BB_D^T B w_0 = Bw_0 = \widehat{B}\widehat{w}.$$

□

Note that Lemma 6.5 and Lemma 6.6 provide all the tools to prove identical condition numbers for FETI-DP with a generalized transformation of basis and FETI-DP with deflation or balancing: From Lemma 6.5, we have $|\widehat{P}_D\widehat{w}|^2_{\widehat{\widetilde{S}}} = |P_D w_0|^2_{\widetilde{S}}$. The relation $|\widehat{w}|_{\widehat{\widetilde{S}}} = |w_0|_{\widetilde{S}}$ for $\widehat{w} \in \widetilde{W}_{T,a}$ and $w_0 \in \widetilde{W}_Q$ can also be proven. The standard Rayleigh quotient estimate, e.g., [43, Theorem 2.4.2], [26, Lemma 3.2], and [31, Theorem 8.2], then gives the eigenvalue bound. However, with Theorem 6.7, we give a more general statement on the equality of all eigenvalues of the preconditioned operators where the relation between $|w_0|_{\widetilde{S}}$ and $|\widehat{w}|_{\widehat{\widetilde{S}}}$ is not needed explicitly.

We can now formulate and proof the main theorem of our work.

Theorem 6.7. *Let all vertices be primal. Then,*

$$\sigma(\widehat{M}_D^{-1}\widehat{F}) = \sigma(M_{PP}^{-1}F), \tag{6.10}$$

i.e., the eigenvalues of the preconditioned FETI-DP system matrix ($\widehat{M}_D^{-1}\widehat{F}$) using a generalized transformation of basis are the same as for the preconditioned FETI-DP system matrix ($M_{PP}^{-1}F$) using deflation.

Furthermore,

$$\sigma(\widehat{M}_D^{-1}\widehat{F}) \setminus \{0\} \subset \sigma(M_{BP}^{-1}F), \tag{6.11}$$

i.e., all nontrivial eigenvalues of the preconditioned FETI-DP system matrix ($\sigma(\widehat{M}_D^{-1}\widehat{F})$) using a generalized transformation of basis are equal to eigenvalues of the preconditioned FETI-DP system matrix ($M_{BP}^{-1}F$) using balancing.

Proof. For arbitrary $\widehat{\lambda}$, we define $\widehat{u} := \widehat{\widetilde{S}}^{-1}\widehat{B}^T\widehat{\lambda} \in \widetilde{W}_{T,a}$. Then, we have

$$\langle \widehat{M}_D^{-1}\widehat{F}\widehat{\lambda}, \widehat{F}\widehat{\lambda}\rangle = \langle \widehat{B}_D\widehat{\widetilde{S}}\widehat{B}_D^T\ \widehat{B}\widehat{\widetilde{S}}^{-1}\widehat{B}^T\widehat{\lambda}\,,\ \widehat{B}\widehat{\widetilde{S}}^{-1}\widehat{B}^T\widehat{\lambda}\rangle = \langle \widehat{P}_D\widehat{u}, \widehat{P}_D\widehat{u}\rangle_{\widehat{\widetilde{S}}} \tag{6.12}$$

as, e.g., in [31, Theorem 8.2]; cf. the Definitions in (5.13), (5.12), and (5.11).

With $u_0 := TR\widehat{u} \in \widetilde{W}_Q$ (cf. Lemma 6.4) consider

$$\underbrace{R^TT^T\widetilde{S}TR}_{=\widehat{\widetilde{S}}}R_\mu^TT^Tu_0 = \widehat{\widetilde{S}}\widehat{u} = \widehat{B}^T\widehat{\lambda} = R^TT^TB^T\widehat{\lambda}. \tag{6.13}$$

Now, we argue as in the proof of [29, Theorem 6.8], only the operators are slightly adapted. So, equivalently to (6.13), we may solve the saddle point problem

$$\begin{pmatrix} \widetilde{S} & Q \\ Q^T & 0 \end{pmatrix}\begin{pmatrix} u_0 \\ \mu \end{pmatrix} = \begin{pmatrix} B^T\widehat{\lambda} \\ 0 \end{pmatrix} \tag{6.14}$$

where the assembly was replaced by the constraint $Q^Tu_0 = 0$, i.e., we have $Q^TTR = 0$ which explicitly uses the matrix Q of $\widetilde{W}_Q := \{w \in \widetilde{W}\ :\ Q^Tw = 0\}$. Note that this is connected to the deflation constraint matrix U by $Q = B^TU$; see the beginning of Section 5. From solving the saddle point system (6.14), we obtain

$$\begin{aligned} u_0 &= (I - \widetilde{S}^{-1}Q(Q^T\widetilde{S}^{-1}Q)^+Q^T)\widetilde{S}^{-1}B^T\widehat{\lambda} \\ &= (I - \widetilde{S}^{-1}B^TU(U^TB\widetilde{S}^{-1}B^TU)^+U^TB)\widetilde{S}^{-1}B^T\widehat{\lambda}. \end{aligned}$$

Thus, we obtain

$$Bu_0 = (I - FU(U^TFU)^+U^T)F\widehat{\lambda} = (I-P)^TF\widehat{\lambda} = (I-P)^TF(I-P)\widehat{\lambda} \tag{6.15}$$

with $P = U(U^TFU)^+U^TF$; see (4.7) and (4.9).

Using Lemma 6.5, (6.12), and (6.15), we obtain

(6.16)

$$\begin{aligned} \langle \widehat{M}_D^{-1}\widehat{F}\widehat{\lambda}, \widehat{F}\widehat{\lambda}\rangle &\overset{(6.12)}{=} \langle \widehat{P}_D\widehat{u}, \widehat{P}_D\widehat{u}\rangle_{\widehat{\widetilde{S}}} \\ &\overset{\text{Lemma 6.5}}{=} \langle P_Du_0, P_Du_0\rangle_{\widetilde{S}} \overset{(4.10),(6.15)}{=} \langle M_{PP}^{-1}F(I-P)\widehat{\lambda}, F(I-P)\widehat{\lambda}\rangle. \end{aligned}$$

Then, using (6.16) and following the Courant-Fischer-Weyl min-max principle, we obtain for the eigenvalues of $\widehat{M}_D^{-1}\widehat{F}$ and $M_{PP}^{-1}F$, the equality

$$\mu_k(\widehat{M}_D^{-1}\widehat{F}) = \min_{\dim(V)=k} \max_{\widehat{\lambda}\in V\,:\,\|\widehat{\lambda}\|=1} \langle \widehat{M}_D^{-1}\widehat{F}\widehat{\lambda}, \widehat{F}\widehat{\lambda}\rangle$$
$$= \min_{\dim(V)=k} \max_{\widehat{\lambda}\in V\,:\,\|\widehat{\lambda}\|=1} \langle M_{PP}^{-1}F(I-P)\widehat{\lambda}, F(I-P)\widehat{\lambda}\rangle = \mu_k(M_{PP}^{-1}F)$$

The relation between the eigenvalues of $M_{PP}^{-1}F$ and $M_{BP}^{-1}F$ can be found in [38] or, in our notation, in [29]. □

Note that we have $0 \in \sigma(\widehat{M}_D^{-1}\widehat{F})$ also for the case of nonredundant Lagrange multipliers if U is not an empty matrix; cf. Remark 6.1. This is a difference to the classic FETI-DP methods using a transformation of basis and results from the fact that constraints in B are applied to vectors which are already continuous in the a posteriori primal variables. These Lagrange multipliers are not discarded since they allow to implement an interaction of a posteriori primal and a posteriori dual variables through the scaling in B_D; see the preconditioned system in (5.13).

7. Modified Operators and Condition Number Estimate for BDDC with a Generalized Transformation of Basis

In the previous sections, we have shown that we have to use a generalized transformation of basis in order to derive a FETI-DP transformation-of-basis approach with the same condition number as a corresponding FETI-DP deflation or balancing approach. Given the close relations between FETI-DP and BDDC methods a corresponding BDDC method using a transformation of basis can also be constructed.

We will use the assembly operator $R_{\Delta'}^T$, that assembles all degrees of freedom on $\Delta' = \Pi \cup \Delta$, i.e., all a posteriori primal (Π) and remaining dual (Δ) degrees of freedom; cf. the presentation of standard BDDC at the end of Section 4.1. Then, we introduce the short notation

$$R' := \begin{pmatrix} I_{\Pi'} & 0 \\ 0 & R_{\Delta'} \end{pmatrix}, \tag{7.1}$$

where R' leaves the initial coarse space variables unchanged and performs the assembly in all other interface variables. The BDDC system matrix is thus the Schur complement on the interface

$$\mathcal{S} = R'^T \widetilde{S} R'. \tag{7.2}$$

Note that, since the transformations are chosen consistently for every equivalence class element, and since $R'R'^T$ assembles and redistributes information in both, a posteriori primal and remaining dual degrees of freedom, we have

$$TR'R'^T = R'R'^T T \text{ and } T^T R'R'^T = R'R'^T T^T. \tag{7.3}$$

We now introduce the scaling matrix D_u for the untransformed degrees of freedom u in BDDC corresponding to the untransformed scaling D of the Lagrange multipliers in FETI-DP. The transformed BDDC scaling is then given by $\widehat{D}_u := T^T D_u T$. Then, the BDDC preconditioner for the system matrix (7.2) is defined by

(7.4)
$$\widehat{M}_{\text{BDDC}}^{-1} := R'^T D_u T R\,(R^T T^T \widetilde{S} T R)^{-1}\, R^T T^T D_u R' = R'^T T \widehat{D}_u R \widehat{\widetilde{S}}^{-1} R^T \widehat{D}_u T^T R',$$

where R' was introduced in (7.1) and R, defined in (5.7), replicates the a posteriori primal variables. Thus, the preconditioned system is

$$\widehat{M}_{\mathrm{BDDC}}^{-1}\mathcal{S} = \Big(R'^T T \widehat{D}_u R \widehat{\widetilde{S}}^{-1} R^T \widehat{D}_u T^T R'\Big)\Big(R'^T \widetilde{S} R'\Big). \tag{7.5}$$

Since the scaling D_u affects dual and a posteriori primal variables, the method is clearly different from standard BDDC with a transformation of basis, which can be written as follows

$$M_{\mathrm{BDDC}}^{-1}\mathcal{S} = \Big(R'^T D_\Delta T R \widehat{\widetilde{S}}^{-1} R^T T^T D_\Delta R'\Big)\Big(R'^T \widetilde{S} R'\Big). \tag{7.6}$$

where D_Δ is a scaling acting only on the a posteriori dual variables; cf. the BDDC preconditioner in (7.4) and (7.6). In our preconditioner, however, an interaction between a posteriori dual and primal variables can be implemented by using a nondiagonal D_u. This interaction can be necessary; cf. Section 2.2.

The E_{D_u} operator, which is central to the condition number proof of BDDC, is given by $E_{D_u} := R' R'^T D_u$. We now define

$$\widehat{E}_{D_u} := R_\mu^T T^T R' R'^T D_u T R. \tag{7.7}$$

Definition 7.1. (Transformed Degree of Freedom Scaling) For a scaling matrix $D^{(i)}$ the transformed scaling $\widehat{D}^{(i)}$ matrix is defined by

$$\widehat{D}_u^{(i)} := T^T D_u^{(i)} T \text{ for } i = 1, \dots, N. \tag{7.8}$$

Lemma 7.2. *Then, we have*

$$\begin{aligned} &i) \quad \widehat{E}_{D_u} &&= R_\mu^T E_{\widehat{D}_u} R, \\ &ii) \quad \widehat{P}_D &&= I - \widehat{E}_{D_u}. \end{aligned} \tag{7.9}$$

Proof. i) By (7.3), we obtain

$$\widehat{E}_{D_u} = R_\mu^T T^T R' R'^T D_u T R = R_\mu^T R' R'^T T^T D_u T R = R_\mu^T R' R'^T \widehat{D}_u R = R_\mu^T E_{\widehat{D}_u} R.$$

ii) Since $R_\mu = (R^T R)^{-1} R^T$, we have $R_\mu^T R = I$. Combining the previous statement, the standard relation $P_D = I - E_{D_u}$, and (5.11), we also have

$$\widehat{P}_D = R_\mu^T T^T P_D T R = R_\mu^T (I - E_{\widehat{D}_u}) R = I - \widehat{E}_{D_u}.$$

□

Theorem 7.3. *Let all vertices be primal. Then,*

$$\sigma(\widehat{M}_{BDDC}^{-1}\mathcal{S}) \setminus \{0, 1\} \subset \sigma(\widehat{M}^{-1}\widehat{F}) = \sigma(M_{PP}^{-1} F), \tag{7.10}$$

i.e., except for zeros and ones, the preconditioned BDDC system matrix $\widehat{M}_{BDDC}^{-1}\mathcal{S}$ has the same eigenvalues as the preconditioned FETI-DP system matrix using either a generalized transformation of basis or deflation.

Proof. The proof is based on the known relation between BDDC and FETI-DP; see [34]. The preconditioned BDDC system operator is given by

$$\widehat{M}_{\mathrm{BDDC}}^{-1}\mathcal{S} = (R'^T T \widehat{D}_u R \widehat{\widetilde{S}}^{-1} R^T \widehat{D}_u T^T R')\,(R'^T \widetilde{S} R')$$

which, except for zeros, has the same eigenvalues as

$$\widehat{\widetilde{S}}^{-1} R^T \widehat{D}_u T^T R' R'^T \widetilde{S} R' R'^T T \widehat{D}_u R.$$

From (7.3) and $RR_\mu^T R' = R'$, we obtain

$$\widehat{\widetilde{S}}^{-1} R^T \widehat{D}_u T^T R' R'^T \widetilde{S} R' R'^T T \widehat{D}_u R = \widehat{\widetilde{S}}^{-1} R^T \widehat{D}_u R' R'^T R_\mu R^T T^T \widetilde{S} T R R_\mu^T R' R'^T \widehat{D}_u R$$
$$= \widehat{\widetilde{S}}^{-1} \widehat{E}_{D_u}^T \widehat{\widetilde{S}} \widehat{E}_{D_u}$$

which then has the same eigenvalues as

$$\widehat{E}_{D_u} \widehat{\widetilde{S}}^{-1} \widehat{E}_{D_u}^T \widehat{\widetilde{S}}.$$

By using $\widehat{P}_D = I - \widehat{E}_{D_u}$ from Lemma 7.2 and the estimate from Theorem 6.7, we obtain that the eigenvalues (except for zero and one) of the BDDC method are identical to that of FETI-DP using a generalized transformation of basis or deflation. □

Implementational remarks for BDDC. Let us note that, as in the case of adaptive FETI-DP, the a posteriori set of primal degrees of freedom (given by the index set Π) have to be scaled by the transformed scaling $\widehat{D}_u$, too. Thus, compared to the standard BDDC preconditioner, we replace $\widetilde{S}^{-1}$ by $\widehat{\widetilde{S}}^{-1}$, D_u by $\widehat{D}_u$ and assemble, using R^T, the a posteriori primal degrees of freedom between the application of the scaling $\widehat{D}_u$ and the solution of the system of equations associated with $\widehat{\widetilde{S}}$, that is, in the preconditioner, we solve systems of the form $\widehat{\widetilde{S}} x = R^T \widehat{D}_u w$ for the unknown $x \in \widetilde{W}_{T,a}$; see (7.5).

8. Application to an Example for the Diffusion Equation

We now present results for the diffusion equation with highly varying coefficients $\rho \in [1, 1e+06]$ on the unit cube $\Omega = [0,1]^3$ and an irregular METIS (see [18, 19]) decomposition for N subdomains. For the face with $x = 0$, we enforce homogeneous Dirichlet boundary conditions, for all other faces, we enforce homogeneous Neumann boundary conditions. We consider two materials. First, we consider a soft matrix material with $\rho_1 = 1$ and an embedded stiff material in the form of $N^{2/3}$ beams with $\rho_2 = 1e+06$ running from the face with $x = 0$ to the face with

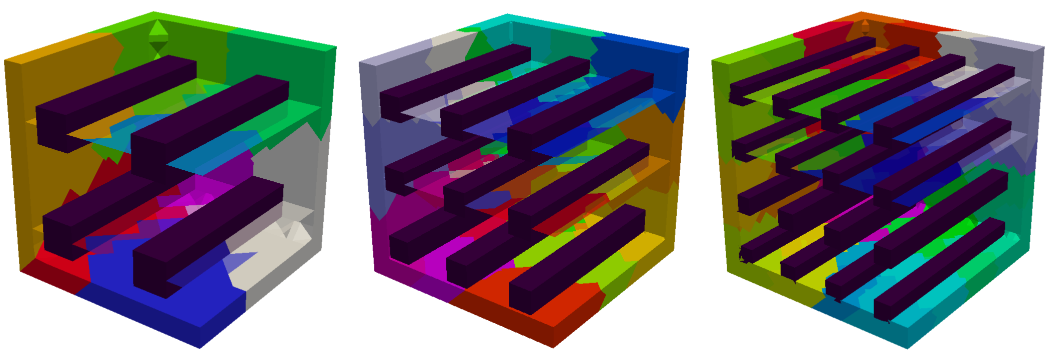

Figure 2. Irregular decomposition of the unit cube with a composite material using METIS [18, 19]. High coefficients $\rho_2 = 1e+06$ are shown in dark purple in the picture, subdomains shown in different colors in the background and by half-transparent slices. Visualization for $N = 8$ subdomains and $1/h = 12$ (left), $N = 27$ subdomains and $1/h = 18$ (center), and $N = 64$ subdomains and $1/h = 24$ (right).

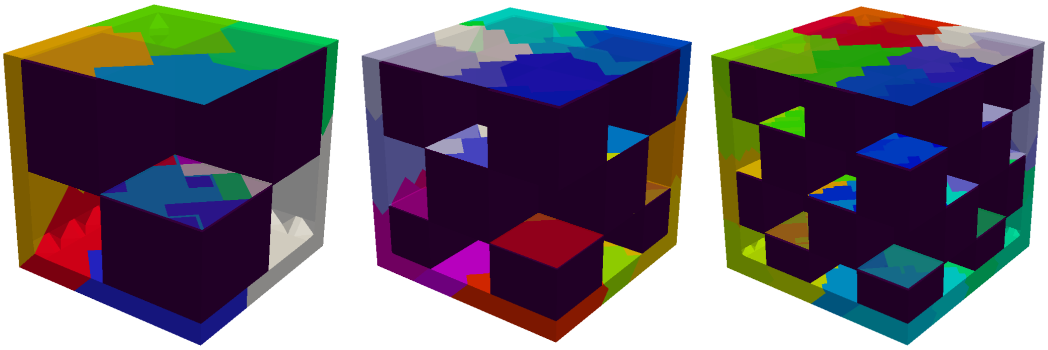

Figure 3. Irregular decomposition of the unit cube with a regular checkerboard material using METIS [18, 19]. High coefficients $\rho_2 = 1e{+}06$ are shown in dark purple in the picture, subdomains shown in different colors in the background and by half-transparent slices. Visualization for $N = 8$ subdomains and $1/h = 12$ (left), $N = 27$ subdomains and $1/h = 18$ (center), and $N = 64$ subdomains and $1/h = 24$ (right).

3D Composite material ($N^{2/3}$ beams), METIS partitioning, and $1/h = 6N^{1/3}$.														
		FETI-DP (Deflation/PP)				**FETI-DP** (ToB)				**BDDC** (ToB)				
N	$\lvert\Pi'\rvert$	$\lambda_{\min}$	$\lambda_{\max}$	its	$\lvert U\rvert$	$\lambda_{\min}$	$\lambda_{\max}$	its	$\lvert\Pi\rvert$	$\lambda_{\min}$	$\lambda_{\max}$	its	$\lvert\Pi\rvert$
2^3	30	1.00	**7.59**	14	32	1.00	**7.59**	14	20	1.00	**7.59**	13	20
3^3	165	1.00	**8.19**	18	203	1.00	**8.19**	18	135	1.00	**8.19**	14	135
4^3	468	1.00	**10.27**	23	545	1.00	**10.27**	23	336	1.00	**10.27**	18	336
5^3	1066	1.00	**10.88**	23	1071	1.00	**10.88**	22	645	1.00	**10.88**	18	645
6^3	1878	1.00	**9.20**	23	1837	1.00	**9.20**	23	1099	1.00	**9.20**	18	1099

Table 1. Diffusion equation with $\rho_1 = 1$, $\rho_2 = 1e + 06$. Coarse spaces for TOL $= 10$ for all generalized eigenvalue problems. $|\Pi'|$: size of a priori coarse space, $|\Pi|$: number of additional a posteriori constraints in the transformation of basis (ToB) approach, $|U|$: number of additional constraints in the deflation approach.

$x = 1$; see Figure 2. In the second material, the Young modulus is distributed in a regular checkerboard pattern; see Figure 3. We apply the adaptive coarse space approach from [22] to obtain a method which is independent of the coefficient jump. Our convergence criterion for the preconditioned conjugate gradients is a relative reduction of the preconditioned residual of $1e - 06$. Our results in Tables 1 and 2 show identical estimates for $\lambda_{\min}$ and $\lambda_{\max}$ for all three methods in accordance with the theory. In Figure 4, all eigenvalues of the three preconditioned operators were computed numerically for $1/h = 12$. We see that indeed all eigenvalues other than 0 and 1 are identical, as predicted by the theory.

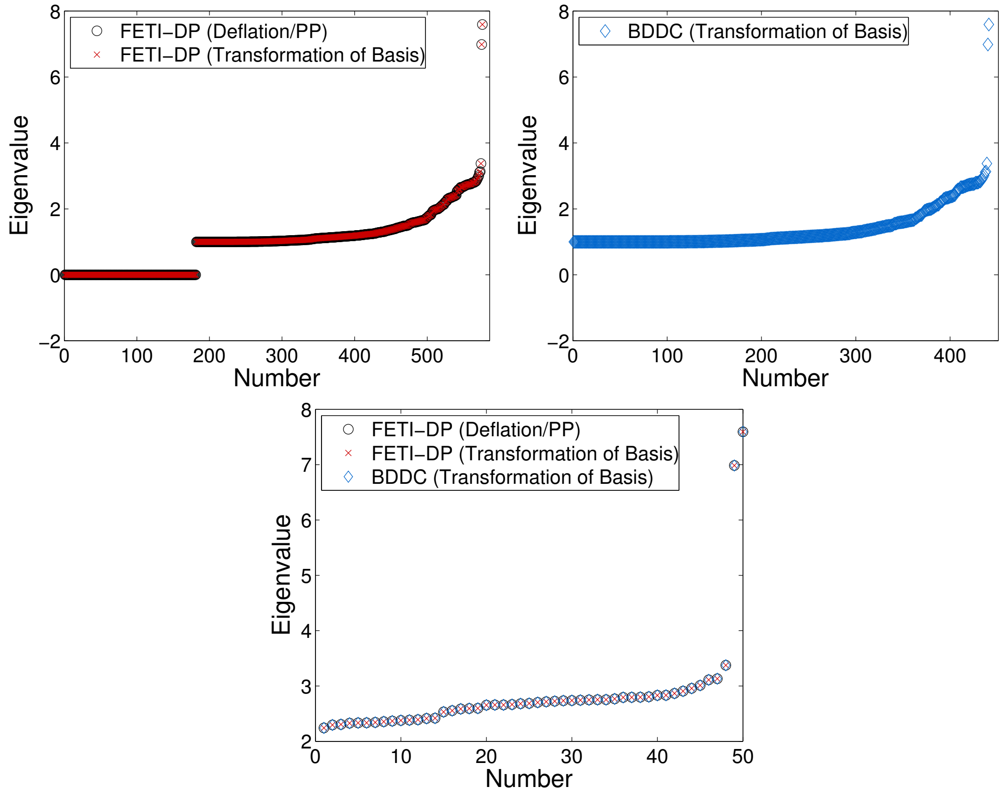

Figure 4. Plot of the eigenvalues of the preconditioned operators for FETI-DP with projector preconditioning ($M_{PP}^{-1}F$) and a transformation of basis ($\widehat{M}_D^{-1}\widehat{F}$) (top left), BDDC with a transformation of basis ($\widehat{M}_{BDDC}^{-1}\mathcal{S}$) (top right) and the largest 50 eigenvalues of the preconditioned operators (bottom center) for the composite material, an irregular decomposition of the unit cube into eight subdomains, and $1/h = 12$. The eigenvalues greater than one are identical for all three algoritms.

3D checkerboard coefficients ($\lceil N/2 \rceil$ cubes with high coefficients), METIS partitioning, and $1/h = 6N^{1/3}$.													
		FETI-DP (Deflation/PP)				**FETI-DP** (ToB)				**BDDC** (ToB)			
N	$\lvert\Pi'\rvert$	$\lambda_{\min}$	$\lambda_{\max}$	its	$\lvert U\rvert$	$\lambda_{\min}$	$\lambda_{\max}$	its	$\lvert\Pi\rvert$	$\lambda_{\min}$	$\lambda_{\max}$	its	$\lvert\Pi\rvert$
2^3	30	1.00	**7.31**	17	14	1.00	**7.31**	17	9	1.00	**7.31**	14	9
3^3	165	1.00	**8.35**	20	45	1.00	**8.35**	20	29	1.00	**8.35**	16	29
4^3	468	1.00	**8.93**	22	188	1.00	**8.93**	22	120	1.00	**8.93**	18	120
5^3	1066	1.00	**12.36**	22	245	1.00	**12.36**	22	150	1.00	**12.36**	18	150
6^3	1878	1.00	**9.72**	23	545	1.00	**9.72**	23	326	1.00	**9.72**	19	326

Table 2. Diffusion equation with $\rho_1 = 1$, $\rho_2 = 1e+06$. Coarse spaces for TOL $= 10$ for all generalized eigenvalue problems. $|\Pi'|$: size of a priori coarse space, $|\Pi|$: number of additional a posteriori constraints in the transformation of basis (ToB) approach, $|U|$: number of additional constraints in the deflation approach.

9. Conclusion

The focus of this paper is to give a generalized transformation-of-basis approach for FETI-DP and BDDC with essentially the same eigenvalues as known FETI-DP methods using deflation or balancing.

A known disadvantage of deflation and balancing methods is that the coarse space has to be solved quite exactly; cf. [29]. The transformation-of-basis approach or the use of local saddle point problems can be a remedy. Moreover, adaptive multi-level extensions are easier to construct.

We have presented FETI-DP and BDDC methods using the deflation vectors in the construction of the transformation of basis. As in [29], we have to assume that the deflation vectors do not span several edges or faces. Moreover, it emerges that for general scalings (as necessary for heterogeneous problems including coefficient jumps inside subdomains) the classical approaches to FETI-DP and BDDC with a transformation of basis have to be revisited and modified. The modifications results from the fact that in deflation or balancing, in general, an interaction between primal and dual variables can occur. This interaction is not present in traditional FETI-DP and BDDC with a transformation of basis but is possible in our generalized approach. As a result of this interaction, a standard assumption, i.e., that $P_D w$ and $(I - E_{D_u})w$ are zero in the second set of primal variables (a posteriori coarse space), is not valid anymore. In our theory, this traditional argument is replaced by Lemma 6.3. Building on this lemma, the equivalence between the generalized transformation-of-basis approach can be shown. Thus, finally, our FETI-DP and BDDC methods using a generalized transformation-of-basis approach satisfy the same condition number bound, and essentially have the same eigenvalues, as the FETI-DP method using deflation or balancing. Some first numerical results for scalar elliptic problems have been presented.

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