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Products of digraphs and their niche hypergraphs

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Abstract. If \( D = (V, A) \) is a digraph, its \( \text{niche hypergraph} \: NH(D) = (V, E) \) has the edge set \( E = \{ e \subseteq V \mid |e| \geq 2 \land \exists v \in V : e = N_D^-(v) \lor e = N_D^+(v) \} \). Niche hypergraphs generalize the well-known niche graphs (cf. [2]) and are closely related to competition hypergraphs (cf. [19]) as well as common enemy hypergraphs. For several products \( D_1 \circ D_2 \) of digraphs \( D_1 \) and \( D_2 \), we investigate the relations between the niche hypergraphs of the factors \( D_1, D_2 \) and the niche hypergraph of their product \( D_1 \circ D_2 \).

Keywords. Niche hypergraph, product of digraphs, competition hypergraph

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1. Introduction and definitions

All hypergraphs \( \mathcal{H} = (\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H})) \), graphs \( G = (\mathcal{V}(G), \mathcal{E}(G)) \) and digraphs \( D = (\mathcal{V}(D), \mathcal{A}(D)) \) considered in the following may have isolated vertices but no multiple edges. Moreover, in digraphs loops are forbidden. With \( N_D^-(v), N_D^+(v), d_D^-(v) \) and \( d_D^+(v) \) we denote the in-neighborhood, out-neighborhood, in-degree and out-degree of \( v \in \mathcal{V}(D) \), respectively. In standard terminology we follow Bang-Jensen and Gutin [1].

In 1968, Cohen [3] introduced the \( \text{competition graph} \: C(D) = (\mathcal{V}, E(C(D))) \) of a digraph \( D = (\mathcal{V}, \mathcal{A}) \) representing a food web of an ecosystem. Here the vertices correspond to the species and different vertices \( v_1, v_2 \) are connected by an edge if and only if they compete for a common prey \( w \), i.e.

\[
E(C(D)) = \{ \{v_1, v_2\} \mid v_1 \neq v_2 \land \exists w \in V : v_1 \in N_D^-(w) \land v_2 \in N_D^+(w) \}.
\]

Surveys of the large literature around competition graphs (and its variants) can be found in [5, 6, 11]; for (a selection of) recent results see [4, 7–10, 12–17, 21].

Meanwhile the following variants of \( C(D) \) have been investigated:

The \( \text{common enemy graph} \: CE(D) \) (cf. [11]) with the edge set

\[
E(CE(D)) = \{ \{v_1, v_2\} \mid v_1 \neq v_2 \land \exists w \in V : v_1 \in N_D^+(w) \land v_2 \in N_D^-(w) \},
\]

the \( \text{double competition graph} \) or \( \text{competition-common enemy graph} \: DC(D) \) with the edge set

\[
E(DC(D)) = E(C(D)) \cap E(CE(D)) \tag{cf. [18]},
\]

and the \( \text{niche graph} \: N(D) \) with

\[
E(N(D)) = E(C(D)) \cup E(CE(D)) \tag{cf. [2]}.
\]
In 2004, the concept of competition hypergraphs was introduced by Sonntag and Teichert [19]. The competition hypergraph \( CH(D) \) of a digraph \( D = (V, A) \) has the vertex set \( V \) and the edge set
\[
E(CH(D)) = \{ e \subseteq V \mid |e| \geq 2 \land \exists v \in V : e = N_D^-(v) \}.
\]

As a second hypergraph generalization recently Park and Sano [16] defined the double competition hypergraph \( DCH(D) \) of a digraph \( D = (V, A) \), which has the vertex set \( V \) and the edge set
\[
E(DCH(D)) = \{ e \subseteq V \mid |e| \geq 2 \land \exists v_1, v_2 \in V : e = N_D^-(v_1) \cap N_D^-(v_2) \}.
\]

Our paper [5] is a third step in this direction; there we consider the niche hypergraph \( NH(D) \) of a digraph \( D = (V, A) \), again with the vertex set \( V \) and the edge set
\[
E(NH(D)) = \{ e \subseteq V \mid |e| \geq 2 \land \exists v \in V : e = N_D^+(v) \lor e = N_D^+(v) \}.
\]

Note that \( NH(D) = NH(\overrightarrow{D}) \) holds for any digraph \( D \), if \( \overrightarrow{D} \) denotes the digraph obtained from \( D \) by reversing all arcs.

In [5] we present results on several properties of niche hypergraphs and the so-called niche number \( n \) of hypergraphs. In most of the investigations in [5] the generating digraph \( D \) of \( NH(D) \) is assumed to be acyclic.

For technical reasons, we define another hypergraph generalization. The common enemy hypergraph \( CEH(D) \) of a digraph \( D = (V, A) \) has the vertex set \( V \) and the edge set
\[
E(CEH(D)) = \{ e \subseteq V \mid |e| \geq 2 \land \exists v \in V : e = N_D^+(v) \lor e = N_D^+(v) \}.
\]

In the hypergraphs \( CH(D), CEH(D) \) and \( NH(D) \) no loops are allowed. Therefore, per definition the in-neighborhoods and out-neighborhoods of cardinality 1 in the digraph \( D \) play no role in the corresponding hypergraphs. This loss of information proved to be disadvantageous in the investigation of competition hypergraphs of products of digraphs (cf. [20]). So, considering niche hypergraphs of products of digraphs, it seems to be consequent to allow loops in niche hypergraphs, too. Therefore, we define the \( l \)-competition hypergraph \( CH^l(D) \), the \( l \)-common enemy hypergraph \( CEH^l(D) \) and the \( l \)-niche hypergraph \( NH^l(D) \) (with loops) having the edge sets
\[
E(CH^l(D)) = \{ e \subseteq V \mid \exists v \in V : e = N_D^-(v) \neq \emptyset \},
\]
\[
E(CEH^l(D)) = \{ e \subseteq V \mid \exists v \in V : e = N_D^+(v) \neq \emptyset \} \quad \text{and}
\]
\[
E(NH^l(D)) = \{ e \subseteq V \mid \exists v \in V : e = N_D^-(v) \neq \emptyset \lor e = N_D^+(v) \neq \emptyset \} = E(CH^l(D)) \cup E(CEH^l(D)).
\]

For the sake of brevity, in the following we often use the term \( (l) \)-competition hypergraph (sometimes in connection with the notation \( CH^{(l)}(D) \)) for the competition hypergraph \( CH(D) \) as well as for the \( l \)-competition hypergraph \( CH^l(D) \), analogously for \( (l) \)-common enemy and \( (l) \)-niche hypergraphs with the notations \( CEH^{(l)}(D) \) and \( NH^{(l)}(D) \), respectively.

For five products \( D_1 \circ D_2 \) (Cartesian product \( D_1 \times D_2 \), Cartesian sum \( D_1 + D_2 \), normal product \( D_1 \ast D_2 \), lexicographic product \( D_1 \cdot D_2 \) and disjunction \( D_1 \lor D_2 \)) of digraphs \( D_1 = (V_1, A_1) \) and \( D_2 = (V_2, A_2) \) we investigate the construction of the \( (l) \)-niche hypergraph \( NH^{(l)}(D_1 \circ D_2) = (V, E_\circ^{(l)}) \) from \( NH^{(l)}(D_1) = (V_1, E_1^{(l)}) \), \( NH^{(l)}(D_2) = (V_2, E_2^{(l)}) \) and vice versa.

The products considered here have always the vertex set \( V := V_1 \times V_2 \); using the notation \( \widetilde{A} := \{ ((a, b), (a', b')) \mid a, a' \in V_1 \land b, b' \in V_2 \} \) their arc sets are defined as follows:
\[
A(D_1 \times D_2) := \{ ((a, b), (a', b')) \in \widetilde{A} \mid (a, a') \in A_1 \land (b, b') \in A_2 \},
\]
\[
A(D_1 \ast D_2) := \{ ((a, b), (a', b')) \in \widetilde{A} \mid ((a, a') \in A_1 \land b = b') \lor (a = a' \land (b, b') \in A_2) \}.
\]
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\[ A(D_1 \ast D_2) := A(D_1 \times D_2) \cup A(D_1 + D_2), \]
\[ A(D_1 \cdot D_2) := \{((a, b), (a', b')) \in \tilde{A} \mid (a, a') \in A_1 \lor (a = a' \land (b, b') \in A_2)\}, \]
\[ A(D_1 \lor D_2) := \{((a, b), (a', b')) \in \tilde{A} \mid (a, a') \in A_1 \lor (b, b') \in A_2\}. \]

It follows immediately that \( A(D_1 + D_2) \subseteq A(D_1 \ast D_2) \subseteq A(D_1 \cdot D_2) \subseteq A(D_1 \lor D_2) \) and \( A(D_1 \times D_2) \subseteq A(D_1 \ast D_2) \). Except the lexicographic product all these products are commutative in the sense that \( D_1 \circ D_2 \cong D_2 \circ D_1 \), where \( \circ \in \{\times, +, *, \lor\} \).

Usually we number the vertices of \( D \) and arrange the vertices of \( D \) in the sense that \( \langle i \rangle \subseteq \{v_1, v_2, \ldots, v_n\} \). For niche hypergraphs such strong results are not expectable.

2. Construction of \( \mathcal{NH}^{(l)}(D_1 \circ D_2) \) from \( \mathcal{NH}^{(l)}(D_1) \) and \( \mathcal{NH}^{(l)}(D_2) \)

In the following let \( D_1 = (V_1, A_1) \) and \( D_2 = (V_2, A_2) \) be digraphs.

The digraphs \( D = (V, A) \) and \( D' = (V, A') \) are \((l)\)-niche equivalent if and only if \( D \) and \( D' \) have one and the same \((l)\)-niche hypergraph: \( \mathcal{NH}^{(l)}(D) = \mathcal{NH}^{(l)}(D') \).

**Theorem 1.** In general, for \( \circ \in \{\times, +, *, \lor\} \), the niche hypergraph \( \mathcal{NH}(D_1 \circ D_2) = (V, \mathcal{E}_o) \) of \( D_1 \circ D_2 \) cannot be obtained from the \( l \)-niche hypergraphs \( \mathcal{NH}^{(l)}(D_1) = (V_1, \mathcal{E}_1) \) and \( \mathcal{NH}^{(l)}(D_2) = (V_2, \mathcal{E}_2) \) of \( D_1 \) and \( D_2 \).

**Proof.** It suffices to present digraphs \( D_1 = (V_1, A_1) \), \( D'_1 = (V_1, A'_1) \), \( D_2 = (V_2, A_2) \) such that \( D_1 \) and \( D'_1 \) are \( l \)-niche equivalent, but the niche hypergraphs of \( D_1 \circ D_2 \) and \( D'_1 \circ D_2 \) are distinct: \( \mathcal{NH}(D_1 \circ D_2) \neq \mathcal{NH}(D'_1 \circ D_2) \).

So let us consider the following digraphs and their niche hypergraphs:
\[ D_1 = (V_1, A_1) \text{ with } V_1 = \{1, 2, 3, 4, 5\} \text{ and } A_1 = \{(1, 2), (3, 2), (4, 5), (2, 4)\}, \]
\[ D'_1 = (V_1, A'_1) \text{ with } A'_1 = \{(1, 2), (3, 2), (4, 5)\} \text{ and } \]
\[ D_2 = (V_2, A_2) \text{ with } V_2 = \{1, 2, 3\} \text{ and } A_2 = \{(1, 3), (2, 3)\}. \]

Obviously, \( D_1 \) and \( D'_1 \) are \( l \)-niche equivalent, they have the \( l \)-niche hypergraph \( \mathcal{NH}^{(l)}(D_1) = \mathcal{NH}^{(l)}(D'_1) = (V_1, \mathcal{E}_1) \), where \( \mathcal{E}_1 = \{\{1, 3\}, \{2\}, \{4\}, \{5\}\}. \)
In detail, looking at $D_1$ we have
\[
\mathcal{E}_1 = \mathcal{E}(\mathcal{NH}(D_1)) = \{(1, 3), N_{D_1}(2), \{2\}, N_{D_1}^+(4) = N_{D_1}^+(1) = N_{D_1}(3), \{4\} = N_{D_1}^+(5) = N_{D_1}(2), \{5\} = N_{D_1}^+(4)\};
\]
regarding $D'_1$ we get
\[
\mathcal{E}'_1 = \mathcal{E}(\mathcal{NH}(D'_1)) = \{(1, 3) = N_{D'_1}(2), \{2\} = N_{D'_1}(1) = N_{D'_1}^+(3), \{4\} = N_{D'_1}^+(5)\}, \{5\} = N_{D'_1}^+(4)\).
\]
Note that $D_1$ and $D'_1$ – despite having one and the same $l$-niche hypergraph – are significantly different in the sense that $D'_1 \neq \overline{D}_1$, $D_1 \neq D'_1$ and, moreover, $D_1$ is connected but $D'_1$ consists of two components. Of course, using $D_1$ and $D_1$ instead of $D_1$ and $D'_1$ could be an alternative approach for proving Theorem 1.

For the sake of completeness, we give the $l$-niche hypergraph $\mathcal{NH}(D_2) = (V_2, \mathcal{E}'_2)$, with
\[
\mathcal{E}'_2 = \{(1, 2) = N_{D_2}(3), \{3\} = N_{D_2}^+(1) = N_{D_2}^+(2)\}.
\]
Now we compare the niche hypergraphs of the products $D_1 \circ D_2$ and $D'_1 \circ D_2$.

**- Cartesian product $D_1^{(i)} \times D_2$:**
Since the Cartesian product has not so many arcs and, consequently, its niche hypergraph $\mathcal{NH}(D_1^{(i)} \times D_2)$ includes only few hyperedges, we present the whole edge sets $\mathcal{E}(\mathcal{NH}(D_1^{(i)} \times D_2))$ here (in case of the other four products the edge sets of $\mathcal{NH}(D_1^{(i)} \circ D_2)$ will be considerably larger, hence in these cases we will give up on writing down these sets completely):
\[
\mathcal{E}(\mathcal{NH}(D_1 \times D_2)) = \{(1, 1), (1, 2), (3, 1), (3, 2)\} = N_{D_1 \times D_2}((2, 3)),
\]
\[
\{(2, 1), (2, 2)\} = N_{D_1 \times D_2}((4, 3)), \{(4, 1), (4, 2)\} = N_{D_1 \times D_2}((5, 3))\}
\]
and
\[
\mathcal{E}(\mathcal{NH}(D'_1 \times D_2)) = \{(1, 1), (1, 2), (3, 1), (3, 2)\} = N_{D'_1 \times D_2}((2, 3)),
\]
\[
\{(4, 1), (4, 2)\} = N_{D'_1 \times D_2}((5, 3))\}
\]
**- Cartesian sum $D_1^{(i)} + D_2$, normal product $D_1^{(i)} \ast D_2$ and lexicographic product $D_1^{(i)} \cdot D_2$:**
Since $D_1$ is connected, the Cartesian sum $D_1 + D_2$, the normal product $D_1 \ast D_2$ as well as the lexicographic product $D_1 \cdot D_2$ are connected, too. Considering the (disconnected) digraph $D'_1$, obviously $D'_1 + D_2$, $D'_1 \ast D_2$ and $D'_1 \cdot D_2$ are disconnected. In detail, each of the products $D'_1 \circ D_2$ ($\circ \in \{+, \cdot, \cdot\}$) consists of the two components $\langle Z_1 \cup Z_2 \cup Z_3 \rangle_{D'_1 \circ D_2}$ and $\langle Z_4 \cup Z_5 \rangle_{D'_1 \circ D_2}$.

Therefore, in the niche hypergraph $\mathcal{NH}(D'_1 \circ D_2)$ hyperedges containing vertices of both components cannot exist:
\[
\forall e \in \mathcal{E}(\mathcal{NH}(D'_1 \circ D_2)) : e \cap (Z_1 \cup Z_2 \cup Z_3) = \emptyset \lor e \cap (Z_4 \cup Z_5) = \emptyset.
\]
Consequently, to show $\mathcal{NH}(D_1 \circ D_2) \neq \mathcal{NH}(D'_1 \circ D_2)$, it suffices to find a hyperedge $e \in \mathcal{E}(\mathcal{NH}(D_1 \circ D_2))$ such that both $e \cap (Z_1 \cup Z_2 \cup Z_3)$ and $e \cap (Z_4 \cup Z_5)$ are nonempty.

For each of the three products $D_1 \circ D_2$ we will obtain such a hyperedge by considering the set of the predecessors of the vertex $(4, 3) \in V(D_1 \circ D_2)$, i.e. $e = N_{D_1 \circ D_2}^-(4, 3)$. Clearly, $e$ results from $N_{D_1}^-(4) = \{2\}$ and $N_{D_2}^-(3) = \{1, 2\}$.

For the Cartesian sum $D_1 + D_2$, we have $e = \{(2, 3), (4, 1), (4, 2)\} = N_{D_1 + D_2}^-(4, 3)$.

In case of the normal product $D_1 \ast D_2$, we obtain
\[
e = \{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2)\} = N_{D_1 \ast D_2}^-(4, 3).
\]
It is easy to see that in the lexicographic product $D_1 \cdot D_2$ the vertex $(4, 3)$ has the same predecessors as in the normal product, hence
\[
e = N_{D_1 \cdot D_2}^-(4, 3) = N_{D_1 \cdot D_2}^-(4, 3) = \{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2)\}.
\]
Remark 1.

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Onto their positive construction results can be derived. For this end we have to make use of Products of digraphs and their niche hypergraphs without the extra information on points (a) – (d).

Disjunction $D_1 \lor D_2$:

Now both $D_1 \lor D_2$ and $D'_1 \lor D_2$ are connected. Nevertheless, as in the previous cases, we consider the predecessors of the vertex $(4,3)$ and get the hyperedge

$$e = N_{D_1 \lor D_2}^{-}(4,3)$$

$$= \{ (1,1), (1,2), (2,1), (2,2), (2,3), (3,1), (3,2), (4,1), (4,2), (5,1), (5,2) \}$$

$$= S_1 \cup S_2 \cup \{ (2,3) \}$$

$$= S_1 \cup S_2 \cup Z_2 \quad \in \mathcal{E}(N\mathcal{H}(D_1 \lor D_2)).$$

Note that $S_1 \cup S_2$ in $e$ result from $N_{D_2}^{-}(3) = \{ 1,2 \}$ and $Z_2$ from $N_{D_1}^{-}(4) = \{ 2 \}$.

We search for this hyperedge $e$ in $N\mathcal{H}(D'_1 \lor D_2)$.

Assume $e = N_{D'_1 \lor D_2}^{-}(i,j)$ or $e = N_{D'_1 \lor D_2}^{-}(i,j)$. Since $D'_1$ and $D_2$ are loopless digraphs, we obtain $(i,j) \notin e$ and $(i,j) \in \{ (1,3), (3,3), (4,3), (5,3) \}$, i.e. $j = 3$.

Let $e = N_{D'_1 \lor D_2}^{+}(i,3)$. Because of $N_{D_2}^{+}(3) = \emptyset$ and $S_1 \subseteq e$, all vertices of $S_1$ have to be successors of $(i,3)$ in $D'_1 \lor D_2$ and $\{ 1,2,\ldots,5 \} = N_{D'_1}^{+}(i)$, where $i \in \{ 1,2,\ldots,5 \}$. This contradicts the fact that $D'_1$ is loopless.

Consequently, $e = N_{D'_1 \lor D_2}^{-}(i,3)$. Then, $S_1 \cup S_2 \subseteq e$ holds trivially. Owing to $(2,3) \in e$ we get $(2,3) \in N_{D'_1 \lor D_2}^{-}(i,3)$, i.e. $2 \in N_{D_1}^{-}(i)$ with $i \in \{ 1,2,\ldots,5 \}$. This contradicts $N_{D_1}^{-}(2) = \emptyset$.

Hence, $e \notin \mathcal{E}(N\mathcal{H}(D'_1 \lor D_2))$, thus $D_1 \lor D_2$ and $D'_1 \lor D_2$ are not niche equivalent. Therefore, the niche hypergraph of the disjunction $D_1 \lor D_2$ cannot be constructed from the niche hypergraphs of $D_1$ and $D_2$ in general.

Using Theorems 1 and 2 from [20], for the Cartesian product and the disjunction some positive construction results can be derived. For this end we have to make use of $\mathcal{E}(N\mathcal{H}^{(l)}(D)) = \mathcal{E}(\mathcal{H}^{(l)}(D)) \cup \mathcal{E}(CE\mathcal{H}^{(l)}(D))$ and $CE\mathcal{H}^{(l)}(D) = \mathcal{H}^{(l)}(\overrightarrow{D})$.

Remark 1. The l-niche hypergraph $N\mathcal{H}^{(l)}(D_1 \times D_2)$ of the Cartesian product can be obtained from the l-competition hypergraphs $\mathcal{H}^{(l)}(D_1), \mathcal{H}^{(l)}(D_2)$ and the l-common enemy hypergraphs $CE\mathcal{H}^{(l)}(D_1), CE\mathcal{H}^{(l)}(D_2)$:

$$\mathcal{E}(N\mathcal{H}^{(l)}(D_1 \times D_2)) = \mathcal{E}(\mathcal{H}^{(l)}(D_1 \times D_2)) \cup \mathcal{E}(CE\mathcal{H}^{(l)}(D_1 \times D_2))$$

$$= \{ e_1 \times e_2 | e_1 \in \mathcal{E}(\mathcal{H}^{(l)}(D_1)) \land e_2 \in \mathcal{E}(\mathcal{H}^{(l)}(D_2)) \} \cup \{ e_1 \times e_2 | e_1 \in \mathcal{E}(CE\mathcal{H}^{(l)}(D_1)) \land e_2 \in \mathcal{E}(CE\mathcal{H}^{(l)}(D_2)) \}.$$

Remark 2. The l-niche hypergraph $N\mathcal{H}^{(l)}(D_1 \lor D_2)$ of the disjunction can be obtained from the l-competition hypergraphs $\mathcal{H}^{(l)}(D_1), \mathcal{H}^{(l)}(D_2)$ and the l-common enemy hypergraphs $CE\mathcal{H}^{(l)}(D_1), CE\mathcal{H}^{(l)}(D_2)$ if for each of the following conditions is known whether it is true or not:

(a) $\exists v_2 \in V_2 : N_{D_2}^{-}(v_2) = \emptyset$ and

(b) $\exists v_1 \in V_1 : N_{D_1}^{-}(v_1) = \emptyset$ and

(c) $\exists v_2 \in V_2 : N_{D_2}^{+}(v_2) = \emptyset$ and

(d) $\exists v_1 \in V_1 : N_{D_1}^{+}(v_1) = \emptyset$.

In general, $N\mathcal{H}^{(l)}(D_1 \lor D_2)$ cannot be obtained from $\mathcal{H}^{(l)}(D_1), \mathcal{H}^{(l)}(D_2), CE\mathcal{H}^{(l)}(D_1)$ and $CE\mathcal{H}^{(l)}(D_2)$ without the extra information on points (a) – (d).

3. Reconstruction of $N\mathcal{H}^{(l)}(D_1)$ and $N\mathcal{H}^{(l)}(D_2)$ from $N\mathcal{H}^{(l)}(D_1 \circ D_2)$

In the following, for a set $e = \{ i_1, j_1, \ldots, i_k, j_k \} \subseteq V_1 \times V_2$ we define $\pi_1(e) := \{ i_1, \ldots, i_k \}$ and $\pi_2(e) := \{ j_1, \ldots, j_k \}$, respectively, i.e. $\pi_i$ denotes the projection of vertices of $N\mathcal{H}^{(l)}(D_1 \circ D_2)$ onto their $i$-th components, for $i \in \{ 1,2 \}$. 


Theorem 2. For the Cartesian product $D_1 \times D_2$ it holds:
(a) If $E(NH(D_1 \times D_2)) \neq \emptyset$, then $NH(D_1)$ and $NH(D_2)$ can be obtained from $NH(D_1 \times D_2)$.
(b) If $E(NH^l(D_1 \times D_2)) \neq \emptyset$, then $NH^l(D_1)$ and $NH^l(D_2)$ can be obtained from $NH^l(D_1 \times D_2)$.

Proof. Note that $E(NH(D_1 \times D_2)) \neq \emptyset$ implies $A_1 \neq \emptyset \neq A_2$ and $\max(|A_1|, |A_2|) \geq 2$.

Moreover, $E(NH^l(D_1 \times D_2)) \neq \emptyset$ is equivalent to $A_1 \neq \emptyset \neq A_2$ and, consequently, to $E(NH^l(D_1)) \neq \emptyset \neq E(NH^l(D_2))$.

To (b): Let $e \in E(NH^l(D_1 \times D_2))$. This is equivalent to $e \in E(CH^l(D_1 \times D_2))$ or $e \in E(CEH^l(D_1 \times D_2))$, i.e. $e = N_{D_1 \times D_2}^1((i,j))$ or $e = N_{D_2 \times D_1}^2((i,j))$, with a certain $(i,j) \in V_1 \times V_2$.

This holds if and only if there is a vertex $(i,j) \in V_1 \times V_2$ such that $\pi_1(e) = N_{D_1}^1(i) \land \pi_2(e) = N_{D_2}^1(j)$ or $\pi_1(e) = N_{D_1}^2(i) \land \pi_2(e) = N_{D_2}^2(j)$, which implies $\pi_1(e) \in E(NH^l(D_1))$ and $\pi_2(e) \in E(NH^l(D_2))$.

Clearly, this way we can get all hyperedges $e_1 \in E(NH^l(D_1))$ and $e_2 \in E(NH^l(D_2))$.

To (a): An analog argumentation holds if we consider the niche hypergraphs $NH$ instead of the l-niche hypergraphs $NH^l$, since hyperedges $e \in E(NH^l(D_1 \times D_2))$ of cardinality 1 can be omitted if we are interested only in hyperedges $e_i \in E(NH(D_i))$ (which have cardinality greater than 1), for $i = 1, 2$.

Theorem 3. For the Cartesian sum $D_1 + D_2$ it holds:
(a) $NH(D_1)$ and $NH(D_2)$ can be obtained from $NH(D_1 + D_2)$.
(b) $NH^l(D_1)$ and $NH^l(D_2)$ can be obtained from $NH^l(D_1 + D_2)$, if one of the following conditions is true:

\begin{enumerate}
\item $E(NH^l(D_1 + D_2)) = \emptyset$;
\item $(\forall e \in E(NH^l(D_1 + D_2)) : |\pi_1(e)| = 1) \land (\exists e \in E(NH^l(D_1 + D_2)) : |\pi_2(e)| \geq 2)$;
\item $(\forall e \in E(NH^l(D_1 + D_2)) : |\pi_2(e)| = 1) \land (\exists e \in E(NH^l(D_1 + D_2)) : |\pi_1(e)| \geq 2)$;
\item $\exists (i,j) \in V_1 \times V_2 \forall e \in E(NH^l(D_1 + D_2)) : (i,j) \notin e$.
\end{enumerate}

Proof. To (a): Let $e \in E(NH(D_1 + D_2))$ and $(i,j) \in V_1 \times V_2$ with $e = N_{D_1 + D_2}^-(i,j)$ or $e = N_{D_1 + D_2}^+(i,j))$. Then $e = \{i,j_1,\ldots,j_k,(i,j),(i,j),\ldots,(i,j)\}$, where $i, i_1, \ldots, i_l$ and $j, j_1, \ldots, j_k$ are pairwise distinct vertices in $V_1$ and $V_2$, respectively.

To construct $E(NH(D_1))$, we need only those hyperedges $e \in E(NH(D_1))$ which contain $l \geq 2$ vertices with one and the same second component:

\[E(NH(D_1)) = \{\pi_1(e) \setminus I \mid e \in E(NH(D_1 + D_2)) \land e = \{(i,j_1),\ldots,(i,j_k),(i_1,j),\ldots,(i_l,j)\}\]
\[\land l \geq 2 \land I = \{\{i\}, k \geq 1 \lor k = 0\} \}

Analogously, we obtain $E(NH(D_2))$:

\[E(NH(D_2)) = \{\pi_2(e) \setminus J \mid e \in E(NH(D_1 + D_2)) \land e = \{(i,j_1),\ldots,(i,j_k),(i_1,j),\ldots,(i_l,j)\}\]
\[\land k \geq 2 \land J = \{\{j\}, l \geq 1 \lor l = 0\} \}

To (b): (1) – (3) can be verified analogously to (1) – (3) of Proposition 2 of [20]. Here is a short remark to case (2): in this case it can be easily seen that $E(NH^l(D_1)) = \emptyset$ (which is equivalent to $A_1 = \emptyset$) and $E(NH^l(D_2)) = \{\pi_2(e) \mid e \in E(NH^l(D_1 + D_2))\}$.

Now we consider (4). Since $\forall e \in E(NH^l(D_1 + D_2)) : (i,j) \notin e$, the vertex $i \in V_1$ is an isolate in $NH^l(D_1)$ and in $D_1$. For the same reason, $j \in V_2$ is an isolate in $NH^l(D_2)$ and in $D_2$.

We discuss only the construction of $NH^l(D_2)$, the rest follows analogously.
Since $i$ is isolated, in $D_1 + D_2$ there is no arc between the $i$-th row $Z_i$ and any other row. Therefore, all arcs with an initial or a terminal vertex in $Z_i$ result from arcs in $D_2$ and $\forall a \in A(D_1 + D_2): V(a) \cap Z_i \neq \emptyset \Rightarrow V(a) \subseteq Z_i$.

Hence, denoting by $(Z_i)_{D_1 + D_2}$ and by $(Z_i)_{NH^l(D_1 + D_2)}$ the subdigraph of $D_1 + D_2$ and the subhypergraph of $NH^l(D_1 + D_2)$ generated by the vertices of $Z_i$, respectively, we obtain

\[-\langle Z_i \rangle_{D_1 + D_2} \simeq D_2,\]

\[-\langle Z_i \rangle_{NH^l(D_1 + D_2)} \simeq NH^l(D_2)\]

and

\[-\mathcal{E}(NH^l(D_2)) = \{\pi_2(e) \mid e \in \mathcal{E}(NH^l(D_1 + D_2)) \land e \subseteq Z_i\}.\]

Note that, being interested in $l$-niche hypergraphs, loops $e = \{(i, j)\} \in \mathcal{E}(NH^l(D_1 + D_2))$ could lead to the problem that $\{(i, j)\}$ can be a loop in $NH^l(D_1 + D_2)$ either because of $\{i\} \in \mathcal{E}(NH^l(D_1))$ and $j$ is an isolate in $D_2$ or because of $i$ is an isolate in $D_1$ and $\{j\} \in \mathcal{E}(NH^l(D_2))$ – and without further information it cannot be decided which of these cases occurs.

In comparison with Proposition 2 of our paper [20] we see that for the reconstruction of the $l$-competition graphs $CH^l(D_1)$ and $CH^l(D_2)$ from $CH^l(D_1 + D_2)$ there is another sufficient condition, namely:

\[\exists e \in \mathcal{E}(CH^l(D_1 + D_2)): |\pi_1(e)| \geq 3 \land |\pi_2(e)| \geq 3.\]

We will show that in the case of niche hypergraphs the analog condition could cause problems. So assume \((*) \exists e \in \mathcal{E}(NH^l(D_1 + D_2)): |\pi_1(e)| \geq 3 \land |\pi_2(e)| \geq 3.\)

Without loss of generality, let $e = \{(i, j_1), \ldots, (i, j_k), (i_1, j), \ldots, (i_l, j)\}$ with $k \geq 2$ and $l \geq 2$.

There are two possibilities for the hyperedge $e$, namely $e = \left\{ N_{D_1 + D_2}^{-1}(i, j) \right\}$, i.e. $\pi_1(e) \setminus \{i\} = \left\{ i_1, \ldots, i_l \right\}$ and $\pi_2(e) \setminus \{j\} = \left\{ j_1, \ldots, j_k \right\}$.

Then we have $e \in \mathcal{E}(CH^l(D_1 + D_2))$, which is equivalent to $e = N_{D_1 + D_2}^{-1}(i, j)$, or otherwise $e \in \mathcal{E}(CH^l(D_1 + D_2))$, i.e. $e = N_{D_1 + D_2}^{-1}(i, j)$. In the first case it follows $\pi_1(e) \setminus \{i\} = N_{D_1}^{-1}(i) \land \pi_2(e) \setminus \{j\} = N_{D_2}^{-1}(j)$, in the second case $\pi_1(e) \setminus \{i\} = N_{D_1}^{-1}(i) \land \pi_2(e) \setminus \{j\} = N_{D_2}^{-1}(j)$ is valid.

In both cases we obtain $\pi_1(e) \setminus \{i\} \in \mathcal{E}(NH^l(D_1))$ and $\pi_2(e) \setminus \{j\} \in \mathcal{E}(NH^l(D_2))$ and both sets $\pi_1(e) \setminus \{i\}$ and $\pi_2(e) \setminus \{j\}$ are hyperedges in the corresponding competition hypergraph $CH^l(D_\tau) \ (\tau \in \{1, 2\})$ or both are hyperedges in the common enemy hypergraph $CEH^l(D_\tau) \ (\tau \in \{1, 2\})$.

The following problem occurs: if in $NH^l(D_1 + D_2)$ all hyperedges fulfilling $(*)$ are edges of the competition hypergraph $CH^l(D_1 + D_2)$ but not edges of the common enemy hypergraph $CEH^l(D_1 + D_2)$ (in general, in $NH^l(D_1 + D_2)$ this cannot be found out), then the argument in the last five lines of the proof of Case 4 of Proposition 2 in [20] does work only for the "competition hyperedges" of $NH^l(D_1 + D_2)$, i.e. the hyperedges of $CH^l(D_1 + D_2)$. Therefore, in general, it would be impossible to obtain all "common enemy hyperedges" of $NH^l(D_1)$ and $NH^l(D_2)$, i.e. all hyperedges in $CEH^l(D_1)$ and $CEH^l(D_2)$, from $NH^l(D_1 + D_2)$.

Obviously, an analog argumentation holds if in $NH^l(D_1 + D_2)$ all hyperedges fulfilling $(*)$ are edges of the common enemy hypergraph $CEH^l(D_1 + D_2)$ but not of the competition hypergraph $CH^l(D_1 + D_2)$.

So $(*)$ does not seem to be useful in the case of $l$-niche hypergraphs.
Theorem 4. For the normal product $D_1 \ast D_2$ it holds:
(a) $NH(D_1)$ and $NH(D_2)$ can be obtained from $NH(D_1 \ast D_2)$.

(b) If there is a hyperedge $e \in \mathcal{E}(NH(D_1 \ast D_2))$ with $|\pi_1(e)| \geq 2$ and $|\pi_2(e)| \geq 2$, then $NH^l(D_1)$ and $NH^l(D_2)$ can be obtained from $NH(D_1 \ast D_2)$.

Proof. To (b): The existence of a hyperedge $e \in \mathcal{E}(NH(D_1 \ast D_2))$ with $|\pi_1(e)| \geq 2$ and $|\pi_2(e)| \geq 2$ is equivalent to $A_1 \neq \emptyset \neq A_2$.

Concerning the reconstruction of the hyperedges of $NH^l(D_1)$ and $NH^l(D_2)$, we can use the proof of Case 2 of Corollary 2 from our paper [20] with the only modification:
The argumentation given in [20] for the $l$-competition hypergraph $CH^l$, the sets of predecessors $N^-$ and indegrees of vertices has to be completed by the analog considerations for the common enemy hypergraph $CEH^l$, the sets of successors $N^+$ and outdegrees of vertices in order to obtain the above result for the niche hypergraph $NH^l$.

This modification can be done without causing any problems.

By contrast, in the case of niche hypergraphs it is impossible to reconstruct the digraphs $D_1$ and $D_2$ themselves in general. The reason had been mentioned before: although for a hyperedge $e \in \mathcal{E}(NH^l(D_1 \ast D_2))$ we can find out the vertex $(i, j)$ with $e = N^l_{D_1 \ast D_2}(i, j)$ or $e = N^+_l_{D_1 \ast D_2}(i, j)$ (cf. the proof of Case 2 of Corollary 2 in [20]), it will be impossible to determine whether $e$ is the set of predecessors or the set of successors of the vertex $(i, j)$ in $D_1 \ast D_2$ in general.

To (a): Because of (b) it suffices to consider the case $A_1 = \emptyset \vee A_2 = \emptyset$.

Replacing "+" by "+" in (1)-(3) of Theorem 3, we see that (1) $\vee$ (2) $\vee$ (3) is equivalent to $A_1 = \emptyset \vee A_2 = \emptyset$ and we can use an analog argumentation as in the corresponding part of the proof of Theorem 3. So using (2) we obtain $\mathcal{E}(NH^l(D_2)) = \{\pi_2(e) \mid e \in \mathcal{E}(NH^l(D_1 \ast D_2))\}$ and $\mathcal{E}(NH(D_2)) = \{\pi_2(e) \mid e \in \mathcal{E}(NH^l(D_1 \ast D_2)) \wedge |\pi_2(e)| \geq 2\}$, respectively. \hfill $\square$

Note that in the case $A_1 = \emptyset \vee A_2 = \emptyset$ we get $D_1 \ast D_2 = D_1 + D_2$. Therefore, the last part of the above proof in connection with Theorem 3 lead to the following consequence.

Corollary 1. $NH^l(D_1)$ and $NH^l(D_2)$ can be obtained from $NH^l(D_1 \ast D_2)$, if one of the following conditions is true:

(1) $\mathcal{E}(NH^l(D_1 \ast D_2)) = \emptyset$;

(2) $(\forall e \in \mathcal{E}(NH^l(D_1 \ast D_2)) : |\pi_1(e)| = 1) \wedge (\exists e \in \mathcal{E}(NH^l(D_1 \ast D_2)) : |\pi_2(e)| \geq 2)$;

(3) $(\forall e \in \mathcal{E}(NH^l(D_1 \ast D_2)) : |\pi_2(e)| = 1) \wedge (\exists e \in \mathcal{E}(NH^l(D_1 \ast D_2)) : |\pi_1(e)| \geq 2)$.

Theorem 5. For the lexicographic product $D_1 \cdot D_2$ it holds:
(a) $NH(D_1)$ and $NH(D_2)$ can be obtained from $NH(D_1 \cdot D_2)$.

(b) If $|V_2| \geq 2$, then $NH^l(D_1)$ can be obtained from $NH(D_1 \cdot D_2)$.

(c) $NH^l(D_1)$ and $NH^l(D_2)$ can be obtained from $NH^l(D_1 \cdot D_2)$.

Proof. First we will show (c), i.e. $NH^l(D_1)$ and $NH^l(D_2)$ can be reconstructed from $NH^l(D_1 \cdot D_2)$. Then we obtain (b) and (a) as follows:

Since for $|V_2| \geq 2$ every loop $e_1 = \{i\}$ in $NH^l(D_1)$ leads to a non-loop $e$ in $NH^l(D_1 \cdot D_2)$ (containing at least all vertices of the row $Z_i$), we will see that we need no loops of $NH^l(D_1 \cdot D_2)$ in order to obtain $NH^l(D_1)$, this includes (b).

Analogously, it is obvious that non-loops $e_2$ of $NH^l(D_1)$ and $NH^l(D_2)$, respectively, result in non-loops in $NH^l(D_1 \cdot D_2)$. In our considerations it will become clear that for the reconstruction of $NH(D_1)$ and $NH(D_2)$ we do not need the loops in $NH^l(D_1 \cdot D_2)$, so we get (a).
In order to prove (c), we consider a hyperedge \( e \in \mathcal{E}(NH^l(D_1 \cdot D_2)) \). Then there is a vertex \((i, j) \in V_1 \times V_2\) such that \( e = N_{D_1 \cdot D_2}^{-}(i, j) \) or \( e = N_{D_1 \cdot D_2}^{+}(i, j) \). In order to simplify our depictions, we write down the considerations only for the case \( e = N_{D_1 \cdot D_2}^{-}(i, j) \in \mathcal{E}(CH^l(D_1 \cdot D_2)) \); the hyperedges \( e = N_{D_1 \cdot D_2}^{+}(i, j) \in \mathcal{E}(CEH^l(D_1 \cdot D_2)) \) can be treated analogously.

In \( NH^l(D_1 \cdot D_2) \) there are two possibilities for the hyperedge \( e \).

**Case 1:** \( \exists l \geq 1 \ \exists i_1, \ldots, i_l \in V_1 : e = Z_{i_1} \cup \ldots \cup Z_{i_l} \).

Without loss of generality let \( i_1, \ldots, i_l \) be pairwise distinct.

Hence, \( e \) is the union of the complete rows \( Z_{i_1}, \ldots, Z_{i_l} \) of \( D_1 \cdot D_2 \) and from the definition of \( D_1 \cdot D_2 \) it follows \( i \notin \{i_1, \ldots, i_l\} \) and \( N_{D_1}(i) = \{i_1, \ldots, i_l\} \cap N_{D_2}(j) = \emptyset \).

Therefore, Case 1 does not provide any hyperedges of \( NH^l(D_2) \) but with \( \pi_1(e) = \{i_1, \ldots, i_l\} = N_{D_1}^{-}(i) \in \mathcal{E}(NH^l(D_1)) \) we obtain a hyperedge of \( NH^l(D_1) \).

Note that the vertex \( i \in V_1 \) is unknown if \( l < |V_1| - 1 \). Moreover, Case 1 occurs if and only if there exists a vertex \( j \in V_2 \) with \( N_{D_2}^{-}(j) = \emptyset \).

**Case 2:** \( \forall l \geq 0 \ \exists i_1, \ldots, i_l, i' \in V_1 : \exists Z' \subset Z_{i'} : e = Z_{i_1} \cup \ldots \cup Z_{i_l} \cup Z' \land Z' \neq \emptyset \).

We get \( i = i' \in V_1 \setminus \{i_1, \ldots, i_l\} \) as well as \( N_{D_1}^{-}(i') = \{i_1, \ldots, i_l\} = \pi_1(e) \setminus \{i'\} \in \mathcal{E}(NH^l(D_1)) \) and \( N_{D_2}^{-}(j) = \pi_2(e) \cap Z' = \pi_2(Z') \in \mathcal{E}(NH^l(D_2)) \) with a certain \( j \in V_2 \). In general, if \( |Z'| < |V_2| - 1 \) holds, the vertex \( j \) cannot be determined.

Again, for any hyperedge \( e \in \mathcal{E}(NH^l(D_1 \cdot D_2)) \) it cannot be found out whether \( e \) is a competition hyperedge (i.e. \( e \in \mathcal{E}(CH^l(D_1 \cdot D_2)) \)) or \( e \) is a common enemy hyperedge (i.e. \( e \in \mathcal{E}(CEH^l(D_1 \cdot D_2)) \)) in general. But for the reconstruction of \( NH^l(D_1) \) and \( NH^l(D_2) \) this plays no role, since the considerations of Case 1 and Case 2 are valid for competition hyperedges (i.e. sets of predecessors) as well as, analogously, for common enemy hyperedges (i.e. sets of successors).

Moreover, we remark that Cases 1 and 2 (together with their analogs for the common enemy hyperedges) provide all hyperedges of the \((l)\)-niche hypergraphs \( NH^{(l)}(D_1) \) and \( NH^{(l)}(D_2) \).

Now we discuss the disjunction \( D_1 \lor D_2 \). The case \(|V_1| = 1 \) or \(|V_2| = 1 \) implies \( D_1 \lor D_2 = D_1 \cdot D_2 \). Therefore, because of Theorem 5 it suffices to investigate the case \(|V_1|, |V_2| \geq 2 \).

**Theorem 6.** If \(|V_1|, |V_2| \geq 2 \), then \( NH^l(D_1) \) and \( NH^l(D_2) \) can be obtained from \( NH(D_1 \lor D_2) \).

**Proof.** Since both \( V_1 \) and \( V_2 \) contain at least two vertices, in \( NH^l(D_1 \lor D_2) \) there are no loops and \( NH^l(D_1 \lor D_2) = NH(D_1 \lor D_2) \).

Moreover, for every hyperedge \( e \in \mathcal{E}(NH(D_1 \lor D_2)) \) it holds

\[
\exists l \geq 0 \ \exists i_1, \ldots, i_l \in V_1 \ \exists j_1, \ldots, j_k \in V_2 : e = Z_{i_1} \cup \ldots \cup Z_{i_l} \cup S_{j_1} \cup \ldots \cup S_{j_k}
\]

and, clearly, \( \min(l, k) > 0 \).

By analogy with the proof of Theorem 5 let \((i, j) \in V_1 \times V_2\) be a vertex such that \( e = N_{D_1 \lor D_2}^{-}(i, j) \) or \( e = N_{D_1 \lor D_2}^{+}(i, j) \). Now we follow the idea of the proof of Proposition 2 in [20], subsection 3.5, and use the abbreviations \( \mathcal{E}_i^l := \mathcal{E}(NH^l(D_i)) \), \( \mathcal{E}_2^l := \mathcal{E}(NH^l(D_2)) \) and \( \mathcal{E}_\lor := \mathcal{E}(NH(D_1 \lor D_2)) \).

In case of \( \mathcal{E}_\lor = \emptyset \) both \( \mathcal{E}_1^l \) and \( \mathcal{E}_2^l \) are empty, too.

So let \( \mathcal{E}_\lor \neq \emptyset \). Additionally, for an arbitrary hyperedge \( e \in \mathcal{E}_\lor \) we define \( \pi_1^l(e) := \{i \mid (i, j) \in e\} \) (for \( j \in \pi_2(e) \)) and \( \pi_2^l(e) := \{j \mid (i, j) \in e\} \) (for \( i \in \pi_1(e) \)).

In \( NH(D_1 \lor D_2) \) we have three types of hyperedges:

\[
A := \{e \in \mathcal{E}_\lor \mid \pi_1(e) \subset V_1\},
\]

\[
B := \{e \in \mathcal{E}_\lor \mid \pi_2(e) \subset V_2\}
\]

and

\[
C := \{e \in \mathcal{E}_\lor \mid \pi_1(e) = V_1 \land \pi_2(e) = V_2\}.
\]
We obtain
\[ A = C = \emptyset \text{ if and only if } A_1 = \emptyset, E_1^1 = \emptyset \text{ and } E_2^1 = \{ \pi_2(e) \mid e \in E \}; \]
\[ B = C = \emptyset \text{ if and only if } A_2 = \emptyset, E_2 = \emptyset \text{ and } E_1^2 = \{ \pi_1(e) \mid e \in E \}; \]
\[ C \neq \emptyset \text{ if and only if } A_1 \neq \emptyset \neq A_2. \]

It remains to investigate the case \( C \neq \emptyset \). Here we see that, to determine \( E_1^1 \) and \( E_2^1 \), it suffices to make use of the hyperedges in \( C \):
\[ E_1^1 = \{ \{ i \in V_1 \mid \pi_1(e) = V_2 \} \mid e \in C \} \text{ and } E_2^1 = \{ \{ j \in V_2 \mid \pi_1(e) = V_1 \} \mid e \in C \}. \]
(Note that in case \( A = \emptyset \) we have \( E_1^1 = \{ \pi_1(e) \mid e \in A \} \) and, analogously, if \( B = \emptyset \) it follows \( E_2^1 = \{ \pi_2(e) \mid e \in B \} \).)

References


