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A COMPARISON OF ADAPTIVE COARSE SPACES FOR
ITERATIVE SUBSTRUCTURING IN TWO DIMENSIONS

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Abstract. The convergence rate of iterative substructuring methods generally deteriorates when
large discontinuities occur in the coefficients of the partial differential equations considered to be
solved. In dual-primal Finite Element Tearing and Interconnecting (FETI-DP) and Balancing Do-
main Decomposition by Constraints (BDDC) methods, sophisticated scalings, e.g., deluxe scaling,
can improve the convergence rate when large coefficient jumps occur. For more general cases, ad-
ditional information has to be added to the coarse space. One possibility is to enhance the coarse
space by local eigenvectors associated with subsets of the interface, e.g., edges. At the center of
the condition number estimates for FETI-DP and BDDC methods is an estimate related to the $P_D$
operator which is defined by the product of the transpose of the scaled jump operator $B_D^T$ and
the jump operator $B$ of the FETI-DP algorithm. Some enhanced algorithms directly spud in at the
$P_D$ operator, using related local eigenvalue problems, and some replace a local extension theorem
and local Poincaré inequalities by appropriate local eigenvalue problems. Three different strategies
are discussed, suggested by different authors, for adapting the coarse space together with suitable
scalings. Complete proofs and numerical results comparing the methods are provided.

Key words. FETI-DP, BDDC, eigenvalue problem, coarse space, domain decomposition, multiscale

AMS subject classifications. 65F10, 65N30, 65N55

1. Introduction. Iterative substructuring methods are known to be efficient
preconditioners for the large linear systems resulting from the discretization of sec-
ond order elliptic partial differential equations, e.g., diffusion and linear elasticity.
However, it is also known that the convergence rate of domain decomposition meth-
ods can severely deteriorate when large coefficient jumps occur. Besides certain spe-
cial coefficient distributions, e.g., constant coefficients in each subdomain and jumps
only across the interface, which can be treated with special scalings, the coarse
space has to be enhanced appropriately. One possible approach consists of, given
a user defined tolerance, adaptively solving certain local eigenvalue problems and
enhancing the coarse space appropriately based on the computed eigenvectors; see,
e.g., [3, 4, 15, 14, 10, 37, 38, 11, 18, 23, 31, 7].

We compare different adaptive coarse spaces that have been proposed by different
authors for the FETI-DP and BDDC domain decomposition methods, in particular
our approach in [23], the classic method in [31], a recent method in [7], and a variant
thereof in [21], are compared. Additionally, a proof of the condition number bound
for the method in [31] for the two dimensional case is given. We also introduce cost
efficient variants of the methods in [23] and [7] that are based on the ideas of an eco-

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local eigenvalue problems directly related to the $P_D$ operator, and the approach in [23] replaces a local extension theorem and local Poincaré inequalities by appropriate local eigenvalue problems. All the adaptive methods have in common to start from an initial coarse space that guarantees a nonsingular system matrix followed by adding additional constraints that are computed solving local generalized eigenvalue problems. In this paper, we implement the additional constraints with a method based on projections known as projector preconditioning [17, 26]. Nevertheless, the constraints could alternatively be computed using a transformation of basis [24, 30, 20].

The remainder of the paper is organized as follows: In Section 2, we introduce the model problems, their finite element discretization, and the domain decomposition. In Section 3, a short introduction to the FETI-DP algorithm and to projector preconditioning and deflation is given. The latter techniques are used to add the additional local eigenvectors to the coarse problem. A new, more general and direct proof for the condition number estimate of FETI-DP using deflation is given. In Section 4, the first approach considered here, see [7], to construct a coarse problem adaptively is considered. A proof for the condition number estimate is provided, different scalings are considered and a new economic variant is introduced and analyzed. In Remark 4.9 it is shown that the use of a certain scaling (deluxe scaling) allows to weaken the requirements on the domain from Jones to John domains in the analysis of FETI-DP and BDDC methods. In Section 5, as a second approach considered in this paper, the adaptive coarse space construction suggested in [31], is described and a new condition number estimate for two dimensions is proven. In Section 6, our approach from [23], which is the third coarse space analyzed here, is briefly described and a new variant with a modified deluxe scaling is introduced and analyzed. In Section 7 a brief analysis comparing the computational cost of the three coarse spaces with different scalings is provided. In Section 8, results of numerical experiments of the three different coarse spaces using different scalings applied to diffusion and almost incompressible linear elasticity are given. Finally, in Section 9, a conclusion is given.

2. Elliptic model problems, finite elements, and domain decomposition.

In this section, we introduce the elliptic model problems and their discretization by finite elements. We consider a scalar diffusion equation discretized by linear finite elements. Additionally, we consider a displacement formulation for almost incompressible linear elasticity which is obtained from a mixed finite element formulation with discontinuous pressure variables by static condensation of the pressure. Let $\Omega \subset \mathbb{R}^2$ be a bounded polyhedral domain, $\partial \Omega_D \subset \partial \Omega$ be a subset of positive surface measure, and $\partial \Omega_N := \partial \Omega \setminus \partial \Omega_D$. We consider the following diffusion problem: Find $u \in H^1_0(\Omega, \partial \Omega_D)$, such that

\begin{equation}
 a(u, v) = f(v) \quad \forall v \in H^1_0(\Omega, \partial \Omega_D),
\end{equation}

where $a(u, v) := \int_{\Omega} \rho(x) \nabla u \cdot \nabla v \, dx$, $f(v) := \int_{\Omega} fv \, dx + \int_{\partial \Omega_N} g_N v \, ds$,

here $g_N$ is the boundary data defined on $\partial \Omega_N$. We assume $\rho(x) > 0$ for $x \in \Omega$ and $\rho$ piecewise constant on $\Omega$. This problem is discretized using piecewise linear finite elements. As a second model problem, we consider the mixed displacement-pressure saddle-point system of almost incompressible linear elasticity. With the Lamé Parameters $\lambda$ and $\mu$ and the bilinear forms

\begin{align*}
 a(u, v) &= \int_{\Omega} 2\mu \varepsilon(u) : \varepsilon(v) \, dx, \\
 b(v, p) &= \int_{\Omega} \text{div}(v)p \, dx, \quad \text{and} \quad c(p, q) = \int_{\Omega} \frac{1}{\lambda} pq \, dx
\end{align*}
the saddle-point variational formulation is of the form: Find \((u, p) \in H^1_0(\Omega, \partial \Omega_D)^d \times L^2(\Omega)\), such that

\[
\begin{align*}
    a(u, v) + b(v, p) &= f(v) \quad \forall \, v \in H^1_0(\Omega, \partial \Omega_D)^d, \\
    b(u, q) - c(p, q) &= 0 \quad \forall \, q \in L^2(\Omega).
\end{align*}
\]

For almost incompressible materials, we use a discretization by mixed finite elements, e.g., P2-P0 elements. In our computations, the pressure variables are statically condensed element-by-element which yields again a variational formulation in the displacement variables. In the following, with a slight abuse of notation, we will make no distinction between a finite element function and its coordinate vector.

We decompose the domain \(\Omega\) into \(N\) nonoverlapping subdomains \(\Omega_i, i = 1, \ldots, N\), where each \(\Omega_i\) is the union of shape-regular triangular elements of diameter \(\mathcal{O}(h)\). We assume that the decomposition is such that the finite element nodes on the boundaries of neighboring subdomains are matching across the interface \(\Gamma := \bigcup_{i=1}^N \partial \Omega_i \setminus \partial \Omega\). The interface \(\Gamma\) is the union of edges and vertices where edges are defined as open sets that are shared by two neighboring subdomains and vertices are endpoints of edges. For a more general definition in three dimensions, see [28, 24]. We denote the edge belonging to the subdomains \(\Omega_i\) and \(\Omega_j\) by \(E_{ij}\). By \(W^h(\Omega_i)\) we denote the standard piecewise linear finite element space on \(\Omega_i\). We assume that these finite element functions vanish on \(\partial \Omega_D\) and that the finite element triangulation is quasi-uniform on each subdomain. By \(H_i\) or generically \(H\), we denote the subdomain diameter of \(\Omega_i\).

The local stiffness matrices on each subdomain are denoted by \(K^{(i)}\), \(i = 1, \ldots, N\).

Let \(a_l(u, v)\) be the bilinear form corresponding to the local stiffness matrix on a subdomain \(\Omega_i\) obtained by a finite element discretization of an elliptic problem. The respective coefficients are denoted by \(\rho_l\) in the case of diffusion and by \(\lambda_l\) and \(\mu_l\) in the case of linear elasticity. For almost incompressible linear elasticity the subdomain stiffness matrices are defined as \(K^{(i)} := A^{(i)} + B^{(i)T} C^{(i)}^{-1} B^{(i)}\), where the matrices

\[
\begin{align*}
    u^T A^{(i)} v &= \int_{\Omega_i} 2 \mu_l \varepsilon(u) : \varepsilon(v) \, dx, \\
    p^T B^{(i)} u &= \int_{\Omega_i} \text{div}(u) \, p \, dx, \\
    p^T C^{(i)} q &= \int_{\Omega_i} \frac{1}{\mu_l} \varepsilon(q) : \varepsilon(q) \, dx
\end{align*}
\]

result from a discretization with inf-sup stable \(P2-P0\) finite elements. Other inf-sup stable elements with discontinuous pressures are possible as well. After the elimination of the pressure we define \(a_l(u, v) = u^T K^{(i)} v\).

3. The FETI-DP algorithm and deflation/projector preconditioning.

In this section we will briefly describe the FETI-DP algorithm and a recently introduced method with a second coarse level incorporated by deflation. For more details on the FETI-DP algorithm, see, e.g., [13, 12, 39, 28, 27] and for FETI-DP with deflation and projector preconditioning, see [26, 17].

We start with the local stiffness matrices \(K^{(i)}\) associated with the subdomains \(\Omega_i\).

Let the variables further be partitioned into variables in the interior of the subdomain \(u^{(i)}\), dual variables on the interface \(u^{D^{(i)}}\), and primal degrees of freedom on the interface \(u^{L^{(i)}}\). As primal variables, vertices associated with subdomain corners can be chosen; other choices are possible. For the local stiffness matrices, unknowns, and right-hand
sides this yields

\[
K^{(i)} = \begin{bmatrix}
K_{II}^{(i)} & K_{\Delta I}^{(i)} & K_{\Delta I}^{(i)T} \\
K_{\Delta I}^{(i)} & K_{\Delta \Delta}^{(i)} & K_{\Delta \Delta}^{(i)T} \\
K_{\Delta I}^{(i)} & K_{\Delta \Delta}^{(i)} & K_{\Delta \Delta}^{(i)T}
\end{bmatrix}, \quad u^{(i)} = \begin{bmatrix}
u^{(i)}_I \\ \nu^{(i)}_\Delta \\ \nu^{(i)}_\Delta
\end{bmatrix}, \quad \text{and } f^{(i)} = \begin{bmatrix} f^{(i)}_I \\ f^{(i)}_\Delta \\ f^{(i)}_\Delta
\end{bmatrix}.
\]

Coupling the local blocks together we obtain the block diagonal matrices \(K_{II} = \text{diag}_{i=1}^N K_{II}^{(i)}, K_{\Delta \Delta} = \text{diag}_{i=1}^N K_{\Delta \Delta}^{(i)}, \text{ and } K_{\Pi \Pi} = \text{diag}_{i=1}^N K_{\Pi \Pi}^{(i)}. \) Interior and dual degrees of freedom can be combined as remaining degrees of freedom.

The associated matrices and vectors are then of the form \(K_{BB}^{(i)} = \begin{bmatrix} K_{II}^{(i)} & K_{\Delta I}^{(i)T} \\ K_{\Delta I}^{(i)} & K_{\Delta \Delta}^{(i)} \end{bmatrix}, \) and \(f_B^{(i)} = \begin{bmatrix} f_I^{(i)T} \\ f_\Delta^{(i)} \end{bmatrix}. \) We define a block matrix and a block vector \(K_{BB} = \text{diag}_{i=1}^N K_{BB}^{(i)}, \) \(u_B = [u_B^{(i)T}, \ldots, u_B^{(N)T}]^T, \) and a block right-hand side vector \(f_B = \begin{bmatrix} f_B^{(i)T}, \ldots, f_B^{(N)T} \end{bmatrix}^T. \) We introduce partial assembly operators in the primal variables \(R_B^{(i)T}. \) These matrices consist of zeros and ones only. After partial assembly, to enforce continuity in the primal variables we obtain the matrices \(\tilde{K}_{\Pi \Pi} = \sum_{i=1}^N R_B^{(i)T}\tilde{K}_{\Pi \Pi}^{(i)}R_B^{(i)}, \) \(\tilde{K}_{BB} = \left[ R_B^{(i)T}K_{BB}^{(i)} \ldots, R_B^{(N)T}K_{BB}^{(N)} \right] \) and right-hand side \(\tilde{f} = \begin{bmatrix} f_B, \ldots, f_B^{(N)} \end{bmatrix}^T. \) After elimination of all but the primal degrees of freedom we obtain the Schur complement \(\tilde{S}_{\Pi \Pi} = \tilde{K}_{\Pi \Pi} - \tilde{K}_{BB}K_{BB}^{-1}\tilde{K}_{BB}. \) We define a jump matrix \(B_B = [B_B^{(1)}, \ldots, B_B^{(N)}] \) that connects the dual degrees of freedom on the interface, such that \(B_Bu_B = 0 \) if \(u_B \) is continuous. The FETI-DP system is then given by \(F\lambda = d, \) with

\[
F = B_BK_{BB}^{-1}B_B^T + B_BK_{BB}^{-1}\tilde{K}_{BB}^{-1}\tilde{K}_{BB}K_{BB}^{-1}B_B^T, \\
\text{and } d = B_BK_{BB}^{-1}\tilde{f} + B_BK_{BB}^{-1}\tilde{K}_{BB}^{-1}\tilde{S}_{\Pi \Pi}\left(\tilde{f} - \tilde{K}_{BB}K_{BB}^{-1}B_B\tilde{f}\right).
\]

The FETI-DP algorithm is the preconditioned conjugate gradients algorithm applied to \(F\lambda = d, \) with the Dirichlet preconditioner

\[
M^{-1} = B_{B,D} \begin{bmatrix} 0 & I_{\Delta} \end{bmatrix}^T \left( K_{\Delta \Delta} - K_{\Delta I}K_{II}^{-1}K_{\Delta I}^T \right) \begin{bmatrix} 0 & I_{\Delta} \end{bmatrix} B_{B,D}^T.
\]

Here \(B_{B,D} \) is a scaled variant of \(B_B, \) in the simplest case it is scaled by the inverse multiplicity of the nodes, e.g., 1/2 in two dimensions. Alternatively, we use the approach in, e.g., [25, 35] and introduce scaling weights by

\[
\delta_j(x) := \left( \sum_{i \in N_x} \hat{\rho}_j(x) \right) / \hat{\rho}_j(x),
\]

where \(\hat{\rho}_j(x) = \max_{x \in \omega(x) \cap \Omega_j \cap T_j} \rho_j(x). \) Here \(\omega(x)\) is the support of the finite element basis function associated with the node \(x \in \partial \Omega_j \cap \Gamma_h, j = 1, \ldots, N. \) The pseudoinverses are defined by

\[
\delta_j(x)^\dagger := \hat{\rho}_j(x) / \sum_{i \in N_x} \hat{\rho}_i(x)
\]
for $x \in \partial \Omega_{j,h} \cap \Gamma_{h}$. Each row of $B^{(i)}$ with a nonzero entry connects a point of $\Gamma^{(i)}_{h}$ with the corresponding point of a neighboring subdomain $x \in \Gamma^{(i)}_{h} \cap \Gamma^{(j)}_{h}$. Multiplying each such row with $\delta_{i}(x)^{T}$ for each $B^{(i)}$, $i = 1, \ldots, N$ results in the scaled operator $B_{D}$. We will refer to this scaling by $\rho$-scaling. For coefficients that are constant on each subdomain this approach reduces to classical $\rho$-scaling; see, e.g., [39].

Another set of primal constraints can be aggregated as columns of a matrix $U$; see, e.g., [17, 26]. To enforce $U^{T}Bu = 0$, e.g., averages of the jump with weights defined by the columns of $U$, we introduce the $F$-orthogonal projection $P = U(U^{T}FU)^{-1}U^{T}F$. Instead of solving $F\lambda = d$ the deflated and singular but consistent system $(I - P)^{T}F\lambda = (I - P)^{T}d$ can be solved. Denoting by $\lambda^{*}$ the exact solution of $F\lambda = d$ we define

$$\bar{\lambda} = U(U^{T}FU)^{-1}U^{T}d = PF^{-1}d = P\lambda^{*}. \quad (3.2)$$

Let $\lambda$ be the solution of $M^{-1}(I - P)^{T}F\lambda = M^{-1}(I - P)^{T}d$ by PCG, where $M^{-1}$ is the classical Dirichlet preconditioner. Then, we can compute $\lambda^{*} = \bar{\lambda} + (I - P)\lambda \in \ker (I - P) \oplus \text{range} (I - P)$. The matrices $P^{T}F = FP$ and $(I - P)^{T}F = F(I - P)$ are symmetric. We do not change the spectrum by projecting the correction onto $\text{range} (I - P)$ in every iteration [26]. Therefore, we obtain the symmetric projector preconditioner

$$M_{BP}^{-1} = (I - P)M^{-1}(I - P)^{T}. \quad (3.3)$$

Adding the correction we compute $\lambda^{*} = \bar{\lambda} + \lambda$, where $\lambda$ is the PCG solution of $M_{BP}^{-1}F\lambda = M_{BP}^{-1}Fd$. Additionally, we can include the computation of $\bar{\lambda}$ into the preconditioner. This results in the balancing preconditioner

$$M_{BP}^{-1} = (I - P)M^{-1}(I - P)^{T} + PF^{-1}. \quad (3.3)$$

Since $PF^{-1} = U(U^{T}FU)^{-1}U^{T}$ this preconditioner is symmetric and can be efficiently computed. Here $U^{T}FU$ is usually much smaller than $F$.

For each subdomain, we introduce local finite element trace spaces $W_{i} := W^{h}(\partial \Omega_{i} \cap \Gamma_{i})$, $i = 1, \ldots, N$. We define the product space $W := \Pi_{i=1}^{N}W_{i}$ and denote the subspace of functions $w \in W$ that are continuous in the primal variables by $\bar{W}$.

The following lemma is an alternative to the proof provided in [26] for projector preconditioning or deflation applied to FETI-DP methods. It directly applies to a larger class of scalings.

**Lemma 3.1.** Let $P_{D} = B_{D}^{T}B$. Assuming that $||P_{D}w||_{\bar{W}}^{2} \leq C||w||_{\bar{W}}^{2}$ for all $w \in \{w \in \bar{W} | U^{T}Bu = 0\}$ with a constant $C > 0$, we have

$$\kappa (M_{BP}^{-1}F) \leq C.$$

Here the constant $C$ can depend on $H/h$ or $\eta/h$, and possibly on a prescribed tolerance from local generalized eigenvalue problems.

**Proof.** Similar to [28, p. 1553] we use $(I - P)^{T}F = F(I - P)$ with the standard Dirichlet preconditioner $M^{-1}$. Observing that $\tilde{S}^{-1}B^{T}(I - P)\lambda \in \bar{W}$ and

$$(I - P)U = 0 \Rightarrow U^{T}B(\tilde{S}^{-1}B^{T}(I - P)\lambda) = U^{T}(I - P)^{T}BS^{-1}B^{T}\lambda = 0,$$
we obtain for the upper bound
\[
\langle M_{pp}^{-1}F\lambda,\lambda \rangle_{F} = \langle (I-P)M^{-1}(I-P)^{T}F\lambda,\lambda \rangle = \langle M^{-1}F(I-P)\lambda,F(I-P)\lambda \rangle \\
= \langle B^{-1}_{D}BS^{-1}B^{T}(I-P)\lambda,B^{-1}_{D}BS^{-1}B^{T}(I-P)\lambda \rangle_{\tilde{S}} = |P_{D}(S^{-1}B^{T}(I-P)\lambda)|^{2}_{\tilde{S}} \\
\leq |P_{D}w|^{2}_{\tilde{S}} \leq C|w^{2}_{\tilde{S}} = |\tilde{S}^{-1}B^{T}(I-P)\lambda|^{2}_{\tilde{S}} \\
= \langle \tilde{S}^{-1}B^{T}(I-P)\lambda,\tilde{S}^{-1}B^{T}(I-P)\lambda \rangle_{\tilde{S}} = \langle (I-P)\lambda,(I-P)\lambda \rangle_{F}. \tag{3.4}
\]

Since \( \lambda \in \text{range}(I-P) \) we have \( \lambda = (I-P)\lambda \). With \( E_{D}w(x) := \sum_{j \in N_{x}}D^{(j)}w_{j}(x) \) we see that \( P_{D}w = B^{-1}_{D}Bw = (I-E_{D})w \). Since \( E_{D}w \) is continuous over the interface, \( P_{D} \) preserves the jump of any function \( w \in \tilde{W} \) in the sense that \( Bw = Bw - 0 = B(I-E_{D})w = BP_{D}w \). Analogously to [28, p. 1552], we obtain for the lower bound
\[
\langle \lambda,\lambda \rangle_{F}^{2} = \langle \lambda,B\tilde{S}^{-1}B^{T}\lambda \rangle^{2} = \langle \lambda,B\tilde{S}^{-1}P_{D}^{T}B^{T}\lambda \rangle^{2} = \langle \lambda,B\tilde{S}^{-1}B^{T}B_{D}B^{T}\lambda \rangle^{2} \\
= \langle \lambda,B_{D}B^{T}\lambda \rangle_{F}^{2} = \langle F\lambda,B_{D}\tilde{S}^{1/2}\tilde{S}^{-1/2}B^{T}\lambda \rangle^{2} \]
\[
\leq \langle \tilde{S}^{1/2}B_{D}^{T}F\lambda,\tilde{S}^{1/2}B_{D}^{T}F\lambda \rangle \langle \tilde{S}^{-1/2}B^{T}\lambda,\tilde{S}^{-1/2}B^{T}\lambda \rangle \\
= \langle M^{-1}F\lambda,F\lambda \rangle \langle (I-P)\lambda,(I-P)\lambda \rangle \langle F\lambda,\lambda \rangle \]
\[
= \langle M_{pp}^{-1}F\lambda,\lambda \rangle_{F} \langle F\lambda,\lambda \rangle.
\]

4. First coarse space. In this approach, the general eigenvalue problems are based on a localization of the \( P_{D} \)-estimate in contrast to [23] where an edge lemma and a Poincaré-Friedrichs inequality are used; see also Section 6. The section is organized as follows. In Section 4.1, we introduce the relevant notation and in Section 4.2 we show how the energy of the \( P_{D} \) operator can be bounded by local estimates. In Section 4.3 we collect some known information on the parallel sum of matrices and show some related spectral estimates. In Sections 4.4 and 4.5 we introduce two approaches to enhance the coarse space with adaptively computed constraints. In both approaches the constraints are computed by solving local generalized eigenvalue problems. The first approach has been proposed in [7] and relies on deluxe scaling. In the second approach first proposed in [21], every kind of scaling is possible as long it satisfies the partition of unity property (4.2). For the special case of deluxe scaling the second approach is the same as the first approach. In Section 4.6, we consider an economic variant solving eigenvalue problems on slabs. Finally, in Section 4.7 we prove a condition number bound for the FETI-DP algorithm with adaptive constraints as described in Sections 4.4.2, 4.5.2, or 4.6.2.

4.1. Notation. We define the energy minimal extension of \( v \) from the local interface to the interior of the subdomain \( \Omega_{l} \) as
\[
\mathcal{H}^{(l)}v := \arg\min_{u \in V_{\Omega_{l}}} \{ a_{l}(u,u) : u|_{\partial\Omega_{l}} = v \} \quad \text{for} \quad l = i,j.
\]
Let \( \theta_{E_{ij}} \) be the standard finite element cut-off function which equals 1 on the edge \( E_{ij} \) and is zero on \( \partial\Omega_{l} \setminus E_{ij} \). With \( l^{h} \) we denote the standard finite element interpolation operator. We will make use of the seminorm
\[
|v|^{2}_{E_{ij}} := a_{l}(v,v). \tag{4.1}
\]
We will also make use of an energy minimal extension from an edge \( E_{ij} \) to the interface \( \Gamma^{(i)} \), \( l = i,j \). The following lemma describes how this extension is obtained. Its proof follows from a standard variational argument.

**Lemma 4.1.** Let \( E \subset \Gamma^{(i)} := \partial \Omega_i \) be an edge and \( \mathcal{E}^c \subset \Gamma^{(i)} \) be the complement of \( \mathcal{E} \) with respect to \( \Gamma^{(i)} \). Define an extension from an edge \( E \subset \Gamma^{(i)} \) to \( \Gamma^{(i)} \) by

\[
v_i = \begin{bmatrix} v_E \\ -S_{E,E^c}^{-1} S_{E^c,E} v_E \end{bmatrix}, \quad \text{where } S^{(i)} = \begin{bmatrix} S_{E,E}^{(i)} & S_{E^c,E}^{(i)T} \\ S_{E,E}^{(i)} & S_{E^c,E}^{(i)} \end{bmatrix}.
\]

Then, for all \( w_i \in V_h(\Gamma^{(i)}) \) with \( w_{ij}E = v_E \), we have \( |v_i|_{S^{(i)}} \leq |w_i|_{S^{(i)}} \).

**Definition 4.2.** We define the extension operator \( H^{(i)}_E v_E := \begin{bmatrix} v_E \\ -S_{E,E^c}^{-1} S_{E^c,E} v_E \end{bmatrix} \) and the matrices \( S^{(i)}_{E_{ij},0} := S^{(i)}_{E_{ij},E_{ij}} \) and \( S^{(i)}_{E_{ij}} := S^{(i)}_{E_{ij},E_{ij}} - S^{(i)}_{E_{ij},E_{ij}}^{-1} S^{(i)}_{E_{ij},E_{ij}} \).

With Definition 4.2 we have the following correspondences between (semi)norms and the matrices defined in Definition 4.2:

\[
|H^{(i)}_E v_E|^2_{E_{ij}} = |v_E|^2_{E_{ij},0} \quad l = i,j
\]

\[
|H^{(i)}_E v_E|^2_{E_{ij}} = |v_E|^2_{E_{ij},E_{ij}} \quad l = i,j.
\]

Let \( D^{(i)} \), \( l = i,j \), be scaling matrices, such that

\[
D^{(i)} + D^{(j)} = I,
\]

where \( I \) is the identity matrix; this is a partition of unity.

**4.2. Bounding the energy of the jump operator by local contributions.**

In the following, we will assume that vectors are restricted to the edge \( E_{ij} \) if they are multiplied by a matrix with the index \( E_{ij} \) or \( E_{ij,0} \) to avoid excessive use of restriction operators. As a classical result in the analysis of iterative substructuring, see, e.g., [28, 39], we have

\[
|P_D w|^2_{S_i} = |R P_D w|^2_{S_i} = \sum_{i=1}^{N} |R^{(i)} P_D w|^2_{S_i}.
\]

Let \( N_E \) denote the maximum number of edges of a subdomain. Under the assumption that all vertices are primal, we obtain

\[
|R^{(i)} P_D w|^2_{S_i} \leq N_E \sum_{j \in N_i} |H^{(i)} v_E, D^{(j)}(w_i - w_j)|^2_{E_j},
\]

where \( N_i \) denotes the set of indices of subdomains that share an edge with \( \Omega_i \). Hence, we are interested to obtain bounds for the local contributions on the edges \( E_{ij} \) of the form:

\[
|H^{(i)} v_E, D^{(j)}(w_i - w_j)|^2_{E_j} + |H^{(j)} v_E, D^{(i)}(w_j - w_i)|^2_{E_j} \leq C \left( |w_i|^2_{E_i} + |w_j|^2_{E_j} \right).
\]

Using Definition 4.2, this is equivalent to

\[
(w_i - w_j)^T D_{E_{ij}}^{(i)T} S_{E_{ij},0} D_{E_{ij}}^{(j)} (w_i - w_j) + (w_j - w_i)^T D_{E_{ij}}^{(j)T} S_{E_{ij},0} D_{E_{ij}}^{(i)} (w_j - w_i) \leq C \left( |w_i|^2_{E_i} + |w_j|^2_{E_j} \right).\]
4.3. Parallel sum of matrices and spectral estimates. The next lemma introduces the notion of parallel sum of matrices for two symmetric positive semidefinite matrices and properties of that operation. The definition of a parallel sum of matrices was first given in [1] and for the first time used in our context in [7]. The first two properties of the next lemma are given and proven in [1]. The third property is given, without a proof, in [7].

Remark 4.3. Using that \( \text{Ker}(A + B) \subset \text{Ker}(A) \) and \( \text{Ker}(A + B) \subset \text{Ker}(B) \) for symmetric positive semidefinite \( A \) and \( B \) and that \( U \subset V \) implies \( V^\perp \subset U^\perp \), we obtain \( \text{Range}(A) \subset \text{Range}(A + B) \) and \( \text{Range}(B) \subset \text{Range}(A + B) \). With [33, Theorem 2.1] we conclude that \( A : B := (A + B)^+ B \) is invariant under the choice of the pseudoinverse \( (A + B)^+ \).

Lemma 4.4 (Parallel sum of matrices). Let \( A, B \) be symmetric positive semidefinite and define \( A : B = A(A + B)^+ B \) as in Remark 4.3 where \( (A + B)^+ \) denotes a pseudoinverse with \( (A + B)(A + B)^+(A + B) = (A + B) \) and \( (A + B)^+(A + B)(A + B)^+ = (A + B)^+ \). Then, we have

1. \( A : B \leq A \) and \( A : B \leq B \) (spectral estimate).
2. \( A : B \) is symmetric positive semidefinite.
3. Defining \( D_A := (A + B)^+ A \) and \( D_B := (A + B)^+ B \), we additionally have

\[
D_A^T B D_A \leq A : B \quad \text{and} \quad D_B^T A D_B \leq A : B \quad (4.3)
\]

Proof. For the proof of 1. and 2., see [1]. Next, we provide a proof of 3. Since \( A \) and \( B \) are s.p.s.d., \( D_A^T B D_B \) and \( D_B^T A D_A \) are also s.p.s.d. and we obtain

\[
\]

Since \( A \) and \( B \) are s.p.s.d., \( x^T (A + B) x = 0 \) implies \( x^T A x = -x^T B x = 0 \). Thus, we have \( \text{Ker}(A + B) = \text{Ker}(A) \cap \text{Ker}(B) \). For any \( x \) we can write \( x = x_R + x_K \) with \( x_R \in \text{Range}(A + B)^+ \) and \( x_K \in \text{Ker}(A + B) = \text{Ker}(A) \cap \text{Ker}(B) \). Using that \( (A + B)^+(A + B) \) is a projection onto \( \text{Range}(A + B)^+ \), we obtain

\[
x^T D_A^T B D_A x + x^T D_B^T A D_B x = x^T (A : B)(A + B)^+(A + B)x
\]

\[
= x^T (A : B)x_R
\]

\[
= x^T (A : B)x.
\]

\( \blacksquare \)

Furthermore, we need some properties of projections on eigenspaces of generalized eigenvector problems. The next lemma is a well known result from linear algebra.

Lemma 4.5. Let \( A \in \mathbb{R}^{n \times n} \) be symmetric positive semidefinite and \( B \in \mathbb{R}^{n \times n} \) be symmetric positive definite. Consider the generalized eigenvector problem

\[
Ax_k = \lambda_k Bx_k \quad \text{for } k = 1, \ldots, n. \quad (4.4)
\]

Then the eigenvectors can be chosen to be \( B \)-orthogonal and such that \( x_k^T B x_k = 1 \). All eigenvalues are positive or zero.

The proof of the next lemma is based on arguments from classical spectral theory, thus the proof is omitted here. A related abstract lemma, also based on classical spectral theory, can be found in [37, Lemma 2.11].
LEMMA 4.6. Let $A, B$ be as in Lemma 4.5 and define $\Pi_m^B := \sum_{i=1}^m x_i x_i^T B$. Let the eigenvalues be sorted in an increasing order $0 = \lambda_1 \leq \ldots \leq \lambda_m < \lambda_{m+1} \leq \ldots \leq \lambda_n$. Then, $x = \Pi_m^B x$ and
\[
|x - \Pi_m^B x|^2 = (x - \Pi_m^B x)^T B (x - \Pi_m^B x) \leq \lambda_{m+1} x^T A x = \lambda_{m+1}^r |x|^2_A.
\]
Additionally, we have the stability of $\Pi_m^B$ in the $B$-norm
\[
|x - \Pi_m^B x|^2 \leq |x|^2_B.
\]

4.4. First approach [7]. The first approach that we will discuss was proposed in [7].

4.4.1. Notation. In the following we define a scaling for the FETI-DP and BDDC method, denoted as deluxe scaling, which has been first introduced in [8]; for further applications, see [2, 29, 34, 5, 6, 20]. Note that this is not a scaling in the common sense since more than just a multiplication with a diagonal matrix is involved.

DEFINITION 4.7 (Deluxe scaling). Let $\mathcal{E}_{ij} \subset \Gamma^{(i)}$ be an edge and let the Schur complements $S_{E_{ij},0}^{(i)}$, $S_{E_{ij},0}^{(j)}$ be as in Definition 4.2. We define the following scaling matrices
\[
D_{E_{ij}}^{(i)} := (S_{E_{ij},0}^{(i)} + S_{E_{ij},0}^{(j)})^{-1} S_{E_{ij},0}^{(i)}, \quad l = i, j.
\]
Let $R_{E_{ij}}^{(i)}$ be the restriction operator restricting the degrees of freedom on a subdomain interface $\Gamma^{(i)}$, $l = i, j$ to the degrees of freedom on the open edge $\mathcal{E}_{ij}$. Then, we define the subdomain (deluxe) scaling matrices by
\[
D^{(i)} := \sum_{\mathcal{E}_{ij} \subset \Gamma^{(i)}} R_{E_{ij}}^{(i)} D_{E_{ij}}^{(i)} R_{E_{ij}}^{(i)}.
\]
Each pair of scaling matrices $D^{(i)}$, $D^{(j)}$, $D_{E_{ij}}^{(i)}$, $D_{E_{ij}}^{(j)}$ satisfies property (4.2). The scaled jump operator $B_D$ in the FETI-DP algorithm is then given by $B_D := [D^{(1)^T} B^{(1)}, \ldots, D^{(N)^T} B^{(N)}]$ where the transpose is necessary since the $D^{(i)}$ are not symmetric. With Lemma 4.4 we obtain
\[
D_{E_{ij}}^{(i)^T} S_{E_{ij},0}^{(i)} D_{E_{ij}}^{(j)} \leq S_{E_{ij},0}^{(i)} : S_{E_{ij},0}^{(j)} \quad \text{and} \quad D_{E_{ij}}^{(i)^T} S_{E_{ij},0}^{(j)} D_{E_{ij}}^{(i)} \leq S_{E_{ij},0}^{(i)} : S_{E_{ij},0}^{(j)}.
\]

4.4.2. Generalized eigenvalue problem (first approach). We solve the eigenvalue problem
\[
S_{E_{ij}}^{(i)} : S_{E_{ij},0}^{(j)} x_k = \mu_k S_{E_{ij},0}^{(i)} : S_{E_{ij},0}^{(j)} x_k, \quad \text{for} \quad \mu_k \leq \text{TOL and enforce the constraints} \quad x_k^T (S_{E_{ij},0}^{(i)} : S_{E_{ij},0}^{(j)}) (w_i - w_j) = 0, \text{e.g., as described in Section 3.}
\]
LEMA 4.8. We define $\Pi_k := \sum_{m=1}^k x_m x_m^T S_{E_{ij},0}^{(i)} : S_{E_{ij},0}^{(j)}$ using the eigenvectors $x_m$ of the generalized eigenvalue problem (4.5). Then, we have $\Pi_k (w_i - w_j) = 0$ and the following inequality holds:
\[
(w_i - w_j)^T \left( D_{E_{ij}}^{(i)^T} S_{E_{ij},0}^{(i)} D_{E_{ij}}^{(j)} + D_{E_{ij}}^{(i)^T} S_{E_{ij},0}^{(j)} D_{E_{ij}}^{(i)} \right) (w_i - w_j) \leq 2 (\mu_k + 1)^{-1} (w_i^T S_{E_{ij},0}^{(i)} w_i + w_j^T S_{E_{ij},0}^{(j)} w_j).
\]
Proof. The property $\Pi_k(w_i - w_j) = 0$ follows directly. We have
\[
(w_i - w_j)^T D^{(j)T}_{E_{ij}} S^{(i)}_{E_{ij},0} D^{(j)}_{E_{ij}} (w_i - w_j) + (w_j - w_i)^T D^{(i)T}_{E_{ij}} S^{(j)}_{E_{ij},0} D^{(i)}_{E_{ij}} (w_i - w_j)
\]
\[
= (w_i - w_j)^T S^{(j)}_{E_{ij},0} (S^{(i)}_{E_{ij},0} + S^{(j)}_{E_{ij},0})^{-1} S^{(i)}_{E_{ij},0} D^{(j)}_{E_{ij}} (w_i - w_j)
\]
\[
+ (w_i - w_j)^T S^{(i)}_{E_{ij},0} (S^{(i)}_{E_{ij},0} + S^{(j)}_{E_{ij},0})^{-1} S^{(j)}_{E_{ij},0} D^{(i)}_{E_{ij}} (w_i - w_j)
\]
\[
= (w_i - w_j)^T [(S^{(i)}_{E_{ij},0} : S^{(j)}_{E_{ij},0}) D^{(j)}_{E_{ij}} + (S^{(i)}_{E_{ij},0} : S^{(j)}_{E_{ij},0}) D^{(i)}_{E_{ij}}] (w_i - w_j)
\]
\[
= (w_i - w_j)^T (S^{(i)}_{E_{ij},0} : S^{(j)}_{E_{ij},0}) (w_i - w_j)
\]
\[
\leq 2(\mu_{k+1})^{-1} \left( w_i^T S^{(i)}_{E_{ij},0} w_i + w_j^T S^{(j)}_{E_{ij},0} w_j \right)
\]
\[
\leq 2(\mu_{k+1})^{-1} \left( w_i^T S^{(i)}_{E_{ij},0} w_i + w_j^T S^{(j)}_{E_{ij},0} w_j \right)
\]
(4.6)

For the last two estimates notice that $w_i - w_j = w_i - \Pi_k w_i - (w_j - \Pi_k w_j)$ and apply Lemma 4.6 with $A = S^{(i)}_{E_{ij},0} : S^{(j)}_{E_{ij},0}$ and $B = S^{(i)}_{E_{ij},0} : S^{(j)}_{E_{ij},0}$. Using the first property of Lemma 4.4 we obtain the desired bound.

Remark 4.9. Up to equation (4.6) no generalized eigenvalue problem is used but only deluxe scaling. Since the term in (4.6) is bounded by $2 \left( w_i^T S^{(i)}_{E_{ij},0} w_i + w_j^T S^{(j)}_{E_{ij},0} w_j \right)$ it replaces a classical extension theorem. In [27], the analysis of FETI-DP methods in two dimensions has been extended to uniform domains which are a subset of John domains. Since all tools were provided for John domains with the exception of the extension theorem which requires uniform domains, by using deluxe scaling, the analysis carries over to the broader class of John domains.

4.5. Second approach [21]. In this section, we describe a variant of the first approach that allows different kinds of scalings. In the case of standard deluxe scaling this algorithm is the same as the algorithm in [7]; cf., Section 4.4. A short description of this variant has already been presented in the proceedings article [21].

4.5.1. Notation. We use the notation from Section 4.1.

4.5.2. Generalized eigenvalue problem (second approach). We solve the eigenvalue problem
\[
S^{(i)}_{E_{ij},0} : S^{(j)}_{E_{ij},0} x_k = \mu_k \left( D^{(j)T}_{E_{ij}} S^{(i)}_{E_{ij},0} D^{(j)}_{E_{ij}} + D^{(i)T}_{E_{ij}} S^{(j)}_{E_{ij},0} D^{(i)}_{E_{ij}} \right) x_k,
\]
(4.7)
where $\mu_k \leq \text{TOL}$ and enforce the constraints
\[
x_k^T \left( D^{(j)T}_{E_{ij}} S^{(i)}_{E_{ij},0} D^{(j)}_{E_{ij}} + D^{(i)T}_{E_{ij}} S^{(j)}_{E_{ij},0} D^{(i)}_{E_{ij}} \right) (w_i - w_j) = 0,
\]
e.g., as described in Section 3. Analogously to Lemma 4.8, we obtain the following bound.

Lemma 4.10. We define $\Pi_k := \sum_{m=1}^k x_m x_m^T (D^{(j)T}_{E_{ij}} S^{(i)}_{E_{ij},0} D^{(j)}_{E_{ij}} + D^{(i)T}_{E_{ij}} S^{(j)}_{E_{ij},0} D^{(i)}_{E_{ij}})$ using the eigenvectors $x_m$ of the generalized eigenvalue problem (4.7). Then, we have $\Pi_k (w_i - w_j) = 0$ and the following inequality holds:
\[
(w_i - w_j)^T \left( D^{(j)T}_{E_{ij}} S^{(i)}_{E_{ij},0} D^{(j)}_{E_{ij}} + D^{(i)T}_{E_{ij}} S^{(j)}_{E_{ij},0} D^{(i)}_{E_{ij}} \right) (w_i - w_j)
\]
\[
\leq 2(\mu_{k+1})^{-1} \left( w_i^T S^{(i)}_{E_{ij},0} w_i + w_j^T S^{(j)}_{E_{ij},0} w_j \right).
\]
where $D^{(i)}_{E,ij}$, $i,j$ are arbitrary scaling matrices that provide a partition of unity, i.e., satisfy (4.2).

Proof. Notice that $w_i - w_j = w_i - \Pi_k w_i - (w_j - \Pi_k w_j)$ and apply Lemma 4.6 with $A = S^{(i)}_{E,ij} : S^{(ij)}_{E,ij}$ and $B = D^{(ij)}_{E,ij} S^{(i)}_{E,ij}D^{(j)}_{E,ij} + D^{(ij)}_{E,ij} S^{(ij)}_{E,ij}D^{(ij)}_{E,ij}$. With (4.3) we obtain the desired bound. \[\square\]

4.6. Economic variant of the algorithm. In this section, we introduce a new, more economic variant, solving such eigenvalue problems on slabs. Using such a variant for deluxe scaling but without such eigenvectors in the coarse space was first introduced and numerically tested in [9]; see Remark 4.13. Let us note that with respect to the eigenvalue problems on slabs, this variant is new. Let us first give the definition of an $\eta$-patch; see, e.g., also [39, Lemma 3.10], [23, Def. 6.1], and [35, Def. 2.5 and 2.6].

Definition 4.11. By an $\eta$-patch $\omega \subset \Omega$ we denote an open set which can be represented as a union of shape regular finite elements of diameter $O(h)$ and which has $\text{diam}(\omega) = O(\eta)$ and a measure of $O(\eta^2)$.

The next definition was introduced in 3D in [16]; see also [23, 19].

Definition 4.12. Let $E_{ij} \subset \partial \Omega_i$ be an edge. Then, a slab $\bar{\Omega}_{ij}$ is a subset of $\Omega_i$ of width $\eta$ with $E_{ij} \subset \partial \bar{\Omega}_{ij}$ which can be represented as the union of $\eta$-patches $\omega_k$, $k = 1, \ldots, n$, such that $(\partial \bar{\omega}_k \cap E_{ij})^\circ \neq \emptyset$, $k = 1, \ldots, n$.

4.6.1. Notation. In addition to $|v|_{E_i}$, c.f. (4.1), we define $|v|^2_{E_i,\eta} := a_{\eta,v}(v, v)$, where $a_{\eta}(v, v)$ is the same bilinear form as $a(v, v)$ but with an integral over the slab $\bar{\Omega}_{ij}$. Let $K^{(i)}_{\eta}$ be the locally assembled stiffness matrix of the slab of width $\eta$ corresponding to an edge $E$ in subdomain $\Omega_i$; here, we use homogeneous Neumann boundary conditions on the part of the boundary of the slab which has an intersection with the interior of $\Omega_i$ with nonvanishing measure. Let

$$S^{(i)}_{E,\eta} = S^{(i)}_{E,E,\eta} - S^{(i)}_{E,E,\eta} S^{(i)}_{E,E,\eta}^{-1} S^{(i)}_{E,E,\eta}$$

be the Schur complement of $K^{(i)}_{\eta}$ after elimination of all degrees of freedom except those on the edge. With the discrete $\rho$-harmonic extension operator $H^{(i)}_{\eta}$ from $\Gamma^{(i)} \cap \partial \bar{\Omega}_{ij}$ to the interior we have $|H^{(i)}_{\eta} H^{(i)}_{E,v}|^2_{E,\eta} = v_{E,\eta}^T H^{(i)}_{\eta} v_{E,\eta}$. Let the subdomain stiffness matrices $K^{(i)}$ be partitioned into variables on the edge $E$ and remaining variables $E^c$ by $K^{(i)} = \begin{pmatrix} K^{(i)}_{EE} & K^{(i)}_{Ef} \\ K^{(i)}_{fE} & K^{(i)}_{ff} \end{pmatrix}$. Thus, by removing all columns and rows related with degrees of freedom outside the the closure of the slab and those on $(\partial \bar{\Omega}_{ij} \cap \Gamma^{(i)}) \setminus E$, we obtain a matrix of the form $\begin{pmatrix} K^{(i)}_{EE} & K^{(i)}_{Ef} \\ K^{(i)}_{I,EE} & K^{(i)}_{I,f} \end{pmatrix}$. Here, the index $I_{\eta}$ relates to the degrees of freedom on the closure of the slab except those on $\partial \bar{\Omega}_{ij} \cap \Gamma^{(i)}$. We define another Schur complement by

$$S^{(i)}_{E,0,\eta} = K^{(i)}_{EE} - K^{(i)}_{I,EE} K^{(i)}_{I,I}^{-1} K^{(i)}_{fI}.$$ 

We define an extension operator $H^{(i)}_{\eta}$ from the local interface $\partial \bar{\Omega}_{ij} \cap \Gamma^{(i)}$ of a subdo-
main $\Omega_l$ to the interior by

$$
H^{(i)}_{\eta_0,0} v = \begin{cases} 
v, & \text{minimal energy extension, in $\Omega_l$,} \\
0, & \text{elsewhere.} 
\end{cases}
$$

Then, we have $v^T S^{(i)}_{E,0,\eta_0} v = |H^{(i)}_{\eta_0,0} (\theta v)|^2_{E_0}$. 

**Remark 4.13** (economic deluxe scaling). In [9], the authors proposed an economic variant of deluxe scaling by replacing the Schur complements $S^{(i)}_{E,0}$, $l = i, j$ by $S^{(i)}_{E,0,\eta}$ with $\eta = h$. As in [9] we will denote this variant by e-deluxe scaling.

**4.6.2. Generalized eigenvalue problem (economic version).** We solve the eigenvalue problem

$$
S^{(i)}_{E,i,j} : x^{(j)}_{E,i,j,\eta} = \mu_k \left( D^{(j)T}_{E,i,j} S^{(i)}_{E,i,j,\eta} D^{(j)}_{E,i,j} + D^{(i)T}_{E,i,j} S^{(j)}_{E,i,j,\eta} D^{(i)}_{E,i,j} \right) x^{(j)}_k,
$$

where $\mu_k \leq \text{TOL}$ and where $D^{(i)}_{E,i,j} = \left( S^{(i)}_{E,i,j,0,\eta} + S^{(j)}_{E,i,j,0,\eta} \right)^{-1} S^{(i)}_{E,i,j,0,\eta}$ for $l = i, j$. Then, we enforce the constraints $x^T_k \left( D^{(j)T}_{E,i,j} S^{(i)}_{E,i,j,0,\eta} D^{(j)}_{E,i,j} + D^{(i)T}_{E,i,j} S^{(j)}_{E,i,j,0,\eta} D^{(i)}_{E,i,j} \right) (w_i - w_j) = 0$, e.g., as described in Section 3.

**Lemma 4.14.** We define $\Pi_k := \sum_{m=1}^k x_k x_k^T \left( D^{(j)T}_{E,i,j} S^{(i)}_{E,i,j,0,\eta} D^{(j)}_{E,i,j} + D^{(i)T}_{E,i,j} S^{(j)}_{E,i,j,0,\eta} D^{(i)}_{E,i,j} \right)$ using the eigenvectors $x_m$ of the generalized eigenvalue problem (4.8). Then $\Pi_k (w_i - w_j) = 0$ and the following inequality holds:

$$
(w_i - w_j)^T \left( D^{(j)T}_{E,i,j} S^{(i)}_{E,i,j,0,\eta} D^{(j)}_{E,i,j} + D^{(i)T}_{E,i,j} S^{(j)}_{E,i,j,0,\eta} D^{(i)}_{E,i,j} \right) (w_i - w_j) 
\leq (\mu_k + 1)^{-1} \left( w_i^2_{S^{(i)}_{E,i,j,\eta}} + w_j^2_{S^{(j)}_{E,i,j,\eta}} \right)
$$

**Proof.** Since the discrete harmonic extension $|H^{(i)}_{\eta_0,0} (\theta v)|^2_{E_0} = v^T S^{(i)}_{E,0,\eta} v$ has the lowest energy, we obtain

$$
(w_i - w_j)^T \left( D^{(j)T}_{E,i,j} S^{(i)}_{E,i,j,0,\eta} D^{(j)}_{E,i,j} + D^{(i)T}_{E,i,j} S^{(j)}_{E,i,j,0,\eta} D^{(i)}_{E,i,j} \right) (w_i - w_j) 
\leq (\mu_k + 1)^{-1} \left( w_i^2_{S^{(i)}_{E,i,j,\eta}} + w_j^2_{S^{(j)}_{E,i,j,\eta}} \right)
$$

$$
= 2 (\mu_k + 1)^{-1} \left( |H^{(i)}_{\eta_0,0} h_i|^2_{E,i,j,\eta} + |H^{(j)}_{\eta_0,0} h_j|^2_{E,i,j,\eta} \right)
$$

$$
\leq 2 (\mu_k + 1)^{-1} \left( |H^{(i)}_{\eta_0,0} h_i|^2_{E,i,j,\eta} + |H^{(j)}_{\eta_0,0} h_j|^2_{E,i,j,\eta} \right)
$$

$$
\leq 2 (\mu_k + 1)^{-1} \left( |H^{(i)}_{\eta_0,0} h_i|^2_{E,i,j,\eta} + |H^{(j)}_{\eta_0,0} h_j|^2_{E,i,j,\eta} \right)
$$

$$
= 2 (\mu_k + 1)^{-1} \left( w_i^2_{S^{(i)}_{E,i,j,\eta}} + w_j^2_{S^{(j)}_{E,i,j,\eta}} \right).
$$
4.7. Condition number estimate for the first coarse space. Based on the estimates for $P_D$ for the first coarse space, given in the previous sections, we now present our condition number estimate.

**Theorem 4.15.** Let $N_E$ be the maximum number of edges of a subdomain. The condition number $\kappa(\tilde{M}^{-1}F)$ of the FETI-DP algorithm with adaptive constraints defined as in Sections 4.4.2, 4.5.2, or 4.6.2 either enforced by the projector preconditioner $\tilde{M}^{-1} = M_D^{-1}$, the balancing preconditioner $\tilde{M}^{-1} = M_{BP}^{-1}$, or a transformation of basis and the Dirichlet preconditioner $\tilde{M}^{-1} = M_D^{-1}$ satisfies

$$
\kappa(\tilde{M}^{-1}F) \leq 2N_E^2 \text{TOL}^{-1}.
$$

**Proof.** For $w \in \tilde{W}$ we have the estimate

$$
|P_Dw|_S^2 = \sum_{i=1}^{N} |R^{(i)}P_Dw|_{S_i}^2
\leq N_E \sum_{i=1}^{N} \sum_{j \in N_i} |I^h(\theta_E, E_j(w_i - w_j))|_{S_i}^2
\leq N_E \sum_{E_{ij} \subset \Gamma} (w_i - w_j)^T \left(D^{(E_{ij})}_{E_{ij}} S^{(i)}_{E_{ij},0} D^{(j)}_{E_{ij}} + D^{(E_{ij})}_{E_{ij}} S^{(j)}_{E_{ij},0} D^{(i)}_{E_{ij}}\right) (w_i - w_j).
$$

Using Lemma 4.8 for the coarse space in Section 4.4.2, Lemma 4.10 for the coarse space in Section 4.5.2, and Lemma 4.14 for the coarse space in Section 4.6.2 and using that $\mu_{k+1} \geq \text{TOL}$ we obtain the estimate

$$
|P_Dw|_S^2 \leq 2N_E \sum_{E_{ij} \subset \Gamma} \text{TOL}^{-1} \left( w_i^T S^{(i)}_{E_{ij}} w_i + w_j^T S^{(j)}_{E_{ij}} w_j \right)
\leq 2N_E \sum_{E_{ij} \subset \Gamma} \text{TOL}^{-1} \left( w_i^T S_i w_i + w_j^T S_j w_j \right)
\leq 2N_E^2 \text{TOL}^{-1} \sum_{i=1}^{N} |R^{(i)}w|_{S_i}^2
\leq 2N_E^2 \text{TOL}^{-1} |w|_S^2.
$$

5. Second coarse space. We will now discuss an approach which has been successfully used in FETI-DP and BDDC for some time [31, 32, 36, 22]. Let us note that this approach is also based on eigenvalue estimates related to the $P_D$ operator. In the following, we give a brief description of the algorithm in [31] for the convenience of the reader. In Section 5.1 we introduce the relevant notation and in Section 5.2 the specific eigenvalue problem. In Section 5.3 we also give an estimate of the condition number in the case of a two-dimensional problem where all vertices are primal in the initial coarse space.

5.1. Notation. Let $E_{ij}$ be the edge between subdomains $\Omega_i$ and $\Omega_j$. Let $B_{E_{ij}} = [B^{(i)}_{E_{ij}} B^{(j)}_{E_{ij}}]$ be the submatrix of $[B^{(i)}B^{(j)}]$ with the rows that consist of exactly one 1 and one $-1$ and are zero otherwise; define $B_{D,E_{ij}} = [B^{(i)}_{D,E_{ij}} B^{(j)}_{D,E_{ij}}]$ by removing the...
same rows in the corresponding submatrix of \([B_D^{(i)} B_D^{(j)}]\). We define \(S_{ij} = \begin{bmatrix} S_i & S_j \end{bmatrix} \)
and \(P_{D_{ij}} = B_D^{(i)} E_{ij}, B_{E_{ij}}\). By \(\tilde{W}_{ij}\) we denote the space of functions in \(W_i \times W_j\) that are continuous in the primal variables that the subdomains \(\Omega_i\) and \(\Omega_j\) have in common and by \(\Pi_{ij}\) the \(l_2\)-orthogonal projection from \(W_i \times W_j\) to \(\tilde{W}_{ij}\). Another orthogonal projection from \(W_i \times W_j\) to \(\text{Range}(\Pi_{ij} S_i, \Pi_{ij} + \sigma(I - \Pi_{ij}))\) is denoted by \(\tilde{\Pi}_{ij}\), where \(\sigma\) is a positive constant, e.g., the maximum of the entries of the diagonal of \(S_{ij}\).

5.2. Generalized eigenvalue problem. We solve the eigenvalue problem

\[
\tilde{\Pi}_{ij} P_{D_{ij}}^T S_{ij} P_{D_{ij}} \Pi_{ij} w_{ij}^k = \mu^{(i)}_{ij} \Pi_{ij}(\Pi_{ij} S_i \Pi_{ij} + \sigma(I - \Pi_{ij}))\Pi_{ij} + \sigma(I - \Pi_{ij}))w_{ij}^k,
\]

where \(\mu^{(i)}_{ij} \geq TOL\). We then enforce the constraints \(w_{ij}^k \in W\). Localy, for \(w_{ij} \in W_i \times W_j\) which satisfies the constraints \(w_{ij}^k P_{D_{ij}}^T S_{ij} P_{D_{ij}} w_{ij} = 0\), \(\mu^{(i)}_{ij} \geq TOL\), the estimate

\[
w_{ij}^T \Pi_{ij} P_{D_{ij}}^T S_{ij} P_{D_{ij}} \Pi_{ij} w_{ij} \leq \mu^{(i)}_{ij} w_{ij}^T \Pi_{ij} S_i \Pi_{ij} w_{ij}
\]

holds; cf. [31]. Note that \(\Pi_{ij} (I - \Pi_{ij}) w_{ij} = (I - \Pi_{ij}) w_{ij}\) since \((I - \Pi_{ij})\) is an orthogonal projection onto the space of rigid body modes that are continuous on \(W_i \times W_j\). Hence \(P_{D_{ij}} \Pi_{ij} (I - \Pi_{ij}) w_{ij} = 0\) and \(S_i \Pi_{ij} (I - \Pi_{ij}) w_{ij} = 0\). Thus, from (5.1), we obtain the estimate

\[
w_{ij}^T \Pi_{ij} P_{D_{ij}}^T S_{ij} P_{D_{ij}} \Pi_{ij} w_{ij} \leq \mu^{(i)}_{ij} w_{ij}^T \Pi_{ij} S_i \Pi_{ij} w_{ij}
\]

for all \(w_{ij} \in W_i \times W_j\) with \(w_{ij}^k P_{D_{ij}}^T S_{ij} P_{D_{ij}} w_{ij} = 0\), \(\mu^{(i)}_{ij} \geq TOL\).

5.3. Condition number estimate of the coarse space in 2D.

Theorem 5.1. Let \(N_E\) be the maximum number of edges of a subdomain. The condition number \(\kappa(M^{-1} F)\) of the FETI-DP algorithm with adaptive constraints defined as in Section 5.2 either enforced by the projector preconditioner \(\tilde{M}^{-1} = M_P^{-1}\), the balancing preconditioner \(M^{-1} = M_B^{-1}\), or a transformation of basis and the Dirichlet preconditioner \(M^{-1} = M_D^{-1}\) satisfies

\[
\kappa(M^{-1} F) \leq N_E^2 TOL.
\]

Proof. The local jump operator in the eigenvalue problems is

\[
P_{D_{ij}} = \begin{bmatrix} B_D^{(i)} E_{ij}, B_{E_{ij}} & B_D^{(i)} E_{ij}, P_{D_{ij}}^{(j)} \end{bmatrix}.
\]

Application to a vector yields

\[
P_{D_{ij}} \begin{bmatrix} R^{(i)} w \\ R^{(j)} w \end{bmatrix} = \begin{bmatrix} I^b(\theta_{E_{ij}, D_{ij}}(w_i - w_j)) \\ P^b(\theta_{E_{ij}, D_{ij}}(w_j - w_i)) \end{bmatrix}.
\]

By \(\tilde{W}_{ij} \subset W_i \times W_j\) we denote the subspace of functions that are continuous in those primal variables which the subdomains \(\Omega_i\) and \(\Omega_j\) have in common. Let \(\Pi_{ij}\) be the \(l_2\)-orthogonal projection from \(W_i \times W_j\) to \(\tilde{W}_{ij}\).

For \(w \in \tilde{W}\) we have \(\begin{bmatrix} R^{(i)} w \\ R^{(j)} w \end{bmatrix} \in \tilde{W}_{ij}\), and therefore \(\Pi_{ij} \begin{bmatrix} R^{(i)} w \\ R^{(j)} w \end{bmatrix} = \begin{bmatrix} R^{(i)} w \\ R^{(j)} w \end{bmatrix}\).
All vertices are assumed to be primal. Thus, for \( w \in \tilde{W} \), we obtain

\[
|P_D w|_S^2 = \sum_{i=1}^{N} |R^{(i)} P_D w|_{S_i}^2 \\
\leq N_\varepsilon \sum_{i,j \in \mathcal{N}} |I^h(\theta_{E_{ij}} D_j(w_i - w_j))|_{S_i}^2 \\
= N_\varepsilon \sum_{E_{ij} \subset \Gamma} |I^h(\theta_{E_{ij}} D_j(w_i - w_j))|_{S_i}^2 + |I^h(\theta_{E_{ij}} D_i(w_j - w_i))|_{S_j}^2 \\
= N_\varepsilon \sum_{E_{ij} \subset \Gamma} \begin{bmatrix} w_i \\ w_j \end{bmatrix}^T \Pi_{ij} P_{D_{ij}}^T \begin{bmatrix} S_i \\ S_j \end{bmatrix} P_{D_{ij}} \Pi_{ij} \begin{bmatrix} w_i \\ w_j \end{bmatrix} \\
\leq N_\varepsilon \sum_{E_{ij} \subset \Gamma} \mu_{ij}^{k-1} \begin{bmatrix} w_i \\ w_j \end{bmatrix}^T \Pi_{ij} \begin{bmatrix} S_i \\ S_j \end{bmatrix} \Pi_{ij} \begin{bmatrix} w_i \\ w_j \end{bmatrix} \\
\leq N_\varepsilon \text{TOL} \sum_{E_{ij} \subset \Gamma} |w_i|_{S_i}^2 + |w_j|_{S_j}^2 \\
\leq N_\varepsilon^2 \text{TOL} \sum_{i=1}^{N} |R^{(i)} w|_{S_i}^2 \\
= N_\varepsilon^2 \text{TOL} |w|_{\tilde{S}}^2.
\]

6. Third coarse space \[23\]. We now discuss our approach from \[23\], which is not based on the localization of the \( P_D \)-estimate, and also introduce some improvements. We denote the weighted mass matrix on the closure of the edge \( E_{ij} \) of \( \Omega_l \), \( l = i, j \) by

\[
m_{E_{ij}}^{(l)}(u, v) := \int_{E_{ij}} \rho_l u \cdot v \, ds, \quad \text{for } l = i, j
\]

and the corresponding matrix by \( M_{E_{ij}}^{(l)} \). We will introduce two generalized eigenvalue problems. The first one is related to a replacement of a generalized Poincaré inequality on the edge in cases where the Poincaré inequality is dependent on the jumps in the diffusion coefficient. The second one is related to an extension theorem. For a detailed description of the algorithm including a proof of the condition number estimate, see \[23\]. We introduce the matrix \( S_{E_{ij},c}^{(l)} \) which is obtained by eliminating all degrees of freedom of \( \Gamma^{(l)} \setminus E_{ij} \) in \( S^{(l)} \) where \( E_{ij} \) is the closure of \( E_{ij} \). To avoid restriction operators as before we assume that a vector is restricted to the closed edge if it is multiplied by a matrix with the index \( E_{ij,c} \).

6.1. First generalized eigenvalue problem. We solve

\[
S_{E_{ij},c}^{(l)} x_{kj}^{ij,l} = \mu_{kj}^{ij,l} M_{E_{ij}}^{(l)} x_{kj}^{ij,l}, \quad \text{for } l = i, j
\]

where \( \mu_{kj}^{ij,l} \leq \text{TOL}_\mu \). We then build the vectors \( M_{E_{ij}}^{(l)} x_{kj}^{(l)} \) and discard the entries not associated with dual variables. We denote the resulting vectors by \( u_{kj}^{(l)} \) and enforce the constraints \( u_{kj}^{(l)T}(w_i - w_j) = 0 \), e.g., as described in Section 3.

6.2. Second generalized eigenvalue problem. Following \[23\], we introduce a second eigenvalue problem to ensure that we can extend a function from one subdomain to another without significantly increasing the energy. Depending on the kernels
of the local Schur complements \( S_{E_{ij},c}^{(l)} \) a generalized eigenvalue problem of the form

\[
S_{E_{ij},c}^{(l)} x_k^{ij} = \nu_k^{ij} S_{E_{ij},c}^{(l)} x_k^{ij},
\]

(6.2)
can have arbitrary eigenvalues. We make use of the \( \ell_2 \)-orthogonal projection \( \Pi \), where \( I - \Pi \) is the orthogonal projection from \( V^h(\mathcal{E}_{ij}) \) to \( \text{Ker}(S_{E_{ij},c}^{(l)}) \cap \text{Ker}(S_{E_{ij},c}^{(l)}) \) and compute the generalized eigenvalue problem

\[
\Pi S_{E_{ij},c}^{(l)} \Pi x_k^{ij} = \nu_k^{ij} (\Pi S_{E_{ij},c}^{(l)} \Pi + \sigma (I - \Pi)) x_k^{ij},
\]

(6.3)
where \( \nu_k^{ij} \leq \text{TOL}_\nu \) and \( \sigma > 0 \) is an arbitrary positive constant. In our computations we use \( \sigma = \max(\text{diag}(S_{E_{ij},c}^{(l)})) \). Analogously to the first eigenvalue problem, we build \( (\Pi S_{E_{ij},c}^{(l)} \Pi + \sigma (I - \Pi)) x_k^{ij} \) and discard the entries not associated with dual variables. Denoting the resulting constraint vectors by \( r_k^{ij} \), we enforce \( r_k^{ij}^T (w_i - w_j) = 0 \).

**6.3. Economic variant.** Analogously to Section 4.6 we present an economic version in which modified Schur complements which are cheaper to compute are used. This variant is new and has not been considered in [23]. All Schur complements are only computed on slabs regarding the edges they are associated with. We define \( S_{E_{ij},c}^{(l)} = S_{E_{ij},c}^{(l)} - S_{E_{ij},c}^{(l)} S_{E_{ij},c}^{(l)} T S_{E_{ij},c}^{(l)} (I - \sigma) S_{E_{ij},c}^{(l)} \) as the Schur complement, which is obtained analogously to \( S_{E_{ij},c}^{(l)} \) in Section 4.6.1 with the exception, that the matrix \( S_{E_{ij},c}^{(l)} \) is built with respect to the degrees of freedom on the closed edge \( \overline{E} \) and its complement in \( \Omega_{ij} \), respectively. The eigenvalue problems and constraints in Sections 6.1 and 6.2 are then computed with \( S_{E_{ij},c}^{(l)} \) instead of \( S_{E_{ij},c}^{(l)} \). The proof of the condition number in [23] can be extended to the economic case with the same arguments as in Lemma 4.14.

**6.4. Extension with a modification of deluxe scaling.** In the following, we construct a scaling for the extension which can be used as an alternative to the second eigenvalue problem (6.2). Thus, using this new scaling, we would only need the eigenvalue problem (6.1).

**Definition 6.1 (Extension scaling).** For a pair of subdomains \( \Omega_i \) and \( \Omega_j \) sharing an edge \( \mathcal{E}_{ij} \), let \( D_{E_{ij},c}^{(l)} \) and \( E_{E_{ij},c}^{(l)} \) be defined by

\[
D_{E_{ij},c}^{(l)} = (S_{E_{ij},c}^{(l)} + S_{E_{ij},c}^{(l)}) + S_{E_{ij},c}^{(l)} + A_{ij},
\]

\[
E_{E_{ij},c}^{(l)} = (S_{E_{ij},c}^{(l)} + S_{E_{ij},c}^{(l)}) + S_{E_{ij},c}^{(l)} + A_{ij},
\]

where \( A_{ij} \) is defined by

\[
A_{ij} = \frac{1}{2} \left( I - (S_{E_{ij},c}^{(l)} + S_{E_{ij},c}^{(l)}) + (S_{E_{ij},c}^{(l)} + S_{E_{ij},c}^{(l)}) \right).
\]

By removing in the matrices \( D_{E_{ij},c}^{(l)} \), \( l = i,j \), those columns and rows associated with the primal vertices in the endpoints of \( \mathcal{E}_{ij} \) we obtain matrices \( D_{E_{ij}}^{(l)} \). We define the subdomain scaling matrices by

\[
D^{(l)} = \sum_{\mathcal{E}_{ij} \in \Gamma^{(l)}} R_{E_{ij}}^{(1)} D_{E_{ij}}^{(l)} R_{E_{ij}}^{(1)}.
\]
As in Section 4.4 the scaled jump operator \( B_D \) in the FETI-DP algorithm is then given by \( B_D := \begin{bmatrix} D^{(1)} & D^{(2)} & \cdots & D^{(N)} \end{bmatrix}^T \begin{bmatrix} B^{(1)} & I & \cdots & 0 \\ I & B^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B^{(N)} \end{bmatrix} \), where the transpose is necessary since the \( D^{(i)} \) are not symmetric.

When using the scaling in Definition 6.1 we build the vectors \( D^{(1)}_{E_{ij},c} \) \( x^{(i)}_{E_{ij}} \) and \( D^{(1)}_{E_{ij},c} M^{(1)}_{E_{ij}} x^{(1)}_{E_{ij}} \) instead of \( M^{(1)}_{E_{ij}} x^{(1)}_{E_{ij}} \), \( l = i, j \), where \( x^{(1)}_{E_{ij}} \) are the eigenvectors computed from (6.1). We then discard the entries not associated with dual variables to obtain our constraints \( u^{(i)}_{E_{ij}} \).

**Lemma 6.2.** For an edge \( E_{ij} \) let \( I^{E_{ij},(l)}_{E_{ij},c} \) for \( l = i, j \) be defined by

\[
I^{E_{ij},(l)}_{E_{ij},c} = \sum_{i=1}^{L_i} x^{(i)}_{E_{ij},c} (w_i - w_j) M^{(1)}_{E_{ij}}
\]

where \( x^{(l)}_{E_{ij}} \) are the eigenvectors from (6.1). Let \( D^{(1)}_{E_{ij},c} \) be the scaling matrices in Definition 6.1. With the choice of the constraints \( u^{(1)}_{E_{ij}} (w_i - w_j) = 0 \), \( l = i, j \) where \( u^{(1)}_{E_{ij}} \) and \( u^{(1)}_{E_{ij}} \) are obtained by discarding the entries not associated with dual variables in the vectors \( D^{(1)}_{E_{ij},c} M^{(1)}_{E_{ij}} x^{(1)}_{E_{ij}} \) and \( D^{(1)}_{E_{ij},c} M^{(1)}_{E_{ij}} x^{(1)}_{E_{ij}} \) with \( \mu^{i,j}_k \leq \text{TOL} \) for \( k = 1, \ldots, L_i \) we have

\[
I^{E_{ij},(i)}_{E_{ij},c} D^{(1)}_{E_{ij},c} (w_i - w_j) = 0 \quad \text{and} \quad I^{E_{ij},(j)}_{E_{ij},c} D^{(1)}_{E_{ij},c} (w_j - w_i) = 0.
\]

**Proof.** The entries not associated with dual variables in \( w_i - w_j \) are zero since \( w_i = R^{(i)} w \) with \( w \in W \). Therefore we have

\[
I^{E_{ij},(i)}_{E_{ij},c} D^{(1)}_{E_{ij},c} (w_i - w_j) = \sum_{k=1}^{L_i} x^{(i)}_{E_{ij},c} u^{(i)}_{E_{ij},c} (w_{\Delta,i} - w_{\Delta,j}) = 0.
\]

By an analogous argument we conclude \( I^{E_{ij},(j)}_{E_{ij},c} D^{(1)}_{E_{ij},c} (w_j - w_i) = 0. \) \( \Box \)

For simplicity, we prove the next theorem only for the diffusion problem.

**Theorem 6.3.** The condition number for our FETI-DP method with a scaling, as defined in Definition 6.1, with all vertices being primal, and the coarse space enhanced with solutions of the eigenvalue problem (6.1), satisfies

\[
\kappa(\hat{M}^{-1} F) \leq C \left( 1 + \log \left( \frac{\eta}{h} \right) \right)^2 \left( 1 + \frac{1}{\eta \mu L+1} \right),
\]

where \( \hat{M}^{-1} = M^{-1} \) or \( \hat{M}^{-1} = M^{-1}_B \), or, alternatively, \( \hat{M}^{-1} = M^{-1} \) if all constraints have been enforced by a transformation of basis. Here, \( C > 0 \) is a constant independent of \( \rho, H, h, \) and \( \eta \) and

\[
\frac{1}{\mu L+1} = \max_{k \in \mathbb{Z}} \frac{1}{\mu_{L+1}} \left\{ \frac{1}{\mu_{L+1}} \right\}.
\]

**Proof.** The proof is modeled on the proof of Theorem 6.8 in [23] and uses the notation from [23]. With application of Lemma 6.2 we obtain for each edge \( E_{ij} \) in
\[|P_D w|^2 \leq \frac{1}{\eta^2} \left( 1 + \frac{1}{\eta^2 L_i} \right) \left( |D_{E_{i,j},c}(w_i - w_j)|^2 + |w_i|^2 + |w_j|^2 \right). \]

Here, \( I \) denotes the identity operator. In the penultimate step we used Lemma 4.6 with \( B = M_{E_{i,j}}, x = D_{E_{i,j},c}(w_i - w_j), m = L_i, \frac{m}{\eta^2} = I_{L_i}^{(i)}, \) and \( A = S_{E_{i,j},c}^{(i)} \). For the last step note that each column of \( A_{i,j} \) in Definition 6.1 is in \( \ker(S_{E_{i,j},c}^{(i)} + S_{E_{i,j},c}^{(j)}) \) and with the same argument as in the proof of Lemma 4.4 we have \( \ker(S_{E_{i,j},c}^{(i)} + S_{E_{i,j},c}^{(j)}) = \ker(S_{E_{i,j},c}^{(i)}) \cap \ker(S_{E_{i,j},c}^{(j)}). \) Thus, we obtain \( S_{E_{i,j},c}^{(i)} A_{i,j} = 0 \) and \( S_{E_{i,j},c}^{(i)} D_{E_{i,j},c} = S_{E_{i,j},c}^{(i)} (S_{E_{i,j},c}^{(i)} + S_{E_{i,j},c}^{(j)}) + S_{E_{i,j},c}^{(j)}. \) Applying Lemma 4.4 with \( A = S_{E_{i,j},c}^{(i)}, B = S_{E_{i,j},c}^{(j)} \), and \( D_A = D_{E_{i,j},c} \) we obtain the estimate. \( \square \)

Remark 6.4. The constant in the condition number estimate for the third coarse space (cf. Theorem 6.3 and [23, Theorem 6.8]) depends in the same way on \( N_2^k \) as that for the first coarse space in Theorem 4.15 and that for the second coarse space in Theorem 5.1. Additionally, the constant depends on the constants in a weighted edge lemma (cf. [23, Lemma 6.3]) and in a weighted Friedrichs inequality (cf. [23, Lemma 6.4]).

Remark 6.5. Similar as in Section 6.3 we can replace the matrices \( S_{E_{i,j},c}^{(i)}, l = i, j, \) by the economic version \( S_{E_{i,j},c,\eta}^{(i)} \) in the scaling in Definition 6.1 and in the generalized eigenvalue problem 6.1.

7. A brief comparison of computational cost. In the following, we will give a short comparison of the cost of the algorithms described in this paper. For the algorithm in Section 4.4 (first coarse space) the matrices \( S_{E_{i,j}}^{(i)} \) and \( S_{E_{i,j},0}^{(i)} \) have to be computed. These matrices are usually dense. For their computation a Cholesky factorization of a sparse matrix is required and will usually need \( \mathcal{O}((H/h)^3) \) floating point operations in two space dimensions since the inverse involved in the
Schur complement is of the order of \((H/h)^2 \times (H/h)^2\). If the Schur complements are computed explicitly, which might be necessary depending on the eigenvalue solver that is used, a matrix-matrix multiplication, a matrix-matrix addition and forward-backward substitutions for multiple right-hand sides with the Cholesky factorization have to be performed. If, e.g., LAPACK (or MATLAB, which itself uses LAPACK) is used substitutions for multiple right-hand sides with the Cholesky factorization have to be used, a matrix-matrix multiplication, a matrix-matrix addition and forward-backward substitution. For \(S_{E_{ij}}^{(i)} : S_{E_{ij}}^{(j)}\), depending on the kernel of \(S_{E_{ij}}^{(i)} + S_{E_{ij}}^{(j)}\), a pseudoinverse or a Cholesky factorization is needed. For the scaling matrices a factorization of \(S_{E_{ij},0}^{(i)} + S_{E_{ij},0}^{(j)}\) has to be performed. If no deluxe scaling but \(\rho\)-scaling is used, the matrix \(D_{E_{ij}}^{(i)T} S_{E_{ij},0}^{(j)} D_{E_{ij}}^{(i)} + D_{E_{ij}}^{(j)T} S_{E_{ij},0}^{(i)} D_{E_{ij}}^{(j)}\) has to be computed instead of \(S_{E_{ij},0}^{(i)} : S_{E_{ij},0}^{(j)}\) which is much cheaper since no factorization of \(S_{E_{ij},0}^{(i)} + S_{E_{ij},0}^{(j)}\) is needed. The computations of \(S_{E_{ij},0,\eta}\) and \(S_{E_{ij},\eta}^{0}\) need \((\eta/H)^{3/2}\) times as many floating point operations as the computations of \(S_{E_{ij},0}^{(l)}\) and \(S_{E_{ij}}^{(l)}\).

The eigenvalue problem in Section 5 (second coarse space) is larger but sparser. The left-hand side is not dense because of the structure of the local jump operator \(P_D\) which contains only two non zero entries for each row. The right-hand side consists of two dense blocks and two zero blocks in the dual part. The size of the eigenvalue problem is determined by the number of degrees of freedom on \(\Gamma^{(i)} \times \Gamma^{(j)}\) while the other algorithms are determined by the number of degrees of freedom on an edge \(E_{ij}\), e.g., in two dimensions it can be eight times larger. The computation of the left-hand side of the generalized eigenvalue problem in Section 5.2 also needs applications of the scaling matrices \(D^{(i)}\) and \(D^{(j)}\) which in case of deluxe scaling is more expensive then for multiplicity or \(\rho\)-scaling.

The generalized eigenvalue problem in Section 6.1 is completely local and needs no inter subdomain communication but needs to be solved for both neighboring subdomains on an edge. The generalized eigenvalue problem in Section 6.2 needs to be solved once but needs inter subdomain communication. While the algorithm in Section 4 needs to exchange the matrices \(S_{E_{ij},0}^{(l)}\) and \(S_{E_{ij}}^{(l)}\), \(l = i,j\) and the scaling matrices, the algorithm in Section 5 needs to exchange \(S_{E_{ij}}^{(l)}\), the local jump matrices \(B_E\), \(l = i,j\), and the scaling matrices. Nonetheless, if \(\rho\)-scaling or deluxe scaling is used the scaling data needs to be communicated for the construction of \(B_D\) in the FETI-DP algorithm anyways. The algorithm in Section 6 only needs to exchange \(S_{E_{ij}}^{(l)}\), \(l = i,j\). However, locally, in two dimensions, a one dimensional matrix has to be assembled for each edge of a subdomain. Note that this matrix has tridiagonal form if piecewise linear finite element functions are used. This makes a Cholesky factorization very cheap.

A disadvantage of the algorithm in Section 6 (third coarse space) compared to the other algorithms is that no \(\rho\)-scaling with varying scaling weights inside of a subdomain can be used. In Section 8 on our numerical results, we will see that using multiplicity scaling can lead to a large number of constraints. However, if the extension constant is nicely bounded, e.g., for coefficient distributions which are symmetric with respect to the subdomain interfaces (at least on slabs) and have jumps only along but not across edges only trivially local generalized eigenvalue problems with a tridiagonal mass matrix on the right-hand side need to be solved and the number of constraints stays bounded independently of \(H/h\). If the scaling in Section 6.4
is used, only the scaling matrices of neighboring subdomains have to be exchanged. The eigenvectors in the first eigenvalue problem can be computed completely local. The constraints need an application of the mass matrix and the scaling matrix of a neighbor.

8. Numerical examples. In all our numerical computations we have removed linearly dependent constraints by using a singular value decomposition of $U$. Constraints related to singular values less than a drop tolerance of $1e-6$ were removed. In an efficient implementation this may not be feasible.

In our numerical experiments, whenever we need to compute a pseudoinverse of a symmetric matrix $A$, we first introduce the partition $A = \begin{pmatrix} A_{pp} & A_{rp} \\ A_{rp} & A_{rr} \end{pmatrix}$, where $A_{pp}$ is an invertible submatrix of $A$ and $A_{rr}$ is a small submatrix of $A$ with a size of at least the dimension of the kernel of $A$. Then we compute

$$A^+ = \begin{pmatrix} I & -A_{pp}^{-1}A_{rp} \\ 0 & I \end{pmatrix} \begin{pmatrix} A_{pp}^{-1} & 0 \\ 0 & S_{rr}^+ \end{pmatrix} \begin{pmatrix} I \\ 0 & -A_{rp}A_{pp}^{-1}I \end{pmatrix}$$

with the Schur complement $S_{rr} = A_{rr} - A_{rp}A_{pp}^{-1}A_{rp}^T$. Here, $S_{rr}^+$ denotes the Moore-Penrose pseudoinverse of $S_{rr}$. In the singular value decomposition of $S_{rr}$ we treat all singular values less than $(1e-3) \cdot \min(\text{diag}(A))$ as zero.

We consider different coefficient distributions. In Test Problem I (Fig. 8.1), we consider the simple case of horizontal channels. In Test Problem II (Fig. 8.2), we have a coefficient configuration which is symmetric in a small environment of vertical edges. In Test Problem III (Fig. 8.3), the coefficient configuration is constructed such that we have no symmetry with respect to the vertical edges. In Test Problem IV (Fig. 8.4), we then have a challenging, randomly chosen coefficient distribution. Note that the coefficient distribution does not change when the meshes are refined. For our adaptive method we therefore expect the number of eigenvalues to stay bounded when $H/h$ is increased.

We attempt a fair comparison of the adaptive methods by using suitable tolerances for the different eigenvalues, i.e., we attempt to choose tolerances such that all very large eigenvalues are removed but not more than these. Exemplarily, we present detailed spectra for Test Problem IV; see Figure 8.6.

We present our numerical results for FETI-DP methods but they are equally valid for BDDC methods. The adaptive constraints are incorporated into the balancing preconditioner $M_{BP}^{-1}$ (cf. equation (3.3)) but other methods can also be used.

In Section 8.1, we present numerical results for a scalar diffusion equation and Problems I-IV, using different adaptive coarse spaces. In Section 8.2, we consider the problem of almost incompressible elasticity.

8.1. Scalar diffusion. First, we perform a comparison for our scalar diffusion problem with a variable coefficient for the first (Section 4 [7, 21]), second (Section 5 [31]) and third (Section 6 [23]) coarse space. We use homogeneous Dirichlet boundary conditions on $\Gamma_D = \partial \Omega$ in all our experiments for scalar diffusion.

The first coefficient distribution is depicted in Fig. 8.1 (Test Problem I, horizontal channels). This coefficient distribution is symmetric with respect to vertical edges. Since there are no jumps across the interface, the simple multiplicity scaling is sufficient, and $\rho-$scaling reduces to multiplicity scaling. The numerical results are shown in Table 8.1. The estimated condition numbers are identical for all cases and the number of eigenvectors is similar.
Fig. 8.1. Test Problem I with a coefficient distribution consisting of two channels per subdomain for a 3x3 decomposition. In case of diffusion the diffusion coefficient is $10^6$ (black) and 1 (white). In case of elasticity the Young modulus is $10^3$ (black) and 1 (white).

In Table 8.2, we consider the coefficient distribution depicted in Fig. 8.2 (Test Problem II, horizontal channels on slabs). Here, the coefficient distribution is symmetric on slabs with respect to vertical edges. Again, there are no coefficient jumps across subdomains, and multiplicity scaling is equivalent to $\rho$-scaling. We note that in this test problem e-deluxe scaling with $H/\eta = 14$ is not equivalent to multiplicity scaling, since in the Schur complements $S_{E,0,\eta}^{(l)}$, $l = i, j$ entries on $\partial \tilde{\Omega}_i \setminus (\partial \Omega_l \cap \partial \tilde{\Omega}_i)$ are eliminated. The economic version of extension scaling in this case is equivalent to multiplicity scaling because the Schur complements $S_{E,\eta}^{(l)}$, $l = i, j$ are computed from local stiffness matrices on the slab. In Table 8.2, we report on multiplicity scaling, deluxe scaling, and e-deluxe scaling for the three cases. Using multiplicity scaling, the results are very similar, but not identical, for all three approaches to adaptive coarse spaces. The use of deluxe scaling can improve the results for the first two approaches. The use of extension scaling for the third approach has no significant impact. Where economic variants exist, e.g., versions on slabs, we also report on results using these methods. As should be expected using the economic versions of the eigenvalue problems yields worse results.

Next, we use the coefficient distribution depicted in Fig. 8.3 (Test Problem III, unsymmetric channel pattern). The results are collected in Table 8.3. In this problem, coefficient jumps across the interface are present, in addition to the jumps inside subdomains. Therefore multiplicity scaling is not sufficient and all coarse space approaches are not scalable with respect to $H/h$, i.e., the number of eigenvectors increase when $H/h$ is increased; cf. Table 8.3 (left). Using $\rho$-scaling or deluxe/extension scaling then yields the expected scalability in $H/h$, i.e. the number of eigenvalues remains bounded when $H/h$ is increased. Where deluxe scaling is available it significantly reduces the size of the coarse problem; cf. Table 8.3 (middle and right). The smallest coarse problem is then obtained for the combination of the second coarse space with deluxe scaling. Using extension scaling in the third coarse space approach yields the smaller condition number and iteration count among $\rho$-, deluxe-, or extension scaling but at the price of a much larger coarse space. In Table 8.4, we present results for Test Problem III using the slab variant of the first coarse space. Our results show that saving computational work by using the slab variants can increase the number of eigenvectors significantly, i.e., the number of eigenvectors grows with decreasing $\eta/H$. On the other hand the condition number and iteration count decreases. This implies
that slab variants can be affordable if a good coarse space solver is available. The results may also indicate that scalability of the coarse space size with respect to $H/h$ may be lost.

The results for Test Problem IV are collected in Table 8.5. Also for this difficult problem the number of eigenvalues seems to remain bounded when $H/h$ is increased although for $\rho-$scaling the number of eigenvalues increases slightly with $H/h$. The smallest coarse problem consisting of only four eigenvectors is obtained when the second coarse space approach is combined with deluxe scaling although the difference between $\rho-$scaling and deluxe scaling is not as large as in Test Problem III. The third coarse space using extension scaling is scalable in $H/h$ but, in this current version, yields the largest number of eigenvectors.

Exemplarily, the 50 largest eigenvalues appearing in the adaptive approaches for Test Problem IV using deluxe or extension scaling are presented in Figure 8.6. We can see that the tolerances chosen in Table 8.5 result in the removal of all large eigenvalues. We therefore believe that our comparison is fair.

For the third coarse space, we have also tested the combination of multiplicity scaling with both eigenvalue problems from [23], i.e., $TOL_\mu = 1/10$ and $TOL_\nu = 1/10$. As in the other cases where we use multiplicity scaling, see Table 8.3, this leads to a small condition number but at the cost of a large number of eigenvalues, and the approach thus is not scalable with respect to $H/h$.

Results for the slab variant of the third coarse space are then presented in Table 8.6. In Table 8.5, we consider the distribution from Fig. 8.4 for the different coarse space approaches. Note that we do not show the results for multiplicity scaling here, since the coarse space grows significantly with $H/h$, and this approach is therefore not recommended.

8.2. Almost incompressible elasticity. In this section, we compare the algorithms for almost incompressible elasticity. First we consider a problem with a constant coefficient distribution. Mixed displacement-pressure $P2 - P0$ elements are used for the discretization.

In the first test, we solve a problem with a Young modulus of 1 and a Poisson
Table 8.1

Scalar diffusion. Test Problem I (see Fig. 8.1)

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Fig. 8.4. Test Problem IV has a random coefficient distribution which is constant on squares of size $1/21 \times 1/21$. Diffusion coefficient $10^6$ (black) and 1 (white). Domain decomposition in $3 \times 3$ subdomains.

Fig. 8.5. Plot of the square root of the condition number vs. $H/h$ of the data given in Table 8.5 for the third coarse space with extension scaling using a logarithmic scale on the $x$-axis.

Fig. 8.6. The 50 largest (inverse) eigenvalues of the generalized eigenvalue problems for Test Problem IV for $H/h = 28$ (see Fig. 8.4). First coarse problem using deluxe scaling, second coarse problem using deluxe scaling, and third coarse space using extension scaling (from left to right); cf. Table 8.5.

The results for the approach in Section 4.5.2 with a tolerance $1/10$ and varying $H/h$ are presented in Table 8.7.

For constant $H/h = 20$ and varying Poisson ratio $\nu$, see Table 8.8.
Table 8.2
Scalar diffusion. For the slab variants of the algorithms, we only consider the first and the third coarse space; see Sections 4 and 6. Test Problem II (see Fig. 8.2), $H/\eta = 14$, $1/H = 3$.

In the third case we consider a distribution of Young’s modulus in Fig. 8.1 and a Poisson ratio of $\nu = 0.4999$. The result for the approach in Section 4.5.2 can be found in Table 8.9.

For the related results of the algorithm in [31], see Tables 8.7, 8.8, and 8.9.

Note that the third coarse space algorithm is not suitable for this problem since its eigenvalue problem is not based on a localization of the jump operator but designed to get constants in Poincaré-like inequalities and in an extension theorem that are independent of jumps in the coefficients. It therefore will not find the zero net flux...
First coarse space, TOL = 1/10

| H/h | cond | its | #EV | multiplicity | ρ | #EV | cond | its | #EV | #EV | #dual |
|-----|------|-----|-----|-------------|-------|-----|------|-----|-----|-----|-----|------|
| 14  | 1.2911 | 8 | 44 | 1.3874 | 9 | 15 | 4.8937 | 11 | 5 | 348 |
| 28  | 1.4148 | 9 | 74 | 1.5782 | 11 | 15 | 4.8672 | 12 | 5 | 348 |
| 42  | 1.5167 | 10 | 104 | 1.7405 | 11 | 15 | 4.8891 | 12 | 5 | 516 |

Second coarse space, TOL = 10

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Third coarse space, TOL = −∞

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Table 8.3
Scalar diffusion. Comparison of the coarse spaces for Test Problem III (see Fig. 8.3).

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Table 8.4
Test Problem III (see Fig. 8.3). Results for the slab variant of the first coarse space; also cf. Table 8.3.

9. Conclusion. For the first and second coarse space a condition number estimate is available for symmetric positive definite problems in two dimensions; see
First coarse space, TOL = 1/10

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Second coarse space, TOL = 10

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Third coarse space, $TOL_{\mu} = 1/10$, $TOL_{\nu} = -\infty$

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Table 8.5
Scalar diffusion. Test Problem IV

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Table 8.6
Scalar diffusion. Test Problem IV (see Fig. 8.4). Third coarse space uses extension scaling with different $\eta/h$, see Definition 6.1 and Remark 6.5, only eigenvalue problem 6.1 with $TOL_{\mu} = 1/10$; also cf. the third coarse space in Table 8.5 for $H/h = 28$. On squares of four elements in each direction the coefficient is constant. Consequently, the number of constraints is reduced if the slab size $\eta/h$ is increased such that a multiple of four is exceeded.
Table 8.7
Almost incompressible elasticity using a $P_2 - P_0$ finite elements discretization using $3 \times 3$ subdomains. Homogeneous coefficients with $E = 1$ and $\nu = 0.499999$.

Table 8.8
Almost incompressible elasticity using $P_2 - P_0$ finite elements and $3 \times 3$ subdomains. Homogeneous coefficients with $E = 1$. We vary $\nu$, $H/h = 20$.

Table 8.9
Almost incompressible elasticity using a $P_2 - P_0$ finite elements discretization and $3 \times 3$ subdomains. Channel distribution with $E_1 = 1e3$ (black), $E_2 = 1$ (white) and $\nu = 0.499999$; cf. Fig. 8.1.

Sections 4.4.2 and 4.5.2 for the first coarse space and Section 5.3 for the second coarse space, respectively. A condition number estimate for the third coarse space applied to scalar diffusion problems can be found in [23] for constant $\rho$-scaling and in Section 6 for extension scaling. For this a condition number estimate can be proven for linear elasticity using similar arguments and replacing the $H^1$-seminorms by elasticity seminorms. For all three coarse spaces, to the best of our knowledge, no published theory exists yet for the three dimensional case but the second coarse space was successfully applied to three dimensional problems in [32].

An advantage of the first and third coarse space is that the size of the eigenvalue problems depends only on the number of degrees of freedom on an edge. This has to be seen in comparison to the size of two local interfaces of two substructures in the second adaptive coarse space. In the eigenvalue problem for the first coarse space the matrices involved are dense while in the second coarse space the eigenvalue problem involves a sparse matrix on the left-hand side and a $2 \times 2$ block matrix with dense blocks on the right-hand side. However, the first coarse space needs the factorization of a matrix on
the left-hand side and possibly matrix-matrix multiplications with Schur complements if a direct eigensolver is used. The third coarse space needs the solution of two eigenvalue problems for each of two substructures sharing an edge with a dense matrix on the left-hand side and a tridiagonal matrix in case of piecewise linear elements on the right-hand side. It can be advantageous that these eigenvalue problems can be computed locally on one substructure and that for building the constraints, only in case of extension scaling, information of the neighboring substructure has to be used. Possibly another eigenvalue problem with two dense matrices needs to be solved for the extension if the extension constant is not small, e.g., if no extension scaling is used or if the coefficient is not symmetric with respect to an edge.

A multilevel BDDC variant for the second coarse space can be found in [36].

All coarse spaces require an additional factorization with matrix-matrix multiplications or multiple forward-backward substitutions if deluxe scaling is used. In case of multiplicity scaling and a nonsymmetric coefficient the size of all coarse spaces can depend on the size of the substructures $H/h$ as can be seen in Section 8.

REFERENCES

A. Klawonn, P. Radtke, and O. Rheinbach


