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FETI-DP Methods with an Adaptive Coarse Space

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Abstract. A coarse space is constructed for the Dual-Primal Finite Element Tearing and Interconnecting (FETI-DP) domain decomposition method applied to highly heterogeneous problems by solving local generalized eigenvalue problems. For certain problems with highly varying coefficients, e.g., from multiscale simulations, the coefficient jump will appear in the condition number bound even if standard techniques as scaling and the weighting of constraints are used. The FETI-DP theory is revisited and two central estimates are identified where the dependency on the coefficient contrast can enter the condition number bound. The first is a Poincaré inequality and the second an extension theorem. These estimates are replaced by local eigenvalue problems. Enriching the FETI-DP coarse space by a few numerically computed eigenvectors yields independence of the contrast of the coefficients even in challenging situations.

Key words. FETI-DP, eigenvalue, coarse space, domain decomposition, multiscale

1. Introduction. We consider nonoverlapping domain decomposition methods for highly heterogeneous elliptic partial differential equations of second order. The algorithms belong to the family of Dual-Primal Finite Element Tearing and Interconnecting (FETI-DP) methods; cf., e.g., [45, 11, 27, 34, 26]. We replace bounds from the FETI-DP theory by local eigenvalue problems to obtain independence of the contrast on coefficients in challenging configurations of the coefficient which can not be treated by the standard techniques, see Figure 1.1. Using a few eigenvectors from the local generalized eigenvalue problems we construct a coarse space for the FETI-DP method and obtain robustness for problems with highly heterogeneous coefficients. With the new coarse space we are, e.g., able to replace a Poincaré inequality in cases where the Poincaré constant depends on the coefficient. Similarly, we can improve certain constants resulting from the use of an extension lemma. In our examples, the resulting constraints will be enforced by projector preconditioning, which provides a convenient second, independent coarse problem [17, 23]. Other implementations are also possible.

FETI-DP was first proposed in [11]. In the case of elliptic equations and constant coefficients on each subdomain the condition number is only logarithmically dependent on the unknowns per subdomain [34, 27, 28, 45, 26]. Note that our results are also of concern for BDDC. The BDDC method is a closely related primal alternative to FETI-DP and was introduced in [6, 5]. BDDC and FETI-DP share the same eigenvalues except for eigenvalues equal to zero and one [31, 29, 4].
Robustness of domain decomposition methods for special configurations of highly varying coefficients can be obtained by enhancement of the coarse space [1, 2, 19, 16, 37, 38, 14, 8, 43, 44, 10]. It is known that already the classical coarse spaces perform well in the case where the coefficient jumps are across the interface of the decomposition or if they are sufficiently far away from it [9, 37, 38, 16]. On the other hand, coefficient jumps along the interface need a special treatment. For so called quasimonotone coefficients, adding weighted averages to the coarse space is sufficient in order to obtain a robust algorithm [16, 37, 38, 19, 39]. Already, in [1, 2] eigenvectors are used to enhance the coarse space of the two-level Neumann-Neumann method. In [32, 33], the authors developed successful adaptive coarse spaces for BDDC and FETI-DP but have to rely on some heuristic assumptions to keep the computational cost acceptable. In [41] the authors developed a multilevel variant of this method. More recently, another adaptive algorithm for FETI and BDD has been proposed in [44]. Both approaches use a splitting of the jump and averaging operators to construct generalized eigenvalue problems and calculate coarse constraints. An analysis of Poincaré inequalities for a heterogeneous coefficient in the Poisson problem and sharper bounds for the Poincaré constant in cases of so called quasimonotone coefficients have been developed in [39]. In [14, 8] coarse spaces for overlapping Schwarz methods replacing Poincaré inequalities with generalized eigenvalue problems have been proposed. Some first results on using numerically computed eigenvectors to improve the Poincaré constant have already appeared as part of a proceedings article [15]. In [15] no theory has been given. Also the use of the second eigenvalue problem, which is related to an extension theorem is presented here for the first time.

The remainder of this article is organized as follows: In Section 2, we introduce as our elliptic model problems of second order a scalar diffusion equation and the equations of linear elasticity as well as their discretizations and the domain decomposition. In Section 3, we briefly review the standard FETI-DP algorithm using vertex constraints and in Section 4, FETI-DP using projector preconditioning for the construction of the coarse space. In Section 5, we introduce a first eigenvalue problem which is used to enhance the coarse space to improve robustness. In the first part of Section 6, we analyze the condition number of this new FETI-DP method. In the second part, an additional eigenvalue problem is introduced which improves once more the robustness of the FETI-DP when the related eigenvectors are added to the coarse problem. Finally, in Section 7 and 8, we present numerical results for the diffusion equation and the equations of linear elasticity, both with different challenging coefficient distributions.
2. Elliptic model problem, finite elements and geometry. Let $\Omega \subset \mathbb{R}^2$ be a bounded polyhedral domain, let $\partial \Omega_D \subset \partial \Omega$ be a closed subset of positive measure, and $\partial \Omega_N := \partial \Omega \setminus \partial \Omega_D$ be its complement. Furthermore, we define the Sobolev space $H^1_0(\Omega, \partial \Omega_D) := \{ v \in H^1(\Omega) : v = 0 \text{ on } \partial \Omega_D \}$. We consider the piecewise linear conforming finite element approximation of the scalar second order problem: Find $u \in H^1_0(\Omega, \partial \Omega_D)$, such that
\[
a(u, v) = f(v) \quad \forall v \in H^1_0(\Omega, \partial \Omega_D),
\]
where
\[a(u, v) := \int_\Omega \rho(x) \nabla u \cdot \nabla v \, dx, \quad f(v) := \int_\Omega f v \, dx + \int_{\partial \Omega_N} g_N v \, ds,
\]
here $g_N$ is the boundary data defined on $\partial \Omega_N$. We assume $\rho(x) > 0$ for $x \in \Omega$ and $\rho$ piecewise constant on $\Omega$. As a second model problem, we consider the problem of linear elasticity. For the compressible case we use the standard variational formulation to find a displacement $u \in (H^1_0(\Omega, \partial \Omega_D))^2$, such that
\[a(u, v) = f(v) \quad \forall v \in (H^1_0(\Omega, \partial \Omega_D))^2\]
where
\[a(u, v) := \int_\Omega G(x) \varepsilon(u) : \varepsilon(v) \, dx + \int_\Omega \beta(x) \text{div}(u) \text{div}(v) \, dx.
\]
The material parameters $G$ and $\beta$ will be expressed by $G = \frac{E}{1+\nu}$ and $\beta = \frac{\nu}{1-2\nu}$, using Young’s modulus $E$ and Poisson’s ratio $\nu$.

We decompose $\Omega$ into $N$ non-overlapping subdomains $\Omega_i$, $i = 1, \ldots, N$, where each $\Omega_i$ is the union of shape regular triangular elements of diameter $O(h)$ with finite element nodes on the boundaries of neighboring subdomains matching across the interface $\Gamma := (\bigcup_{i=1}^N \partial \Omega_i) \setminus \partial \Omega$. For our theoretical analysis we will make the assumption that each subdomain $\Omega_i$ is the union of a uniformly bounded number of shape regular finite elements with a diameter of $O(H)$. The interface in two dimensions is composed of subdomain edges and vertices. Here, edges are defined as open sets that are shared by two subdomains, and vertices as endpoints of edges; see [26] or [20] for a more general definition. We will denote the edge between two subdomains $\Omega_i$ and $\Omega_j$ by $E_{ij}$ and its vertices by $V^{ij}$. We will also assume that all the edges of $\Omega_i$ are straight line segments.

In the following the standard piecewise linear finite element space on $\Omega_i$ is denoted by $W^h(\Omega_i)$. We assume that these finite element functions vanish on $\partial \Omega_D$ and that the triangulation on each subdomain is quasi-uniform. The diameter of a subdomain $\Omega_i$ is denoted by $H_i$ or generically by $H$. For a part of the boundary $\Gamma \subset \partial \Omega_i$ with positive measure, we denote by $W^h(\Gamma)$ the corresponding finite element trace space.

3. The basic FETI-DP algorithm. In this section, we briefly review an algorithmic description of FETI-DP; see, e.g., [12, 11, 26, 22, 21] and [6, 5, 30, 29, 31] for the closely related BDDC algorithm. For a more detailed introduction to FETI-DP, see, e.g., [26, 20, 45]. For every subdomain $\Omega_i$, $i = 1, \ldots, N$, we assemble a local matrix $K^{(i)}$ and a local right hand side $f^{(i)}$. We denote the unknowns in $\Omega_i$ with $u^{(i)}$ which we further partition into unknowns $u^{(i)}_I$ in the interior part of the subdomain and unknowns $u^{(i)}_I$ on the interface. We further partition the unknowns on the interface in primal unknowns $u^{(i)}_P$ and dual unknowns $u^{(i)}_A$. Continuity in the primal unknowns is implemented by global subassembly. For the dual unknowns we introduce a jump
operator and Lagrange multipliers. While the continuity in the primal unknowns is guaranteed by subassembly in each iteration, continuity in the dual unknowns is enforced by Lagrange multipliers at convergence of the iterative method. The local stiffness matrices $K^{(i)}$ and right hand sides $f^{(i)}$ are partitioned correspondingly to the unknowns $u^{(i)}$. Combining, for each subdomain, the non-primal unknowns $u^{(i)}_B$ and $u^{(i)}_\Delta$ to a vector $u^{(i)}_B$ and partitioning the coefficients in the local stiffness matrices and right hand sides accordingly, we obtain

$$K^{(i)} = \begin{bmatrix} K_{BB}^{(i)} & K_{PB}^{(i)} \\ K_{PB}^{(i)} & K_{PP}^{(i)} \end{bmatrix}, \quad f^{(i)} = \begin{bmatrix} f_B^{(i)} \\ f_P^{(i)} \end{bmatrix}.$$  

We define

$$K = \text{diag}_{i=1}^N(K^{(i)}), \quad u = \begin{bmatrix} u^{(1)} \ T, \ldots, u^{(N)} \ T \end{bmatrix} \quad \text{and} \quad f = \begin{bmatrix} f^{(1)T}, \ldots, f^{(N)T} \end{bmatrix}.$$  

Partially assembling $K$ in the primal variables, we obtain a saddle point problem of the form

$$\begin{bmatrix} \bar{K} & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \bar{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \bar{f} \\ 0 \end{bmatrix}.$$  

Here the matrix $\bar{K}$ and right hand side $\bar{f}$ are obtained from partially assembly of $K$ and $f$ in the primal variables. The jump operator $B = [B^{(1)}, \ldots, B^{(N)}]$ enforces the continuity in the dual variables. We have, ordering the primal variables last,

$$\bar{K} = \begin{bmatrix} K^{(1)}_{BB} & & & \\ & \ddots & & \\ & & K^{(N)}_{BB} & \\ \bar{K}_{PB}^{(1)} & \cdots & \bar{K}_{PB}^{(N)} & \bar{K}_{PP} \end{bmatrix}, \quad \bar{f} = \begin{bmatrix} f_B^{(1)} \\ \vdots \\ f_B^{(N)} \\ f_P \end{bmatrix}.$$  

The local problems are invertible and the variables $\bar{u}$ can be eliminated. The factorization of $\bar{K}$ involves factoring the Schur complement

$$\bar{S}_{PP} = \bar{K}_{PP} - \sum_{i=1}^N \bar{K}_{PB}^{(i)} \left( K_{BB}^{(i)} \right)^{-1} \bar{K}_{PB}^{(i)T}.$$  

The coupling, and thus $\bar{S}_{PP}$, provides a coarse problem. We reduce the system of equations to an equation in $\lambda$. It remains to solve

$$F \lambda = d,$$

where $F = B \bar{K}^{-1} B^T$. As a preconditioner for $F$, we use the standard Dirichlet preconditioner $M_D^{-1} := B_D R_D^T S R_T B_D^T$. Here, $S$ is the Schur complement obtained by eliminating the interior variables in every subdomain, i.e.,

$$S = \begin{bmatrix} S^{(1)} \\ \vdots \\ S^{(N)} \end{bmatrix}.$$
The restriction matrix \( R \) consists of zeros and ones and removes the interior variable \( s \) from \( \tilde{u} \) when applied to a vector \( \tilde{u} \). The matrices \( B_D \) are scaled variants of the jump operator \( B \) where, in the simplest case the contribution from and to each interface node is scaled by the inverse of the multiplicity of the node. We define the multiplicity of a node as the number of subdomains it belongs to. Let \( N_x \) be the set of indices of subdomains which have \( x \in \Gamma_h \) on its boundary. We define weighted counting functions on the interface by

\[
\delta_j(x_k) := \frac{\sum_{i \in N_{x_k}} \hat{\rho}_i(x_k)}{\hat{\rho}_j(x_k)},
\]

where \( \hat{\rho}_j(x) = \max_{x \in \Omega_j} \rho_j(x) \), for \( x \in \partial \Omega_{j,h} \cap \Gamma_h \), \( j = 1, \ldots, N \). The pseudoinverses are defined by

\[
\delta_j(x_k)^\dagger := \frac{\hat{\rho}_j(x_k)}{\sum_{i \in N_{x_k}} \hat{\rho}_i(x_k)}
\]

for \( x_k \in \partial \Omega_{j,h} \cap \Gamma_h \). Each row of \( B^{(i)} \) with a nonzero entry connects a point of \( \Gamma^{(i)}_h \) with the corresponding point of a neighboring subdomain \( x \in \Gamma^{(i)}_h \cap \Gamma^{(j)}_h \). Multiplying each such row with \( \delta_j(x)^\dagger \) for each \( B^{(i)}_j \), \( i = 1, \ldots, N \) results in the scaled operator \( B_D \). This approach is referred to as \( \rho \)-scaling, see, e.g., [26] for more details. In our approach we choose every node with multiplicity of three or higher as primal. Later we also choose weighted edge averages as primal variables, but we will not enforce additional constraints by subassembly but by a projection method as described in the next section.

4. FETI-DP with projector preconditioning. It is possible to enrich the coarse space of FETI-DP by additional constraints using projections; see, e.g., [23, 17]. Here, we give a brief revision of this approach known as projector preconditioning or deflation. For more details, see [23]. For a given rectangular matrix \( U \) containing the constraints as columns, the constraint \( U^T Bu = 0 \) is enforced in each iteration of the preconditioned conjugate gradient (PCG) method. We define the \( F \)-orthogonal projection \( P \) onto the range of \( U \) by

\[
P = U(U^T FU)^{-1}U^T F.
\]

Then, we will solve the projected system

\[
(I - P)^T F \lambda = (I - P)^T d.
\]

The matrix \( (I - P)^T F \) is singular but the linear system is consistent and can be solved by CG. The preconditioned system is

\[
M^{-1}(I - P)^T F \lambda = M^{-1}(I - P)^T d
\]

with the Dirichlet preconditioner \( M^{-1} \). Let \( \lambda^* \) be the solution of the original system \( F \lambda = d \) and we define \( \bar{\lambda} := PF^{-1} d \). With \( \lambda \) we denote the solution of (4.1). The solution of the original problem can then be written as \( \lambda^* = \bar{\lambda} + (I - P)\lambda \). We will include the projection \( (I - P)^T \) into the preconditioner and project the correction onto \( \text{Range}(I - P) \) in each iteration. We obtain the symmetric preconditioner

\[
M_{PP}^{-1} = (I - P)M^{-1}(I - P)^T
\]

and solve the original problem applying this preconditioner. This preconditioned system is singular but consistent. The solution \( \lambda \) of this system is in the subspace \( \text{Range}(I - P) \). The solution \( \lambda^* \) of the original problem is then computed by \( \lambda^* = \bar{\lambda} + (I - P)\lambda \).
If we include the computation of $\bar{\lambda}$ into the iteration we get the balancing preconditioner

$$M_{BP}^{-1} = (I - P)M^{-1}_0(I - P)^T + U(U^TFU)^{-1}U^T.$$ We will use this preconditioner to solve $F\lambda = d$ by PCG and we directly obtain the solution without an additional correction. Let us briefly describe how the projection in the deflation approach can be built efficiently since the cost of a naive implementation is prohibitive. First $FU = B\tilde{K}^{-1}B^TU$ has to be computed. This is performed by exploiting neighborhood information as in one and two-level FETI methods, see, e.g., [13]. Considering the standard block factorization for

$$\begin{bmatrix}
K_{BB} & \tilde{K}_{BB}^T \\
\tilde{K}_{NB} & \tilde{K}_{PB}\end{bmatrix}^{-1} = \begin{bmatrix}
I & -K_{BB}^{-1}\tilde{K}_{PB}^T \\
0 & I\end{bmatrix} \begin{bmatrix}
K_{BB}^{-1} & 0 \\
0 & \tilde{S}_{PB}^{-1}\end{bmatrix} \begin{bmatrix}
I & 0 \\
-\tilde{K}_{PB}K_{BB}^{-1} & I\end{bmatrix},$$

when computing $K_{BB}^{-1}B^TU$ only a small number of subdomains has to be considered, i.e., the computation involves only neighboring subdomains. For edges in 2D, for each column of $U$, only two subdomain solves are necessary. A coarse solve follows for every column of $U$. Finally, the matrix $U^TFU$ has to be assembled, again exploiting neighborhood information.

Let us note that our strategy is to start with a small and cheap coarse space which is large enough to ensure invertibility. Then, in order to accelerate the convergence, the coarse space is enriched using additional constraints. In the present approach, these are build using certain eigenvectors; see section 5, for details. The new, larger coarse space can, of course, be implemented in many different ways, including transformation of basis, local saddle point problems or deflation. From the discussion above, for the deflation approach, we see that the number of applications of $\tilde{S}_{PB}^{-1}$ is reduced, compared to the small first coarse problem, if an additional constraint (eigenvector) reduces the number of iterations by one. Note that a similar decomposition of the coarse problem into two stages is also used in the two level FETI method; see [13] and [44] where in the latter the second level is also based on certain eigenvectors. The implementation of the second level in two level FETI methods is the same as in projector preconditioning.

5. Enriching the coarse space by eigenvectors computed from local eigenvalue problems. In some special cases of varying coefficients, robustness of FETI-DP and BDDC methods can be obtained at no or almost no additional computational cost.

If the coefficient of the partial differential equation is constant or only slightly varying on every subdomain but possibly has arbitrarily large jumps across the interface, a robust coarse space can be constructed using only vertices as primal variables in combination with a proper scaling in the preconditioner; see, e.g., [40, 24]. Other simple configurations where the coefficient jump is not across the interface can be treated by weighted constraints [19]. Some configurations do not need any modification of the algorithm [37, 16].

In this section, we present coarse spaces which are tailored for more general coefficient distributions. Of course, we obtain the robustness at additional computational cost, i.e., we have to solve local eigenvalue problems and we have to accept a slightly larger coarse space.

Let $E_{ij}$ be an edge between the subdomains $\Omega_i$ and $\Omega_j$ and let $S^{(i)}_{E_{ij},\rho}$ be the Schur complement that is obtained from $K^{(i)}$ after eliminating all variables except of the
degrees of freedom on the closure of the edge. Let
\[ s_{E_{ij},\rho}^{(i)}(u,v) := u^T S_{E_{ij},\rho}^{(i)} v \]
be the corresponding bilinear form; see also Remark 5.2. In addition, we define the weighted \( L^2(E_{ij}) \)-inner product
\[ m_{E_{ij},\rho}^{(i)}(u,v) := \int_{E_{ij}} \rho_i u \cdot v \, ds, \quad \text{for } l = i, j. \]
For \( \rho_{E_{ij}} = 1 \) we have the Poincaré inequality on an edge \( E_{ij} \)
\[ ||v_{(i)} - \bar{v}(i)||_{L^2(E_{ij})}^2 = m_{E_{ij},1}(v_{(i)} - \bar{v}(i), v_{(i)} - \bar{v}(i)) \leq CH_i |v_{(i)}|_{H^1(\Omega)}^2 \quad \forall v_{(i)} \in H^1(\Omega_i), \]
where \( \bar{v}(i) = \frac{1}{|E_{ij}|} \int_{E_{ij}} v_{(i)} \, ds \) is the edge average. However, if the coefficient on a sub-domain has a large variation, the constant in the Poincaré inequality may depend on the ratio of the largest and the smallest value of the coefficient, and the higher the contrast the larger the Poincaré constant becomes; see, e.g., [39, Theorem 2.9, Proposition 3.7]. In case of a quasi-monotone coefficient, see [39] for a definition, it is possible to show the independence of the constant on the contrast by introducing weighted Poincaré inequalities and weighted edge averages. If the coefficient is not quasi-monotone, this approach is not successful. In the following, we will use a different approach to obtain a constant independent of the coefficient jump for coefficients which are not quasi-monotone by solving local eigenvalue problems and enriching the coarse space with certain eigenvectors. Similar approaches have been used for overlapping Schwarz methods in [14, 8, 43, 10]. To obtain a similar estimate for coefficients which are not quasi-monotone, we need to replace the Poincaré inequality by a more general estimate since the Poincaré constant is contrast dependent in this case. However, we need to enforce more constraints on the function to get a contrast independent estimate. The Poincaré constant, in general, can also depend on the geometric scale; for a more detailed discussion of quasi-monotone coefficients and generalized Poincaré inequalities, see [39, 35, 36].

Let \( E_{ij} \) be an edge. We solve the following generalized eigenvalue problem on \( E_{ij} \).

**Eigenvalue Problem 1.** Find \((u_k^{(i)}, \mu_k^{(i)}) \in W^h(E_{ij}) \times \mathbb{R} such that\)
\[ s_{E_{ij},\rho}^{(i)}(u_k^{(i)}, v) = \mu_k^{(i)} m_{E_{ij},\rho}^{(i)}(u_k^{(i)}, v) \quad \forall v \in W^h(E_{ij}), \quad k = 1, \ldots, n_{E_{ij}}. \] (5.1)

We do not need to solve this problem for all but only for a number of small eigenvalues and their corresponding eigenvectors. Let the eigenvalues \( 0 = \mu_1^{(i)} \leq \ldots \leq \mu_{n_{E_{ij}}}^{(i)} \) be sorted in an increasing order. For a given natural number \( L \leq n_{E_{ij}} \) and for every subdomain, we define the projection
\[ I_{L}^{E_{ij},(l)} v := \sum_{k=1}^{L} m_{E_{ij},\rho}^{(i)}(u_k^{(l)}, v) u_k^{(l)}, \quad l = i, j, \]
where \( u_k^{(l)} \) are the eigenvectors of (5.1) corresponding to the eigenvalues \( \mu_k^{(l)} \). Note that the eigenvectors \( u_k^{(l)} \) can be chosen orthonormal w.r.t. \( m_{E_{ij},\rho}^{(i)}(.,.) \). For our
analysis we will use the seminorm

\[ |v|^2_{H_j^1(\Omega_i)} := \int_{\Omega_i} \rho_i (\nabla v)^2 \, dx \]

and the norms

\[ \|v\|^2_{L^2(\varepsilon_{ij})} := \int_{\varepsilon_{ij}} \rho_i v^2 \, ds, \quad \|v\|^2_{L^2(\Omega_i)} := \int_{\Omega_i} \rho_i v^2 \, dx. \]

Furthermore, we define the \( \rho_i \)-harmonic extension of \( v \) as

\[ H_{\rho_i}^l v := \min_{v \in H^l(\Omega_i)} \left\{ \int_{\Omega_i} \rho_i (\nabla u)^2 \, dx : \, u_{|\partial \Omega_i} = v \right\}. \]

By standard variational arguments we obtain the following lemma.

**Lemma 5.1.** Let \( E \subset \Gamma^{(i)} := \partial \Omega_i \) be an edge, \( E \) its closure, and \( E \subset \Gamma^{(i)} \) be the complement of \( E \) with respect to \( \Gamma^{(i)} \). Define an extension from the edge \( E \subset \Gamma^{(i)} \) to \( \Gamma^{(i)} \) by

\[ v^{(i)} := \begin{bmatrix} v_E^{(i)} \\ -S^{(i)} S^{(i)} v_E^{(i)} \end{bmatrix}, \quad \text{where} \quad S^{(i)} := \begin{bmatrix} s^{(i)} & S^{(i)T} \\ S^{(i)} & S^{(i)} \end{bmatrix}. \]

For \( w^{(i)} \in W^h(\Gamma^{(i)}) \) we denote by \( v_E^{(i)} \) the nodal vector of \( w_E^{(i)} \). Then, for all \( w^{(i)} \in W^h(\Gamma^{(i)}) \), we have \( |v_E^{(i)}|^2 \leq |w^{(i)}|^2 \).

**Remark 5.2.** With the extension operator \( H_{\rho_i}^{(i)} v_E^{(i)} := \begin{bmatrix} v_E^{(i)} \\ -S^{(i)} S^{(i)} v_E^{(i)} \end{bmatrix} \) we have \( |H_{\rho_i}^{(i)} H_{\rho_i}^{(i)} v_E^{(i)}|_{H^{1}_{\rho_i}(\Omega_i)}^2 = s_{\rho_i}^{(i)}(v_E^{(i)}, v_E^{(i)}) \) \( \forall v_E^{(i)} \in W^h(\varepsilon) \). Here, \( v_E^{(i)} \) denotes the nodal vector of \( v_E^{(i)} \).

**Lemma 5.3.** For \( v \in W^h(\varepsilon_{ij}) \) and \( w := \left( v - I_{E_{ij}}^{l(v)} \right) \in W^h(\varepsilon_{ij}) \), we have

\[ \|v - I_{E_{ij}}^{l(v)} \|^2_{L^2(\varepsilon_{ij})} = m_{E_{ij}, \rho}^{(i)}(w, w) \leq \frac{1}{\mu_{L+1}^{(i)}} s_{E_{ij}, \rho}^{(i)}(v, v) = \frac{1}{\mu_{L+1}^{(i)}} |H_{\rho_i}^{(i)} H_{\rho_i}^{(i)} v|_{H^{1}_{\rho_i}(\Omega_i)}^2 \]

and

\[ s_{E_{ij}, \rho}^{(i)}(w, w) \leq s_{E_{ij}, \rho}^{(i)}(v, v). \]  

**Proof.** The proof is based on arguments from classical spectral theory. For completeness, we provide the arguments in detail. A related abstract lemma also based on classical spectral theory can be found in [43, Lemma 2.11]. We first prove (5.3). Since \( u_k^{(i)} S^{(i)} E_{ij} u_m^{(i)} = \delta_{km} \) for eigenvectors \( u_k^{(i)} \) and \( u_m^{(i)} \), where \( \delta_{km} \) is the Kronecker symbol, we have

\[ s_{E_{ij}, \rho}^{(i)}(I_{E_{ij}}^{l(v)} v, w) = s_{E_{ij}, \rho}^{(i)}(I_{E_{ij}}^{l(v)} v, v - I_{E_{ij}}^{l(v)} v) = 0. \]
We obtain
\begin{equation}
\sum_{i=1}^{n_{\varepsilon_{ij}}} \left( \sum_{j=L+1}^{n_{\varepsilon_{ij}}} \sum_{l=1}^{m_{\varepsilon_{ij}, \rho}} (v, u_{i,l}^{(l)} u_{j,l}^{(l)}) \right) = \sum_{i=1}^{n_{\varepsilon_{ij}}} \left( \sum_{j=L+1}^{n_{\varepsilon_{ij}}} \sum_{l=1}^{m_{\varepsilon_{ij}, \rho}} \left( u_{i,l}^{(l)} - u_{j,l}^{(l)} \right) \right) (5.6)
\end{equation}
and analogously
\begin{equation}
\sum_{i=1}^{n_{\varepsilon_{ij}}} \left( \sum_{j=L+1}^{n_{\varepsilon_{ij}}} \sum_{l=1}^{m_{\varepsilon_{ij}, \rho}} \left( u_{i,l}^{(l)} - u_{j,l}^{(l)} \right) \right) = \sum_{i=1}^{n_{\varepsilon_{ij}}} \left( \sum_{j=L+1}^{n_{\varepsilon_{ij}}} \sum_{l=1}^{m_{\varepsilon_{ij}, \rho}} \left( u_{i,l}^{(l)} - u_{j,l}^{(l)} \right) \right) . (5.7)
\end{equation}

The inequality (5.3) follows by noting that the terms of the sums in (5.6) and (5.7) are all positive or zero. Next, we prove the inequality (5.2). Here, we replace \( m_{\varepsilon_{ij}, \rho}(u, v) \) by \( \mu_{k}^{(l)} s_{\varepsilon_{ij}, \rho}(v, u_{k}^{(l)}) \), and consider
\begin{equation}
\sum_{i=1}^{n_{\varepsilon_{ij}}} \mu_{k}^{(l)} s_{\varepsilon_{ij}, \rho}(v, u_{k}^{(l)}) \leq \frac{1}{\mu_{L+1}} \sum_{i=1}^{n_{\varepsilon_{ij}}} \mu_{k}^{(l)} s_{\varepsilon_{ij}, \rho}(v, u_{k}^{(l)}) . (5.4)
\end{equation}

A similar inequality for the whole domain \( \Omega_i \) instead of the edge \( \varepsilon_{ij} \) is given in [14, (3.10)]. In [8] an analogous inequality has been shown for \( \partial \Omega \), instead of \( \varepsilon_{ij} \). Let us note that those inequalities are all also related to different eigenvalue problems. The eigenvalue problem considered here, cf. (5.1), is more local than those in [14] and [8]. To take advantage of inequality (5.2) in our FETI-DP algorithm using projector preconditioning, we need to enforce the projected jumps across the interface to be zero to obtain \( \Pi_{L}^{\varepsilon_{ij}, (l)} v^{(i)} = \Pi_{L}^{\varepsilon_{ij}, (l)} v^{(j)} \) and \( \Pi_{L}^{\varepsilon_{ij}, (l)} v^{(i)} = \Pi_{L}^{\varepsilon_{ij}, (l)} v^{(j)} \). Let \( v_{E_{ij}}^{(l)} \) be the restriction of \( v \) to the edge \( \varepsilon_{ij} \). To guarantee this equality, we enforce the constraint \( m_{\varepsilon_{ij}, \rho}(u_{k}^{(l)}, v_{E_{ij}}^{(l)} - v_{E_{ij}}^{(j)} = 0 \) for \( k = 1, \ldots, L \). To enrich the coarse space, we first multiply the eigenvectors by the mass matrix corresponding to \( m_{\varepsilon_{ij}, \rho} \), then discard
the entries associated with primal vertices, and finally extend these vectors by zero on the remaining part of the interface. Then these vectors define the corresponding columns of $U$. We do this for each edge of each subdomain and for each eigenvector of the generalized eigenvalue problem (5.1) for which the corresponding eigenvalue is smaller than or equal to a chosen tolerance $\tau_\mu$, i.e.,

$$\mu_L \leq \tau_\mu. \quad (5.8)$$

Let us note that linearly dependent eigenvectors are removed by using a singular value decomposition; see the discussion at the end of section 6.2. Only this reduced set of eigenvectors are used to define the columns of $U$.

6. Theory and condition number estimate. In Section 6.1 we will show a weighted edge lemma, a weighted Friedrichs inequality and an extension theorem for the case that coefficient functions on the subdomain satisfy certain conditions. In Section 6.2 we introduce a second eigenvalue problem to bound extensions between subdomains in more general cases. Finally, we prove our condition number estimate in Section 6.3.

6.1. Technical Tools. In this section, we provide a few technical tools, together with their proofs, which are needed for the proof of our condition number estimate.

**Definition 6.1.** By an $\eta$-patch $\omega \subset \Omega$ we denote an open set which can be represented as a union of shape regular finite elements of diameter $O(h)$ and which has $\text{diam}(\omega) = O(\eta)$ and a measure of $O(\eta^2)$.

The next definition was introduced in 3D in [16].

**Definition 6.2.** Let $E_{ij} \subset \partial \Omega$, be an edge. Then, a slab $\tilde{\Omega}_{\eta}$ is a subset of $\Omega$, of width $\eta$ with $E_{ij} \subset \partial \tilde{\Omega}_{\eta}$ which can be represented as the union of $\eta$-patches $\omega_{ik}$, $k = 1, \ldots, n$, such that $E_{ij}^{(k)} := (\partial \omega_{ik} \cap E_{ij})^c \neq \emptyset$, $k = 1, \ldots, n$.

Next, we formulate and prove an edge lemma.

**Lemma 6.3.** Let $\tilde{\Omega}_{\eta} \subset \Omega_i$ be a slab of width $\eta$, such that $E_{ij} \subset \partial \tilde{\Omega}_{\eta}$. Let $\omega_{ik} \subset \tilde{\Omega}_{\eta}$, $k = 1, \ldots, n$, be a set of $\eta$-patches such that $\tilde{\Omega}_{\eta} = \cup_{k=1}^n \omega_{ik}$, and the coefficient function $\rho_{\omega_{ik}} = \rho_{ik}$ is constant on each $\omega_{ik}$. Let $\omega_{ik} \cap \omega_{il} = \emptyset$, $k \neq l$. Then there exists a finite element function $\vartheta_{E_{ij}}$, which equals $\theta_{E_{ij}}$ on $\partial \tilde{\Omega}_{\eta}$, such that for $u \in W^h(\Omega_i)$

$$|H^{(1)}_p (\vartheta_{E_{ij}} u)^2_{H^1(\tilde{\Omega}_{\eta})} \leq |H^{(1)}(\vartheta_{E_{ij}} u)^2_{H^1(\tilde{\Omega}_{\eta})} \leq C \left(1 + \log \left(\frac{\eta}{h}\right)^2 \left(|u|^2_{H^1(\tilde{\Omega}_{\eta})} + \frac{1}{\eta^2} \|u\|^2_{L^2(\tilde{\Omega}_{\eta})}\right),

$$

where $C > 0$ is a constant independent of $H, h, \eta$, and the contrast of $p$.

**Proof.** We define $E^{(k)}_{ij} := (\partial \omega_{ik} \cap E_{ij})^c$, where $M^2$ is the interior of the set $M$, i.e., $E^{(k)}_{ij}$ is an open edge without its endpoints, and $E^{(k+1)}_{ij} := E^{(k)}_{ij} \cap E^{(k+1)}_{ij}$ is an endpoint of that edge. For each patch $\omega_{ik}$ and its local edge $E^{(k)}_{ij}$, there exists a finite element function $\vartheta(E^{(k)}_{ij})$, which is one at $E^{(k)}_{ij}$ and zero in all other nodes on the boundary of $\omega_{ik}$.

In the interior of the patch $\omega_{ik}$, the function $\vartheta^{(k)}_{E_{ij}}$ can be defined such that

$$|H^{(1)}(\vartheta^{(k)}_{E_{ij}} u)^2_{H^1(\omega_{ik})} \leq C \left(1 + \log \left(\frac{\eta}{h}\right)^2 \left(|u|^2_{H^1(\omega_{ik})} + \frac{1}{\eta^2} \|u\|^2_{L^2(\omega_{ik})}\right); \quad (6.1)$$
see, e.g., [25, proof of Lemma 4.4] or [45, Lemma 4.24], which is a three-dimensional analogon. For the endpoints $V_{ij}^{k}$ and $V_{ij}^{k+1}$ of $E^{(k)}$, we denote by $\vartheta_{V_{ij}^{k}}$ and $\vartheta_{V_{ij}^{k+1}}$, respectively, the corresponding nodal finite element basis function. Next, we define the finite element function $\vartheta_{E_{ij}} := \sum_{k=1}^{n} \vartheta_{E_{ij}^{(k)}} + \sum_{k=2}^{n} \vartheta_{V_{ij}^{(k)}}$. Note that $\vartheta_{E_{ij}}$ equals 1 in all nodes on $E_{ij}$ and 0 in all nodes on $\partial \Omega_{\eta} \setminus E_{ij}$, the boundary of the slab without the edge. Since the discrete $\rho_{i}$-harmonic function has the smallest energy we have

$$|\mathcal{H}_{\rho_{i}}^{(k)}(\vartheta_{E_{ij}} u)|_{H^{2}_{\rho_{i}}(\Omega)}^{2} \leq |I^{h}(\vartheta_{E_{ij}} u)|_{H^{2}_{\rho_{i}}(\Omega)}^{2}.$$ 

For the function $\vartheta_{ij}^{(k)}$, $t = k, k + 1$, we have

$$|I^{h}(\vartheta_{ij}^{(k)} u)|_{H^{1}_{\rho_{i}}(\Omega_{\eta})}^{2} \leq C \left(1 + \log \left(\frac{\eta}{h}\right)\right) \left(|u|_{H^{1}_{\rho_{i}}(\Omega_{\eta})} + \frac{1}{\eta^{2}} ||u||_{L^{2}(\Omega_{\eta})}^{2}\right),$$

which follows from using an inverse inequality, see, e.g., [3] or [45, Lemma B.5], and a Sobolev inequality for finite element functions; see, e.g., [7, Lemma 3.2] or [3]. Using (6.1), (6.2), and a triangle inequality, we obtain

$$|I^{h}(\vartheta_{E_{ij}} u)|_{H^{2}_{\rho_{i}}(\Omega_{\eta})}^{2} = \sum_{k=1}^{n} |I^{h}(\vartheta_{E_{ij}} u)|_{H^{2}_{\rho_{i}}(\Omega_{\eta})}^{2}$$

$$\leq 3 \sum_{k=1}^{n} |I^{h}(\vartheta_{E_{ij}} u)|_{H^{2}_{\rho_{i}}(\Omega_{\eta})}^{2} + 3 \sum_{k=1}^{n-1} |I^{h}(\vartheta_{ij}^{(k+1)} u)|_{H^{2}_{\rho_{i}}(\Omega_{\eta})}^{2} + 3 \sum_{k=2}^{n} |I^{h}(\vartheta_{ij}^{(k)} u)|_{H^{2}_{\rho_{i}}(\Omega_{\eta})}^{2}$$

$$\leq C \left(1 + \log \left(\frac{\eta}{h}\right)\right)^{2} \left(|u|_{H^{2}_{\rho_{i}}(\Omega_{\eta})}^{2} + \frac{1}{\eta^{2}} ||u||_{L^{2}(\Omega_{\eta})}^{2}\right).$$

Here, we have used that $\rho_{i}$ is constant on each patch $\omega_{ik}, k = 1, \ldots, n$. \n
Let us note that similar techniques using patches for heterogeneous coefficients have been also used in [38]. For inequalities related to the next lemma, see [8, Lemma 2.4] and [45, proof of Lemma 3.10].

**Lemma 6.4 (Weighted Friedrichs inequality).** For $u \in H^{1}(\Omega_{\eta})$, we have

$$||u||_{L^{2}_{\rho_{i}}(\Omega_{\eta})}^{2} \leq C \left(\eta^{2} ||u||_{H^{2}_{\rho_{i}}(\Omega_{\eta})}^{2} + \eta ||u||_{L^{2}_{\rho_{i}}(\Omega_{\eta})}^{2}\right).$$

**Proof.** Let $\omega_{ik} \subset \Omega_{\eta}, k = 1, \ldots, n$ be a set of $\eta$-patches such that $\Omega_{\eta} = \bigcup_{k=1}^{n} \omega_{ik}$, and the coefficient function $\rho_{i|\omega_{ik}} = \rho_{ik}$ is constant on each $\omega_{ik}$. Let $\tilde{\pi}_{E_{ij}^{(k)}} := \frac{1}{|E_{ij}^{(k)}|} \int_{E_{ij}^{(k)}} u \, dx$ be the standard edge average. Then, we obtain by using a standard Poincaré inequality

$$||u||_{L^{2}_{\rho_{i}}(\omega_{ik})}^{2} = \rho_{ik} ||u||_{L^{2}(\omega_{ik})}^{2} \leq 2 \rho_{ik} \left||u - \tilde{\pi}_{E_{ij}^{(k)}}\right||_{L^{2}(\omega_{ik})}^{2} + 2 \rho_{ik} \left\|\pi_{E_{ij}^{(k)}}\right\|_{L^{2}(\omega_{ik})}^{2}$$

$$\leq \rho_{ik} C_{k} \eta^{2} \left(||u||_{H^{1}_{\rho_{i}}(\omega_{ik})}^{2} + \left\|\pi_{E_{ij}^{(k)}}\right\|_{L^{2}(\omega_{ik})}^{2}\right)$$

$$\leq \max\{C_{k}, 1\} \left(\eta^{2} ||u||_{H^{1}_{\rho_{i}}(\omega_{ik})}^{2} + \rho_{ik} \eta ||u||_{L^{2}_{\rho_{i}}(\Omega_{\eta})}^{2}\right) .$$
In the last step we have applied a Cauchy-Schwarz inequality. Summing over \( k \) completes the proof. \( \square \)

In order to obtain a condition number estimate that is independent of the contrast of the coefficient function, it is sufficient to have an extension operator from a slab to a neighboring slab across the edge shared by these two, which is uniformly bounded with respect to the contrast.

**Assumption 6.5 (special case).** We assume that there exists an extension operator

\[
E_{ij} : W^{h}(\tilde{\Omega}_{j\kappa}) \to W^{h}(\tilde{\Omega}_{i\kappa})
\]

with

\[
|E_{ij}u|_{H^{1}((\tilde{\Omega}_{i\kappa}))}^{2} \leq C|u|_{H^{1}((\tilde{\Omega}_{j\kappa}))}^{2},
\]

where \( C > 0 \) is a constant independent of \( H, b, \) and the contrast of \( \rho \).

Assumption 6.5 is satisfied for some special coefficient distributions; see, e.g., Section 7, Figures 7.1, 7.2. In those cases, \( E_{ij} \) can be obtained by a constant translation in \( x \)- or \( y \)-direction.

### 6.2. Bounding the extension between subdomains.

Assumption 6.5 can only be used for special coefficient distributions. To be able to treat more general cases, we consider a second set of primal constraints to bound the terms

\[
|\mathcal{H}^{(i)}_{\rho_{i}}\mathcal{H}_{E_{ij} \rightarrow \Gamma^{(i)}}\hat{w}^{(i)}|_{H^{1}((\tilde{\Omega}_{i}))}^{2} \leq C|\mathcal{H}^{(i)}_{\rho_{i}}\mathcal{H}_{E_{ij} \rightarrow \Gamma^{(i)}}w^{(i)}|_{H^{2}_{\rho_{j}}(\tilde{\Omega}_{j})}^{2}.
\]

Here \( \hat{w}^{(i)} \) is a projection of \( w^{(i)} \) on a subspace; see the discussion below. To compute these additional primal constraints, we consider the following second generalized eigenvalue problem.

**Eigenvalue Problem 2.**

\[
\begin{align*}
&\mathcal{E}^{(i)}_{E_{ij}, \rho_{j}}(v, w_{\kappa}) = \nu_{\kappa}^{(i)}\hat{P}_{\rho}^{(i)}_{E_{ij}, \rho_{i}}(v, w_{\kappa}) \quad \forall v \in W^{h}(\mathcal{E}_{ij}), \quad \kappa = 1, \ldots, n_{E_{ij}}, \quad (6.4)
\end{align*}
\]

We note that the bilinear forms in the generalized eigenvalue problem (6.4) have nontrivial nullspaces. However, if \( \text{Ker}(\mathcal{S}^{(j)}_{E_{ij}, \rho_{j}}) = \text{Ker}(\mathcal{S}^{(i)}_{E_{ij}, \rho_{i}}) \), we can solve the problem on \( \text{Range}(\mathcal{S}^{(j)}_{E_{ij}, \rho_{j}}) = \text{Range}(\mathcal{S}^{(i)}_{E_{ij}, \rho_{i}}) \). To guarantee \( \text{Ker}(\mathcal{S}^{(j)}_{E_{ij}, \rho_{j}}) = \text{Ker}(\mathcal{S}^{(i)}_{E_{ij}, \rho_{i}}) \) we use the corresponding Schur complements without Dirichlet boundary conditions, i.e., the Schur complements are obtained from the positive semidefinite stiffness matrices \( K^{(i)} \) before Dirichlet boundary conditions related to \( \partial \Omega_{D} \) are incorporated. Note that we could also use Schur complements with boundary conditions. Here, we have implemented the version without boundary conditions to avoid having to treat the different cases. Let \( w^{(j)} = w^{(j)}_{K} + w^{(j)}_{R} \) with \( w^{(j)}_{K} \in \text{Ker}(\mathcal{S}^{(j)}_{E_{ij}, \rho_{j}}) \) and \( w^{(j)}_{R} \in \text{Range}(\mathcal{S}^{(j)}_{E_{ij}, \rho_{j}}) \).

Then, we consider the orthogonal projection \( \Pi \) onto \( \text{Range}(\mathcal{S}^{(j)}_{E_{ij}, \rho_{j}}) \) and the positive semidefinite bilinear forms

\[
\begin{align*}
&\mathcal{E}^{(j)}_{E_{ij}, \rho_{j}}(v, w) := s^{(j)}_{E_{ij}, \rho_{j}}(\Pi v, \Pi w) \\
&\mathcal{E}^{(i)}_{E_{ij}, \rho_{i}}(v, w) := s^{(i)}_{E_{ij}, \rho_{i}}(\Pi v, \Pi w).
\end{align*}
\]
The orthogonal projection $\Pi$ can be obtained by $\Pi = \sum_{r=1}^p v_r v_r^T$, where $\{v_1, \ldots, v_p\}$ is an orthonormal basis of $\text{Ker}(S^{(i)}_{E_{ij}, \rho_j})$. In exact arithmetic, we have $s^{(i)}_{E_{ij}, \rho_j} = s^{(i)}_{E_{ij}, \rho_j}$, $l = i, j$ but we use the projection $\Pi$ for stability in our computations. We can formulate a modified problem on $\text{Range}(S^{(j)}_{E_{ij}, \rho_j})$ by

$$s^{(i)}_{E_{ij}, \rho_j}(v, \pi_\kappa) = v_{\kappa}^T \frac{\rho_i}{\rho_j} s^{(i)}_{E_{ij}, \rho_j}(v, \pi_\kappa), \quad \forall v \in \text{Range}(S^{(j)}_{E_{ij}, \rho_j}), \quad \kappa = 1, \ldots, n_{E_{ij}}. \quad (6.5)$$

The eigenpairs $(\nu^{(i)}_{E_{ij}, \rho_j}, \omega^{(i)}_{E_{ij}, \rho_j})$ are also eigenpairs of the original problem $(6.4)$. Let $K \in \{1, \ldots, n_{E_{ij}}\}$ and the corresponding eigenvalues $\nu^{(i)}_{E_{ij}, \rho_j} \leq \ldots \leq \nu^{(i)}_{E_{ij}, \rho_j} \leq \ldots \leq \nu^{(i)}_{E_{ij}, \rho_j}$ be sorted in an increasing order. We define the projection

$$\Pi^{(i)}_K v := \sum_{\kappa=1}^K \frac{\rho_i}{\rho_j} s^{(i)}_{E_{ij}, \rho_j}(v, \pi_\kappa)$$

and obtain the following lemma.

**Lemma 6.6.** We have

$$s^{(i)}_{E_{ij}, \rho_j}(w^{(i)} - \Pi^{(i)}_K w^{(i)}, w^{(i)} - \Pi^{(i)}_K w^{(i)}) \leq \frac{1}{\nu^{(i)}_{K+1} \rho_j} s^{(i)}_{E_{ij}, \rho_j}(w^{(i)}, w^{(i)}) \quad \forall w^{(i)} \in W^h(E_{ij}). \quad (6.6)$$

which is equivalent to

$$|\mathcal{H}^{(i)}_{\rho_j} \mathcal{H}_{E_{ij} \rightarrow \Gamma^{(i)}} (w^{(i)} - \Pi^{(i)}_K w^{(i)}) ||^2_{H^{1/2}_{\rho_j}(\Omega)} \leq \frac{1}{\nu^{(i)}_{K+1} \rho_j} |\mathcal{H}^{(i)}_{\rho_j} \mathcal{H}_{E_{ij} \rightarrow \Gamma^{(i)}} w^{(i)} ||^2_{H^{1/2}_{\rho_j}(\Omega)}. \quad (6.7)$$

Additionally, we have

$$s^{(i)}_{E_{ij}, \rho_j}(w^{(i)} - \Pi^{(i)}_K w^{(i)}, w^{(i)} - \Pi^{(i)}_K w^{(i)}) \leq s^{(i)}_{E_{ij}, \rho_j}(w^{(i)}, w^{(i)}). \quad (6.7)$$

**Proof.** Using the additive decomposition $w^{(i)} = w^{(i)}_K + w^{(i)}_R$ with $w^{(i)}_K \in \text{Ker}(S^{(i)}_{E_{ij}, \rho_j})$, $w^{(i)}_R \in \text{Range}(S^{(i)}_{E_{ij}, \rho_j})$, we have

$$\Pi^{(i)}_K w^{(i)} = \sum_{\kappa=1}^K \frac{\rho_i}{\rho_j} s^{(i)}_{E_{ij}, \rho_j}(w^{(i)}_K, \pi_\kappa) = \sum_{\kappa=1}^K \frac{\rho_i}{\rho_j} s^{(i)}_{E_{ij}, \rho_j}(w^{(i)}_K, \pi_\kappa) = 0.$$

The proof can be completed using $s^{(i)}_{E_{ij}, \rho_j}(\pi_\kappa, \pi_\ell) = \delta_{kl}$ with the eigenvectors $\pi_\kappa$ and $\pi_\ell$, where $\delta_{kl}$ is the Kronecker symbol, and similar arguments as in the proof of Lemma 5.3. □

The additional primal constraints that we enforce for each edge $E_{ij}$ are of the form

$$\Pi^{(i)}_K w^{(i)} = \Pi^{(i)}_K w^{(i)} \quad \text{and} \quad \Pi^{(i)}_K w^{(i)} = \Pi^{(i)}_K w^{(i)}.$$

The following remark is motivated by [32].

**Remark 6.7.** We are only interested in eigenvectors in $\text{Range}(S^{(i)}_{E_{ij}, \rho_j})$ of $(6.5)$. Instead of solving this problem on $\text{Range}(S^{(i)}_{E_{ij}, \rho_j})$, we can consider instead

$$\Sigma^{(i)}_{E_{ij}, \rho_j} \pi = \nu \left( s^{(i)}_{E_{ij}, \rho_j} + \sigma (I - \Pi) \right) \pi,$$
where $E^{(k)}_{i,j,\rho_k}$ is the matrix associated with the bilinear form $\pi^{(k)}_{E_{i,j,\rho_k}}$ and $\sigma$ is any positive constant. In our computations we have chosen $\sigma$ as the maximum diagonal entry of $E^{(k)}_{i,j,\rho_k}$. The right-hand side of this problem is positive definite; see also [32].

To enhance our coarse problem, we consider for a given tolerance $\tau_\nu$ the eigenpairs $(w_k, \nu_k)$, where $k \leq K$ and

$$\nu_K \leq \tau_\nu.$$ (6.8)

For these eigenvectors, we first build $\left( E^{(i)}_{i,j,\rho_i} + \sigma (I - \Pi) \right) w_k$, discard the entries related to primal vertices, and finally extend them by zero on the remaining interface. These vectors are added to the constraints obtained from the eigenvalue problem (5.1).

Let us now consider the set of all constraints obtained from Eigenvalue Problems 1 and 2. From this set we need to remove linearly dependent vectors. In our experiments we orthonormalize all these vectors by a singular value decomposition with a drop tolerance of $10^{-6}$ and remove linearly dependent vectors. The resulting vectors are added as columns to the matrix $U$ from Section 4.

6.3. Condition number estimate. We can now prove our condition number estimate. For simplicity we will restrict ourselves to the case of second order scalar elliptic equations.

**Theorem 6.8.** The condition number for our FETI-DP method with a $\rho$-scaling, as defined at the end of section 3, satisfies

$$\kappa(\hat{M}^{-1}F) \leq C \left( 1 + \log \left( \frac{\eta h}{\mu L} \right) \right)^2 \frac{1}{\nu_{K+1}} \left( 1 + \frac{1}{\eta \mu_{L+1}} \right),$$

where $\hat{M} = M_{PP}^{-1}$ or $\hat{M} = M_{BP}^{-1}$, or alternatively $\hat{M}^{-1} = M^{-1}$ if all constraints have been enforced by a transformation of basis. Here, $C > 0$ is a constant independent of $H, h$, and $\eta$ and

$$\frac{1}{\mu_{L+1}} = \max_{k=1,\ldots,N} \left\{ \frac{1}{\nu^{(k)}_{K+1}} \right\}, \quad \frac{1}{\nu_{K+1}} = \max \left\{ 1, \max_{k=1,\ldots,N} \frac{1}{\nu^{(k)}_{K+1}} \right\}.$$

**Remark 6.9.**

1. Note that in the case of a coefficient distribution where Assumption 6.5 is satisfied, we have an estimate of the form

$$\kappa(\hat{M}^{-1}F) \leq C \left( 1 + \log \left( \frac{\eta h}{\mu L} \right) \right)^2 \left( 1 + \frac{1}{\eta \mu_{L+1}} \right).$$

In general, such an estimate holds for coefficient distributions where an extension operator exists with an upper bound independent of the values of the coefficients.

2. A similar result can be obtained for linear elasticity, e.g., by using the tools provided in [26].

3. An implementation for the related BBDC method using a transformation of basis can be found in [18].

**Proof.** [Theorem 6.8] Since the spectra of FETI-DP with projector preconditioning and FETI-DP using a transformation of basis have the same spectra if the same
constraints are enforced, see [23, Theorem 6.9], we can assume for the proof that a transformation of basis has been carried out to enforce the eigenvector constraints. Then the proof of the condition number estimate can be modeled on the corresponding proof of Lemma 8.5 in Klawonn and Widlund [26]. We briefly repeat the notation of spaces used in that proof. For each subdomain, we introduce local finite element trace spaces \( W_i := W^h(\partial \Omega_i \cap \Gamma), \ i = 1, \ldots, N \) and the product space \( W := \Pi_i W_i \). Furthermore, we define by \( \hat{W} \) the space of functions in \( W \) which are continuous across the interface and introduce an intermediate space \( \hat{W}, \hat{W} \subset W \subset C \), which consists of functions that are continuous in the primal variables, i.e., in all primal vertices and primal constraints. As mentioned before we assume for the proof that this is obtained by using a transformation of basis; see, Klawonn and Widlund [26] for more details. As usual, we always assume functions from these trace spaces to be \( \rho \)-harmonically extended to the interior of the subdomains. All these notations are standard; see, e.g., the monograph by Toselli and Widlund [45, Chapter 6] or Klawonn and Widlund [26, p. 1546].

We consider an arbitrary \( \tilde{w} \in \hat{W} \). Let \( R^{(i)T} \) be the local operator assembling in the primal variables and \( R^F = [R^{(1)T}, \ldots, R^{(N)T}] \); see, e.g., [26, p. 1533]. In the following, we will use the notation \( w^{(i)} := R^{(i)} \tilde{w} \in \hat{W}_i \) and \( w^{(j)} := R^{(j)} \tilde{w} \in \hat{W}_j \). With \( v^{(i)} := R^{(i)} P_D \tilde{w} \) and \( \hat{S}_p = R^T \hat{S}_p R \), we obtain

\[
|P_D \tilde{w}|_{\hat{S}_p}^2 = |R P_D \tilde{w}|_{\hat{S}_p}^2 = \sum_{i=1}^{n} |R^{(i)} P_D \tilde{w}|_{\hat{S}_p}^2 (\tilde{w})_{(i)} = \sum_{i=1}^{n} |v^{(i)}|_{\hat{S}_p}^2.
\]

If all vertices are chosen to be primal we can write \( v^{(i)} = \sum \hat{E}_{ij} I^h(\Theta_{\hat{E}_{ij}}, v^{(i)}) \); here we sum over all edges \( \hat{E}_{ij} \subset \Gamma^{(i)} \). In the following we will develop bounds for the edge contributions

\[
|H_{\hat{E}_{ij}}^{(i)} I^h(\Theta_{\hat{E}_{ij}}, v^{(i)})(\hat{H}_p^h(\Omega_i)).
\]

Obviously, we have \( v^{(i)} = (\delta^j)^{i} (w^{(i)} - w^{(j)}) \). We choose \( L \) such that \( \mu^{(i)}_{L+1} \) is independent of the contrast in the coefficient \( \rho \). Moreover we enforce the equalities

\[
L_{\hat{E}_{ij}}^{(i)} w^{(i)} = L_{\hat{E}_{ij}}^{(j)} w^{(j)} = L_{\hat{E}_{ij}}^{(j)} w^{(i)},
\]

\[
\Pi_K^{(i)} w^{(i)} = \Pi_K^{(j)} w^{(j)}, \quad \text{and} \quad \Pi_K^{(i)} w^{(i)} = \Pi_K^{(j)} w^{(j)},
\]

either with projector preconditioning or a transformation of basis. Further, we define \( \hat{w}^{(i)} := w^{(i)} - \Pi_K^{(i)} w^{(i)} \) and \( \hat{w}^{(j)} := w^{(j)} - \Pi_K^{(j)} w^{(j)} \). Then, we have, using Lemma 6.3,

\[
|H_{\hat{E}_{ij}}^{(i)} I^h(\Theta_{\hat{E}_{ij}}, v^{(i)})(\hat{H}_p^h(\Omega_i)) = |H_{\hat{E}_{ij}}^{(i)} I^h(\Theta_{\hat{E}_{ij}}, H_{\hat{E}_{ij}}^{(i)}(\hat{w}^{(i)}))(\hat{H}_p^h(\Omega_i)).
\]

\[
= |H_{\hat{E}_{ij}}^{(i)} I^h(\Theta_{\hat{E}_{ij}}, H_{\hat{E}_{ij}}^{(i)}(\hat{w}^{(i)} - w^{(j)}))(\hat{H}_p^h(\Omega_i)) = |H_{\hat{E}_{ij}}^{(i)} I^h(\Theta_{\hat{E}_{ij}}, H_{\hat{E}_{ij}}^{(i)}(\hat{w}^{(i)} - w^{(j)}))(\hat{H}_p^h(\Omega_i)).
\]

\[
\leq 2(\delta^j)^2 |H_{\hat{E}_{ij}}^{(i)} I^h(\Theta_{\hat{E}_{ij}}, H_{\hat{E}_{ij}}^{(i)}(\hat{w}^{(i)}))(\hat{H}_p^h(\Omega_i)) + 2(\delta^j)^2 |H_{\hat{E}_{ij}}^{(i)} I^h(\Theta_{\hat{E}_{ij}}, H_{\hat{E}_{ij}}^{(i)}(\hat{w}^{(i)}))(\hat{H}_p^h(\Omega_i)).
\]

\[
\leq C \left( 1 + \log \left( \frac{\eta}{h} \right) \right)^2 (\delta^j)^2 \left( |H_{\hat{E}_{ij}}^{(i)} I^h(\Theta_{\hat{E}_{ij}}, H_{\hat{E}_{ij}}^{(i)}(\hat{w}^{(i)}))(\hat{H}_p^h(\Omega_i)) + \frac{1}{\eta^2} \right)^2 (\hat{H}_p^h(\Omega_i)) (\hat{H}_p^h(\Omega_i)) + 1 + \frac{1}{\eta^2} ||H_{\hat{E}_{ij}}^{(i)} I^h(\Theta_{\hat{E}_{ij}}, H_{\hat{E}_{ij}}^{(i)}(\hat{w}^{(i)}))(\hat{H}_p^h(\Omega_i)) ||^2 (\hat{H}_p^h(\Omega_i))
\]

\[
+ |H_{\hat{E}_{ij}}^{(i)} I^h(\Theta_{\hat{E}_{ij}}, H_{\hat{E}_{ij}}^{(i)}(\hat{w}^{(i)}))(\hat{H}_p^h(\Omega_i)) + \frac{1}{\eta^2} ||H_{\hat{E}_{ij}}^{(i)} I^h(\Theta_{\hat{E}_{ij}}, H_{\hat{E}_{ij}}^{(i)}(\hat{w}^{(i)}))(\hat{H}_p^h(\Omega_i)) ||^2 (\hat{H}_p^h(\Omega_i))
\]

\[
+ \frac{1}{\eta^2} ||H_{\hat{E}_{ij}}^{(i)} I^h(\Theta_{\hat{E}_{ij}}, H_{\hat{E}_{ij}}^{(i)}(\hat{w}^{(i)}))(\hat{H}_p^h(\Omega_i)) ||^2 (\hat{H}_p^h(\Omega_i))
\]

\[
+ \frac{1}{\eta^2} ||H_{\hat{E}_{ij}}^{(i)} I^h(\Theta_{\hat{E}_{ij}}, H_{\hat{E}_{ij}}^{(i)}(\hat{w}^{(i)}))(\hat{H}_p^h(\Omega_i)) ||^2 (\hat{H}_p^h(\Omega_i))
\].
Now, Lemma 6.4 yields with the stability of the projections $I_{t}^{i}(i)$ and $\Pi_{t}^{i}(i)$

$$
\frac{1}{\eta} ||\mathcal{H}_{\mu}(i)\mathcal{H}_{E_{ij}} \rightarrow \Gamma_{t}^{(i)}(\mathcal{W}(i))||_{L^{2}_{H_{L}(\Omega_{\alpha})}}^{2} \leq C \left( ||\mathcal{H}_{\mu}(i)\mathcal{H}_{E_{ij}} \rightarrow \Gamma_{t}^{(i)}(\mathcal{W}(i))||_{L^{2}_{H_{L}(\Omega_{\alpha})}}^{2} + \frac{1}{\eta} ||\mathcal{W}(i)||_{L^{2}_{H_{L}(\Omega_{\alpha})}}^{2} \right)
$$

(6.9)

$$
\leq C \left( ||\mathcal{H}_{\mu}(i)\mathcal{H}_{E_{ij}} \rightarrow \Gamma_{t}^{(i)}(\mathcal{W}(i))||_{L^{2}_{H_{L}(\Omega_{\alpha})}}^{2} + \frac{1}{\eta \mu_{L+1}} ||\mathcal{H}_{\mu}(i)\mathcal{H}_{E_{ij}} \rightarrow \Gamma_{t}^{(i)}(\mathcal{W}(i))||_{L^{2}_{H_{L}(\Omega_{\alpha})}}^{2} \right)
$$

In the penultimate step, we have applied (5.2). In the last step, we have used Remark 5.2, (5.3), and (6.7). Finally, we obtain

$$
(\delta_{j})^{2} ||\mathcal{H}^{(i)}(i)\mathcal{H}_{E_{ij}} \rightarrow \Gamma_{t}^{(i)}(\mathcal{W}(i))||_{H^{1}_{L}(\Omega_{\alpha})}^{2} \leq C \left( 1 + \log \left( \frac{\eta}{h} \right) \right)^{2} \left( 1 + \frac{1}{\eta \mu_{L+1}} \right) \left( \delta_{j} \right)^{2} ||\mathcal{H}^{(i)}(i)\mathcal{H}_{E_{ij}} \rightarrow \Gamma_{t}^{(i)}(\mathcal{W}(i))||_{H^{1}_{L}(\Omega_{\alpha})}^{2}
$$

We can estimate the term $(\delta_{j})^{2} ||\mathcal{H}^{(i)}(i)\mathcal{H}_{E_{ij}} \rightarrow \Gamma_{t}^{(i)}(\mathcal{W}(i))||_{H^{1}_{L}(\Omega_{\alpha})}^{2}$ analogously and obtain

$$
(\delta_{j})^{2} ||\mathcal{H}_{\mu}(i)\mathcal{H}_{E_{ij}} \rightarrow \Gamma_{t}^{(i)}(\mathcal{W}(i))||_{H^{1}_{L}(\Omega_{\alpha})}^{2} \leq C \left( 1 + \log \left( \frac{\eta}{h} \right) \right)^{2} \left( 1 + \frac{1}{\eta \mu_{L+1}} \right) \left( \delta_{j} \right)^{2} ||\mathcal{H}_{\mu}(i)\mathcal{H}_{E_{ij}} \rightarrow \Gamma_{t}^{(i)}(\mathcal{W}(i))||_{H^{1}_{L}(\Omega_{\alpha})}^{2}
$$

Application of Lemma 6.6 yields

$$
(\delta_{j})^{2} ||\mathcal{H}_{\mu}(i)\mathcal{H}_{E_{ij}} \rightarrow \Gamma_{t}^{(i)}(\mathcal{W}(i))||_{H^{1}_{L}(\Omega_{\alpha})}^{2} \leq C \left( 1 + \log \left( \frac{\eta}{h} \right) \right)^{2} \left( 1 + \frac{1}{\eta \mu_{L+1}} \right) \left( \delta_{j} \right)^{2} ||\mathcal{H}_{\mu}(i)\mathcal{H}_{E_{ij}} \rightarrow \Gamma_{t}^{(i)}(\mathcal{W}(i))||_{H^{1}_{L}(\Omega_{\alpha})}^{2}
$$

and we get the estimate

$$
||\mathcal{H}_{\mu}(i)\mathcal{H}_{E_{ij}} \rightarrow \Gamma_{t}^{(i)}(\mathcal{W}(i))||_{H^{1}_{L}(\Omega_{\alpha})}^{2} \leq C \left( 1 + \log \left( \frac{\eta}{h} \right) \right)^{2} \left( 1 + \frac{1}{\eta \mu_{L+1}} \right) \left( ||\mathcal{W}(i)||_{H^{1}_{L}(\Omega_{\alpha})}^{2} + ||\mathcal{W}(i)||_{H^{1}_{L}(\Omega_{\alpha})}^{2} \right)
$$

with $C = C \max \left\{ 1, \frac{1}{\mu_{L+1}} \right\}$.

**7. Numerical results using Eigenvalue Problem 1.** In this section, we present numerical results for our new algorithm applied to scalar diffusion and linear elasticity problems using Eigenvalue Problem 1; see (5.1). We solve the eigenvalue problems in all experiments with the MATLAB function 'eig' which itself uses LAPACK. We always start with a FETI-DP coarse space using only vertices, often denoted as Algorithm A [45]. In this section, we have set the tolerance $\tau_{\mu} = 1$; see (5.8).

**7.1. Diffusion equation.** We present numerical results for different coefficient distributions. We decompose the unit square into square subdomains and consider a coefficient distribution with a different number of channels, cutting through subdomain edges; see Figure 7.1 and 7.2. All experiments for the diffusion equation are
Fig. 7.1. Domain decomposition in nine subdomains (left), coefficient distribution: One channel in black for each subdomain with high coefficient and $\rho = 1$ in the white area (right).

carried out with homogeneous Dirichlet boundary conditions on $\partial \Omega$ and a constant right hand side $f = 1/10$. In case of one channel for each subdomain, we have a quasi-monotone coefficient; cf. [39]. In this case, which is illustrated in Figure 7.1, on each interior edge the eigenvector to the eigenvalue zero is added to the coarse space. On interior edges which do not intersect a channel with a high coefficient, the resulting constraint is a standard edge average. On interior edges intersected by a channel, the constraint is a weighted edge average, cf., also [19], up to a multiplicative constant. See also [35] for an analysis of the scalar elliptic case. This results in 8 additional constraints; see Table 7.1. In case of two disjunct channels with a high coefficient on each edge, which intersect these channels, a maximum of two eigenvectors and in case of three channels, a maximum of three eigenvectors for each edge intersecting the channels are added to the coarse space; see Figure 7.2 and Table 7.1. In Table 7.2, we see that the condition number of the algorithm with an enriched coarse space stays bounded if we change the contrast $\rho_2 \in \{1e03, 1e04, 1e05, 1e06\}$. Moreover, the number of adaptive constraints stays bounded. From Table 7.3 it can be seen that the number of adaptive constraints grows roughly in proportion to the number of subdomains and channels. Note that the adaptive algorithm chooses only constraints on subdomains, where the Dirichlet boundary does not intersect the inclusions. In case of three channels on subdomains with Dirichlet boundary conditions that do not intersect the channels six constraints, and on all inner subdomains, eight constraints are chosen. Linearly dependent constraints are detected using a singular value decomposition with a tolerance of $1e-6$ and removed. The additional constraints are implemented using balancing, i.e., $M_{BP}$. They could also be implemented using a transformation of basis. Our stopping criterion is the reduction of the preconditioned residual to $(1e-10) ||z_0||_2 + 1e - 16$, where $z_0$ is the preconditioned starting residual.

7.2. Elasticity with discontinuous coefficients. We test our algorithm for linear elasticity problems with certain distributions of varying coefficients inside subdomains. We impose homogeneous Dirichlet boundary conditions on the lower part of the boundary where $y = 0$ and a constant volume force $f = (1/10, 1/10)^T$. First, we run a set of experiments for the example above with three channels and with jumps in the Young modulus $E$. We use the balancing preconditioner $M_{BP}$ or the projector preconditioner $M_{PP}$ in our examples.

In our current strategy we need to solve eigenvalue problems on all edges, i.e., also on edges where no heterogeneity appears and thus no additional constraints are nec-
Fig. 7.2. Coefficient distribution: Two channels for each subdomain (left), three channels for each subdomain (right). The black channels correspond to a high coefficient, in the white area the coefficient is $\rho_1 = 1$.

One, two, and three channels for each subdomain; see Figure 7.1 (right) and Figure 7.2. Adaptive method using eigenvalue problem (5.1). We have $\rho_1 = 1e06$ in the channel, and $\rho_2 = 1$ elsewhere. The number of additional constraints is clearly determined by the structure of the heterogeneity and independent of the mesh size. $1/H = 3$. $\tau_{\mu} = 1$.

<table>
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<th>Algorithm A</th>
<th>Adaptive Method</th>
<th># Adaptive</th>
<th># Dual</th>
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<td># its</td>
<td>cond</td>
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Table 7.1

Three channels for each subdomain; see Figure 7.2 (right). Adaptive method using eigenvalue problem (5.1). We have $\rho_2 = 1e06$ in the channels, and $\rho_1 = 1$ elsewhere. $H/h = 28$. The number of additional constraints is bounded for increasing contrast $\rho_2/\rho_1$. $1/H = 3$. $\tau_{\mu} = 1$.

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Algorithm A

Adaptive Method

# Adaptive

# Dual

\( \frac{1}{H} \)

cond

# its

# Adaptive

constraints \( L \)

# Dual

Variables

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Table 7.3

Three channels for each subdomain; see Figure 7.2 (right). Increasing number of subdomains and channels. We have \( \rho_2 = 1\text{e}06 \) in the channel, and \( \rho_1 = 1 \) elsewhere. \( H/h = 28 \). \( \tau_\mu = 1 \).

<table>
<thead>
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<th>( E_2/E_1 )</th>
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<th>( M_{BP} )</th>
<th>( M_{PP} )</th>
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Table 7.4

Linear elasticity, three channels for each subdomain; see Figure 7.2 (right) with coefficient \( E_2 = 10^6 \), outside the channels \( E_1 = 1 \), \( \tau_\mu = 1 \). The number of additional constraints is determined by the structure of the heterogeneity, independent of the mesh size and bounded for increasing contrast \( E_2/E_1 \).

8. Elasticity with discontinuous coefficients and using Eigenvalue Problem 1 and Eigenvalue Problem 2. In our examples considered in Section 7 we have only used Eigenvalue Problem 1, see (5.1), to enhance our coarse space. We will now consider coefficient distributions where this is not sufficient anymore and where the additional Eigenvalue Problem 2, see (6.6), is necessary for the construction of the coarse space. We test a coefficient distribution with jumps across and along the interfaces; see Figures 7.3 and 7.4. In these cases it will be necessary to bound the extensions independently of the jumps in the coefficient; see also proof of Theorem 6.8. We use our strategy from Section 6.2. The numerical results are displayed in the Tables 8.1 and 8.2 for different tolerances for the eigenvalues \( \nu_k^{(i)} \) in Section 6.2. With a tolerance of “−” we denote the case where no additional constraints are chosen based...
Fig. 7.3. Test problem with a coefficient distribution which is unsymmetric with respect to the edges for a 3x3 decomposition; see Tab. 8.2. Young’s modulus $10^6$ (black) and 1 (white).

Fig. 7.4. Test problem with a coefficient distribution which is unsymmetric with respect to the edges for a 3x3 decomposition; see Tab. 8.2. Young’s modulus $10^6$ (black) and 1 (white).

Fig. 7.5. First (left) and second (right) component of the starting residuum of PCG in the test problem in Figure 7.2 for linear elasticity with vertex constraints and $H/h = 28$. The oscillations of the residuum on the edges appear on the difficult edges and indicate additional constraints needed, e.g., from our eigenvalue problems.

on the respective eigenvalue problem, i.e., the tolerance is set to $-\infty$.

Finally, as in the previous section, we use a coefficient distribution obtained from a steel microsection pattern with 150 x 150 pixels; see Figure 1.1. We discretize the problem with $H/h = 50$ and $1/H = 3$; see Table 8.3 for the numerical results, which show the effectiveness of the adaptive algorithm now using both eigenvalue problems (5.1) and (6.6).

REFERENCES

Table 8.1
Results for linear elasticity using the coefficient distribution for the heterogenous problem from the image in Figure 7.3 with a Young’s modulus of $10^6$ (black) and 1 (white) respectively. Decomposition into $3 \times 3$ subdomains. The first column refers to the tolerance for Eigenvalue Problem 1, see (5.1), the second column refers to the tolerance for Eigenvalue Problem 2, see (6.4). With a tolerance of “−” no additional constraints are chosen. L is the number of constraints from (5.1), K is the number of constraints from (6.4).

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Table 8.2
Results for linear elasticity using the coefficient distribution for the heterogenous problem from the image in Figure 7.4 with a Young’s modulus of $10^6$ (black) and 1 (white) respectively. Decomposition into $3 \times 3$ subdomains. The first column refers to the tolerance for Eigenvalue Problem 1, see (5.1), the second column refers to the tolerance for Eigenvalue Problem 2, see (6.4). With a tolerance of “−” no additional constraints are chosen. L is the number of constraints from (5.1), K is the number of constraints from (6.4).

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<th>$#$ Dual</th>
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Adaptive Method

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& \text{left} & \text{H/h} & \mu & L + K & \mu & \nu & \text{Variables} \\
\hline
1 & \tau_\mu & 1e-01 & 50 & 6.75e04 & > 250 & 89 & 1236 \\
1 & \tau_\mu & 1e-02 & 50 & 2.23e04 & 15 & 243 & 1236 \\
1 & \tau_\mu & 1e-02 & 50 & 6.52e04 & 19 & 224 & 1236 \\
1 & \tau_\mu & 1e-03 & 50 & 8.53e08 & 23 & 220 & 1236 \\
\hline
1 & \tau_\nu & \tau_\nu & 50 & 6.46e04 & > 250 & 0 & 1236 \\
\hline
\end{array}
\]

Table 8.3

Results for linear elasticity using the coefficient distribution for the heterogeneous problem from the gray scale image in Figure 1.1. We have set the coefficient \(E_1 = 1\) for white and \(E_2 = 1e06\) for black. An interpolated value is used for the different shades of gray. As above, \(\tau_\mu\) is the tolerance for Eigenvalue Problem 1, see (5.1), \(\tau_\nu\) is the tolerance for Eigenvalue Problem 2, see (6.4).


series.


