M. Sonntag, H.-M. Teichert

Competition graphs of products of digraphs
Martin Sonntag and Hanns-Martin Teichert

Competition graphs of products of digraphs

TU Bergakademie Freiberg
Fakultät für Mathematik und Informatik
Prüferstraße 9
09596 FREIBERG
http://www.mathe.tu-freiberg.de
Abstract. If $D = (V, A)$ is a digraph, its competition graph (with loops) $CG^l(D)$ has the vertex set $V$ and $\{u, v\} \subseteq V$ is an edge of $CG^l(D)$ iff there is a vertex $w \in V$, such that $(u, w), (v, w) \in A$. In $CG^l(D)$, loops $\{v\}$ are allowed only if $v$ is the only predecessor of a certain vertex $w \in V$.

For several products $D_1 \circ D_2$ of digraphs $D_1$ and $D_2$, we investigate the relations between the competition graphs of the factors $D_1$, $D_2$ and the competition graph of their product $D_1 \circ D_2$.

Keywords. Competition graph, Product of digraphs

1. Introduction and definitions

All graphs $G = (V(G), E(G))$, hypergraphs $H = (V(H), E(H))$ and digraphs $D = (V(D), A(D))$ considered here may have isolated vertices but no multiple edges and arcs, respectively. Moreover, in digraphs loops are forbidden. In standard terminology, concerning digraphs we follow Bang–Jensen and Gutin [1]. With $d^-_D(v), d^+_D(v), N^-_D(v)$ and $N^+_D(v)$ we denote the in-degree, out-degree, in-neighbourhood and out-neighbourhood of a vertex $v$ in a digraph $D$, respectively.

In 1968 Cohen [2] introduced the competition graph (without loops) $CG(D)$ associated with a digraph $D = (V, A)$ representing a food web of an ecosystem. $CG(D) = (V, E)$ is the graph with the same vertex set as $D$ (corresponding to the species) and

$$E = \{\{u, v\} \mid u \neq v \land \exists w \in V : (u, w) \in A \land (v, w) \in A\},$$

i.e. $\{u, v\} \in E$ iff $u$ and $v$ compete for a common prey $w \in V$.

Surveys of the large literature around competition graphs can be found in Roberts [6], Kim [4] and Lundgren [5].

In [7] it is shown that in many cases competition hypergraphs yield a better description of the predation relations among the species in $D = (V, A)$ than competition graphs. If $D = (V, A)$ is a digraph its competition hypergraph $CH(D) = (V, E)$ has the vertex set $V$ and $e \subseteq V$ is an edge of $CH(D)$ iff $|e| \geq 2$ and there is a vertex $w \in V$, such that $e = \{v \in V \mid (v, w) \in A\}$. In this case we say $w \in V = V(D)$ corresponds to $e \in E$ and vice versa.

In our paper [7] we dealt with competition hypergraphs without loops, that way we followed the most usual definition of competition graphs. In the case of digraphs $D$ possessing vertices with only one predecessor, a competition hypergraph with loops contains
a more detailed information on $D$ (cf. [8]). For that reason, we also include competition hypergraphs (as well as competition graphs) with loops in our investigations and modify the notions given above.

If $D = (V, A)$ is a digraph, its $l$-competition hypergraph (competition hypergraph with loops) $\mathcal{CH}_l(D) = (V, E^l)$ has the vertex set $V$ and $e \subseteq V$ is an edge of $\mathcal{CH}(D)$ iff $|e| \neq 0$ and there is a vertex $w \in V$, such that $e = \{v \in V \mid (v, w) \in A\}$.

Analogously, the $l$-competition graph (competition graph with loops) $CG^l(D) = (V, E^l)$ has the vertex set $V$ and $E^l = E(CG(D)) \cup \{\{v\} \mid v \in V \land \exists w \in V : N^-_D(w) = \{v\}\}$.

For the sake of brevity, in the following we often use the term competition graph (sometimes in connection with the notation $CG^l(D)$) for the competition graph $CG(D)$ as well as for the $l$-competition graph $CG^l(D)$ (analogously for competition hypergraphs).

Analogically with [8], for five products $D_1 \circ D_2$ (Cartesian product $D_1 \times D_2$, Cartesian sum $D_1 + D_2$, normal product $D_1 \ast D_2$, lexicographic product $D_1 \cdot D_2$ and disjunction $D_1 \lor D_2$) of digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ we investigate the construction of the competition graph $CG^l(D_1 \circ D_2) = (V, E^l)$ from $CG^l(D_1) = (V_1, E^{l_1})$, $CG^l(D_2) = (V_2, E^{l_2})$ and vice versa.

The products considered here have always the vertex set $V := V_1 \times V_2$; using the notation $\tilde{A} := \{(a, b), (a', b')\} \mid a, a' \in V_1 \land b, b' \in V_2\}$ their arc sets $A_o := A(D_1 \circ D_2)$ are defined as follows:

$A_x := \{((a, b), (a', b')) \in \tilde{A} \mid (a, a') \in A_1 \land (b, b') \in A_2\}$,

$A_+ := \{((a, b), (a', b')) \in \tilde{A} \mid ((a, a') \in A_1 \land b = b') \lor (a = a' \land (b, b') \in A_2)\}$,

$A_\lor := A(D_1 \times D_2) \cup A(D_1 + D_2)$,

$A_\lor := \{((a, b), (a', b')) \in \tilde{A} \mid (a, a') \in A_1 \lor (a = a' \land (b, b') \in A_2)\}$,

$A_\land := \{((a, b), (a', b')) \in \tilde{A} \mid (a, a') \in A_1 \land (b, b') \in A_2\}$.

It follows immediately that $A_+ \subset A_\lor \subset A \subset A_\land \subset A_x$. Except the lexicographic product all these products are commutative in the sense that $D_1 \circ D_2 \simeq D_2 \circ D_1$, where $o \in \{x, +, \ast, \lor\}$.

Usually we label the vertices of $V_1$ and $V_2$ by $1, 2, \ldots, r_1$ and by $1, 2, \ldots, r_2$, respectively, and arrange the vertices of $V_1 \times V_2$ according to the places of an $(r_1, r_2)$-matrix. Then, for each $o \in \{+, \ast, \lor\}$, the subdigraph of $D_1 \circ D_2$ generated by the vertices of a column $S_j := \{(i, j) \mid i \in \{1, \ldots, r_1\} \land j \in \{1, \ldots, r_2\}\}$ and a row $Z_i := \{(i, j) \mid j \in \{1, \ldots, r_2\}\}$ of this matrix scheme is isomorphic to $D_1$ and $D_2$, respectively.

The factor decomposition of product graphs is an interesting question (cf. Imrich and Klavzar [3]). Related to this problem the question arises, whether or not $CG^l(D_1 \circ D_2)$ can be obtained from $CG^l(D_1)$ and $CG^l(D_2)$ and vice versa. For competition hypergraphs this problem had been investigated in [8].

Since competition hypergraphs include more information than competition graphs, especially in the case of the reconstruction of $\mathcal{CH}^l(D_1)$ and $\mathcal{CH}^l(D_2)$ from $\mathcal{CH}^l(D_1 \circ D_2)$ we achieved better results (cf. [8]) than for competition graphs (see Section 3 in the present paper). In this context, it is interesting that under certain conditions $D_1 \circ D_2$ and even $D_1$ and $D_2$ can be reconstructed from $\mathcal{CH}^l(D_1 \circ D_2)$ (cf. [8], Corollaries 1-3).

Contrastingly, the results for the construction of $CG^l(D_1 \circ D_2)$ from $CG^l(D_1)$ and $CG^l(D_2)$ (see Section 2) and for the construction of $\mathcal{CH}^l(D_1 \circ D_2)$ from $\mathcal{CH}^l(D_1)$ and $\mathcal{CH}^l(D_2)$ (cf. [8]) are comparable.
2. Determination of $CG^l(D_1 \circ D_2)$ from $CG^l(D_1)$ and $CG^l(D_2)$

In the following let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be digraphs. By $N^+_1(v)$, $N^-_2(v)$ and $N^+_o(v)$ we denote the set of all predecessors of a vertex $v$ in $D_1$, $D_2$ and $D_1 \circ D_2$, respectively, where $\circ \in \{\times, +, *, \cdot, \cup\}$.

**Theorem 1.** The $l$-competition graph $CG^l(D_1 \times D_2) = (V, E'_x)$ of the Cartesian product can be obtained from the $l$-competition graphs $CG^l(D_1) = (V_1, E'_1)$ and $CG^l(D_2) = (V_2, E'_2)$ of $D_1$ and $D_2$:

$$E'_x = \{\{(a, b), (a', b')\} | \exists e_1 \in E'_1 \exists e_2 \in E'_2 : \{a, a'\} \subseteq e_1 \wedge \{b, b'\} \subseteq e_2 \wedge$$

$$(a = a' \wedge b = b' \implies e_1 = \{a\} \wedge e_2 = \{b\})\}. \quad (1)$$

**Proof.** The definition of $A_x$ implies

$$E'_x = \{\{(a, b), (a', b')\} | \exists x \in V_1 \exists y \in V_2 : (a, x), (a', x) \in A_1 \wedge (b, y), (b', y) \in A_2 \wedge$$

$$((a, b) = (a', b') \implies N^+_1(x) = \{a\} \wedge N^-_2(y) = \{b\})\}$$

$$= \{\{(a, b), (a', b')\} | \exists e_1 \in E'_1 \exists e_2 \in E'_2 : \{a, a'\} \subseteq e_1 \wedge \{b, b'\} \subseteq e_2 \wedge$$

$$(a = a' \wedge b = b' \implies e_1 = \{a\} \wedge e_2 = \{b\})\}. \square$$

Clearly, $E_x$ results from $E'_x$ by deleting all loops in $E'_x$.

**Remark 1.** In general, $CG(D_1 \times D_2) = (V, E_x)$ – and therefore $CG^l(D_1 \times D_2) = (V, E'_x)$ – cannot be obtained from $CG(D_1) = (V_1, E_1)$ and $CG(D_2) = (V_2, E_2)$.

**Proof.** Consider $D_1 = (V_1 = \{a, x\}, A_1 = \{(a, x)\})$, $D'_1 = (V_1, A'_1 = \emptyset)$ and $D_2 = (V_2 = \{b, b', y\}, A_2 = \{(y, b'), (b', y)\})$.

On the one hand, $E(CG(D_1 \times D_2)) = \emptyset = E(CG(D'_1 \times D_2))$, but on the other hand $E(CG(D_1)) = \emptyset = E(CG(D'_1))$. \square

**Remark 2.** If both $D_1$ and $D_2$ contain at least 2 vertices, then $CG^l(D_1 \vee D_2) = CG(D_1 \vee D_2)$, i.e. $CG^l(D_1 \vee D_2)$ contains no loops.

**Proof.** Assume, $\{(a, b)\} \in E'_v$ is a loop. Then there is a vertex $(x, y) \in V_1 \times V_2$ with $N^-_{((x, y))} = \{(a, b)\}$. Consequently, $(a, x) \in A_1$ or $(b, y) \in A_2$.

This implies $\{(a, b') | b' \in V_2\} \subseteq N^-_{(x, y)}$ or $\{(a', b) | a' \in V_1\} \subseteq N^-_{(x, y)}$. Both situations contradict $|N^-_{(x, y)}| = 1$. \square

**Theorem 2.** The $l$-competition graph $CG^l(D_1 \vee D_2) = (V, E'_v)$ of the disjunction can be obtained from the $l$-competition graphs $CG^l(D_1) = (V_1, E'_1)$ and $CG^l(D_2) = (V_2, E'_2)$ of $D_1$ and $D_2$.

**Proof.** From the definition of $A_v$ it follows $E'_v = \emptyset$ if and only if $E'_1 = E'_2 = \emptyset$.

In case of $E'_1 \neq \emptyset \wedge E'_2 \neq \emptyset$ we have

$$E'_v = \{\{(a, b), (a', b')\} | (a, b) \neq (a', b') \wedge \exists x \in V_1 \exists y \in V_2 :$$

$$(a, x) \in A_1 \vee (b, y) \in A_2) \wedge ((a', x) \in A_1 \vee (b', y) \in A_2)\}$$

$$= \{\{(a, b), (a', b')\} | (a, b) \neq (a', b') \wedge \exists e_1 \in E'_1 \exists e_2 \in E'_2 :$$

$$\{a, a'\} \subseteq e_1 \vee \{b, b'\} \subseteq e_2 \wedge (a \in e_1 \wedge b \in e_2)$$

$$\vee (a' \in e_1 \wedge b' \in e_2)\}.$$ 

If exactly one of the sets $E'_1, E'_2$ is empty, then

$$E'_v = \{\{(a, b), (a', b')\} | (a, b) \neq (a', b') \wedge \exists e_1 \in E'_1 : \{a, a'\} \subseteq e_1 \vee \exists e_2 \in E'_2 : \{b, b'\} \subseteq e_2\}. \square$$
Note that in the corresponding result for competition hypergraphs (cf. [8], Theorem 2) some little additional supposition is needed.

Considering digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ with $|V_1|, |V_2| \geq 2$, $|A_1| = 1$ and $A_2 = \emptyset$ we obtain the following Remark.

**Remark 3.** In general, $CG(D_1 \lor D_2) = (V, E_\lor)$ cannot be obtained from $CG(D_1) = (V_1, E_1)$ and $CG(D_2) = (V_2, E_2)$.

**Proposition 1.** In general, $CG(D_1 \circ D_2) = (V, E_\circ)$ and therefore $CG^l(D_1 \circ D_2) – cannot be obtained from $CG^l(D_1)$ and $CG^l(D_2)$, for $\circ \in \{+, *, \cdot\}$.

**Proof.** For illustration, we use figures of $CG(D_1 \circ D_2)$. In these figures several large cliques (each of them induced by the in-neighbourhood $N^-(v)$ of a vertex $v$ of $D_1 \circ D_2$) will occur. Some of these cliques contain many edges what could be confusing in the drawings. Therefore we represent such cliques (i.e. cliques of cardinality greater than 2) as closed curves around the vertices of $N^-(v)$, i.e. as a kind of hyperedges in the competition hypergraph $CH(D_1 \circ D_2)$. Of course, if $N^-(v') \subseteq N^-(v)$, it would be sufficient to draw the clique induced by the larger in-neighbourhood $N^-(v)$. But for a better traceability of the structure of $CG(D_1 \circ D_2)$ we decided to draw all "hyperedges" representing such cliques.

We make use of an example from our paper [8].

**Example 1.** Consider the digraphs $D_1 = (V_1, A_1), D'_1 = (V_1, A'_1)$ and $D_2 = (V_2, A_2)$ with $V_1 = \{1, 2, 3, 4\}, V_2 = \{1, 2, 3\}, A_1 = \{(1, 2), (3, 2), (4, 3)\}, A'_1 = \{(1, 4), (3, 4), (4, 2)\}$ and $A_2 = \{(1, 3), (2, 3)\}$, respectively (cf. Fig. 1).

Then $E(CG^l(D_1)) = \{\{1, 3\}, \{4\}\} = E(CG^l(D'_1))$.
On the other hand, 

\[-CG(D_1 + D_2) \neq CG(D'_1 + D_2),\] since the vertices (4,1) and (1,3) are adjacent in 
\[CG(D'_1 + D_2)\] but non-adjacent in \[CG(D_1 + D_2)\] (cf. Fig. 2).

\[D_1 + D_2:\]
\[
\begin{array}{ccc}
1 & 2 & 3 \\
2 & & \\
3 & & \\
4 & & \\
\end{array}
\]

\[D'_1 + D_2:\]
\[
\begin{array}{ccc}
1 & 2 & 3 \\
2 & & \\
3 & & \\
4 & & \\
\end{array}
\]

\[CG(D_1 + D_2):\]
\[
\begin{array}{ccc}
1 & 2 & 3 \\
2 & & \\
3 & & \\
4 & & \\
\end{array}
\]

\[CG(D'_1 + D_2):\]
\[
\begin{array}{ccc}
1 & 2 & 3 \\
2 & & \\
3 & & \\
4 & & \\
\end{array}
\]

Fig. 2

\[CG(D_1 \ast D_2):\]
\[
\begin{array}{ccc}
1 & 2 & 3 \\
2 & & \\
3 & & \\
4 & & \\
\end{array}
\]

\[CG(D'_1 \ast D_2):\]
\[
\begin{array}{ccc}
1 & 2 & 3 \\
2 & & \\
3 & & \\
4 & & \\
\end{array}
\]

Fig. 3

\[-CG(D_1 \ast D_2) \neq CG(D'_1 \ast D_2),\] since the vertices (2,1) and (4,1) are adjacent in 
\[CG(D'_1 \ast D_2)\] but non-adjacent in \[CG(D_1 \ast D_2)\] (cf. Fig. 3).
Competition graphs of products of digraphs

\[ CG(D_1 \cdot D_2) \neq CG(D'_1 \cdot D_2), \] since the vertices (2, 1) and (4, 1) are adjacent in \( CG(D'_1 \cdot D_2) \) but non-adjacent in \( CG(D_1 \cdot D_2) \) (cf. Fig. 4).

Looking at Figures 3 and 4, replacing the “hyperedges” by cliques of ordinary edges (of cardinality 2) and identifying multiple edges we make a nice observation.

**Remark 4.** For the digraphs \( D_1, D'_1, D_2 \), we obtain \( CG(D_1 \ast D_2) = CG(D_1 \cdot D_2) \) and \( CG(D'_1 \ast D_2) = CG(D'_1 \cdot D_2) \).
3. Reconstruction of CG\(^{(l)}\)(D\(_1\)) and CG\(^{(l)}\)(D\(_2\)) from CG\(^{(l)}\)(D\(_1 \circ D_2\))

Whereas in Section 2 the results for constructing CG\(^{(l)}\)(D\(_1 \circ D_2\)) from CG\(^{(l)}\)(D\(_1\)) and CG\(^{(l)}\)(D\(_2\)) are very closely related to the corresponding results for competition hypergraphs (cf. [8]), in the present Section 3 we will find more significant differences between the graph and the hypergraph case. So it is worth mentioning that under certain conditions it is even possible to reconstruct the digraphs D\(_1\) and D\(_2\) themselves from CH\(^l\)(D\(_1 + D_2\)) or CH\(_i\)(D\(_1 \ast D_2\)). In general, being premised on competition graphs, this is impossible.

In the following, for a set e = \{(i\(_1\), j\(_1\)),..., (i\(_k\), j\(_k\))\} ⊆ V \(_1\) × V \(_2\) we define \(\pi_1(e) := \{i\(_1\),..., i\(_k\)\}\) and \(\pi_2(e) := \{j\(_1\),..., j\(_k\)\}\), respectively, i.e. \(\pi_i\) denotes the projection of the vertices of CG\(^{(l)}\)(D\(_1 \circ D_2\)) onto their \(i\)th component, for \(i \in \{1, 2\}\).

3.1. The Cartesian product D\(_1 \times D_2\).

First of all, if \(E^l_x = E(CG^l(D_1 \times D_2)) = \emptyset\) then \(A(D_1 \times D_2) = \emptyset\) and, therefore, \(A_1 = \emptyset\) or \(A_2 = \emptyset\). But considering only \(CG^l(D_1 \times D_2)\) (or even \(D_1 \times D_2\)) it is impossible to detect which of the arc sets \(A_1\) or \(A_2\) is empty. The same holds for \(E^l_\ell = E(CG^l(D_1)) = \emptyset\) and \(E^l_\ell = E(CG^l(D_2)) = \emptyset\), respectively, since \(A_1 = \emptyset\) if and only if \(E(CG^l(D_1)) = \emptyset\) (\(i \in \{1, 2\}\)).

**Theorem 3.** For the Cartesian product \(D_1 \times D_2\) it holds:

(a) If \(E_x \neq \emptyset\), then the competition graphs \(CG(D_1)\) and \(CG(D_2)\) can be reconstructed from \(CG(D_1 \times D_2)\).

(b) In general, the \(l\)-competition graphs \(CG^l(D_1)\) and \(CG^l(D_2)\) cannot be reconstructed from \(CG^l(D_1 \times D_2)\).

(c) If \(CG^l(D_1 \times D_2)\) contains a loop, then the \(l\)-competition graphs \(CG^l(D_1)\) and \(CG^l(D_2)\) can be reconstructed from \(CG^l(D_1 \times D_2)\).

**Proof.**

(a) Let \(e \in E_x\) and \((a, b) \in e\). Then there exists a vertex \((x, y) \in V_1 \times V_2\) with \((a, x) \in A_1\) and \((b, y) \in A_2\).

Suppose, \(\{a', a''\} \in E_1\) and \(x' \in V_1\) such that \((a', x'), (a'', x') \in A_1\). Clearly, \((a', b), (a'', b) \in N^\_x((x', y))\), \(\{a', b\} \in E_x\) and \(\{a', a''\} = \pi_1(\{(a', b), (a'', b)\})\). So it follows

\[E_1 = \{\pi_1(e) \mid e \in E_x \land |\pi_1(e)| = 2\}\]

and analogously,

\[E_2 = \{\pi_2(e) \mid e \in E_x \land |\pi_2(e)| = 2\}\]

(b) **Example 2.** Let \(D_1 = (V_1 = \{1, 2, 3, 4\}, A_1 = \{(1, 2), (3, 2), (3, 4)\})\), \(D_1' = (V_1, A_1' = A_1 \cup \{(1, 4)\})\) and \(D_2 = (V_2 = \{1, 2, 3\}, A_2 = \{(1, 2), (3, 2)\})\).

Then \(E(CG^l(D_1 \times D_2)) = \{\{(1, 1), (1, 3)\}, \{(1, 1), (1, 3)\}, \{(1, 1), (3, 1)\}, \{(1, 3), (3, 1)\}, \{(1, 1), (3, 3)\}, \{(1, 3), (3, 3)\}\}

but \(E(CG^l(D_1)) = \{\{(1, 3), \{(1, 3), (3, 1)\}\} \neq \{\{(1, 3), \{(1, 3), (3, 3)\}\} = E(CG^l(D_1))\). \(\square\)

(c) It suffices to show that all loops in \(CG^l(D_1)\) and \(CG^l(D_2)\) can be reconstructed. Let \(\{(a, b) \in E_x\) be a loop. Consequently, there is a vertex \((x, y) \in V_1 \times V_2\) such that \(N^{-1}_x((x, y)) = \{(a, b)\}\) and we obtain the loops \(N^\_x\((x) = \{(a)\}\) and \(N^\_y\((y) = \{b\}\) in \(E^l_1\) and \(E^l_2\), respectively.

Now let \(\{a'\} \in E^l_x\) be a loop in \(CG^l(D_1)\) and \(x' \in V_1\) with \(N^{-1}_x((x') = \{a'\})\). Clearly, \(N^\_x\((x', y) = \{(a', b)\} \in E^l_x\) is a loop and \(\{a'\} = \pi_1(\{(a', b)\})\). Analogously, every loop \(\{b'\} \in E^l_2\) can be obtained as the projection \(\pi_2(e)\) of a certain loop \(e \in E^l_x\). \(\square\)
Note that there is a loop in $CG^l(D_1 \times D_2)$ if and only if both $CG^l(D_1)$ and $CG^l(D_2)$ contain a loop, which is equivalent to the fact that in $D_1$ as well as in $D_2$ there is at least one vertex with in-degree 1.

### 3.2. The Cartesian sum $D_1 + D_2$.

Based on the definition of $D_1 + D_2 = (V_1 \times V_2, A_+)$ we get for the edge set of $CG(D_1 + D_2)$

$$E_+ = \{(a, b), (a', b')\} | \exists (x, y) \in V_1 \times V_2 : (a, b) \not\in (a', b') \land \{(a, b), (a', b')\} \subseteq N_+((x, y))\}
\begin{align*}
&= \{(a, b), (a', b')\} | \exists (x, y) \in V_1 \times V_2 : (a, b) \not\in (a', b') \land \\
&\quad \quad ((a = x \land (b, y) \in A_2) \lor ((a, x) \in A_1 \land b = y)) \land \\
&\quad \quad ((a' = x \land (b', y) \in A_2) \lor ((a', x) \in A_1 \land b' = y))\}
\end{align*}

$$

$$
= \{(a, b), (a', b')\} | \exists (x, y) \in V_1 \times V_2 : (a = a' = x \land b \not\in b' \land \{(b, y), (b', y)\} \subseteq A_2) \lor \\
(a = x \land (a, x) \in A_1 \land (b, y) \in A_2 \land b = y) \lor \\
(a' = x \land (a', x) \in A_1 \land (b', y) \in A_2 \land b = y)\}
\begin{align*}
&= \{(a, b), (a', b')\} | (a = a' \land \{b, b'\} \in E_2) \lor ((a', a) \in A_1 \land (b, b') \in A_2) \lor \\
&\quad \quad ((a, a') \in A_1 \land (b', b) \in A_2) \lor \{a, a'\} \in E_1 \land b = b')\}.
\end{align*}

#### Theorem 4. For the Cartesian sum $D_1 + D_2$ it holds:

(a) The competition graphs $CG(D_1)$ and $CG(D_2)$ can be reconstructed from $CG(D_1 + D_2)$.

(b) In general, the $l$-competition graphs $CG^l(D_1)$ and $CG^l(D_2)$ cannot be reconstructed from $CG^l(D_1 + D_2)$.

(c) If $CG^l(D_1 + D_2)$ contains an isolated vertex, then the $l$-competition graphs $CG^l(D_1)$ and $CG^l(D_2)$ can be reconstructed from $CG^l(D_1 + D_2)$.

**Proof.**

(a) From the above expression for $E_+$ we obtain

$$E_1 = \{\pi_1(e) | e \in E_+ \land |\pi_1(e)| = 2 \land |\pi_2(e)| = 1\} \quad \text{and, analogously,}
$$

$$E_2 = \{\pi_2(e) | e \in E_+ \land |\pi_2(e)| = 2 \land |\pi_1(e)| = 1\}.
$$

(b) We show (b) by an counterexample.

**Example 3.** Let $D_1 = (V_1 = \{1, 2, 3\}, A_1 = \{(1, 2), (1, 3), (2, 3)\}), \ D'_1 = (V_1, A'_1 = \{(2, 1), (1, 3), (2, 3)\})$ and $D_2 = (V_2 = \{1, 2, 3\}, A_2 = \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\})$. Clearly, $E(CG^l(D_1)) = \{(1, 2), \{1\}\} \not\subseteq \{(1, 2), \{2\}\} = E(CG^l(D'_1))$.

Let's consider $E(CG^l(D_1 + D_2)) = \bigcup \{EN_+(i, j) | (i, j) \in V_1 \times V_2\}$, where $EN_+(i, j)$ includes all edges in $CG^l(D_1 + D_2)$ generated by the predecessors of the vertex $(i, j)$ in $D_1 + D_2$. Denoting the corresponding edge sets in $E(CG^l(D'_1 + D_2))$ by $EN_-(i, j)$, we observe the following:

- In $E(CG^l(D_1 + D_2))$ we have the sets

  $EN_+(1, 1) = \{(1, 2), (1, 3)\}$,

  $EN_+(1, 2) = \{(1, 1), (1, 3)\}$,

  $EN_+(1, 3) = \{(1, 1), (1, 2)\}$,

  $EN_+(2, 1) = \{(1, 1), (2, 2)\}$,

  $EN_+(2, 2) = \{(1, 1), (2, 2)\}$,

  $EN_+(2, 3) = \{(1, 2), (2, 3)\}$,

  $EN_+(3, 1) = \{(1, 2), (2, 3)\}$,

  $EN_+(3, 2) = \{(1, 2), (2, 3)\}$,

  $EN_+(3, 3) = \{(1, 2), (2, 3)\}$.
The deletion of the arc $(1, 2) \in A_1$ induces that in $D_1 + D_2$ the arcs $((1, 1), (2, 1))$, $((1, 2), (2, 2))$ and $((1, 3), (2, 3))$ vanish and, consequently, the edge set

$\tilde{E} = \{(1, 1), (2, 2), (1, 3), (2, 3), (1, 2), (2, 1), (1, 3), (2, 1), (1, 3), (2, 2)\}$

disappears in the $l$-competition graph.

On the other hand, if we add the arc $(2, 1)$ to $D_1 \setminus \{(1, 2)\}$ we obtain $D'_1$ and the same set $\tilde{E}$ of edges emerges in the $l$-competition graph $CG^l(D'_1 + D_2)$. In detail, for $E(CG^l(D'_1 + D_2))$ we get

$EN'_+(1, 1) = \{(1, 2), (1, 3), (1, 2), (2, 1), (1, 3), (2, 1)\}$

$EN'_+(1, 2) = \{(1, 1), (1, 3), (1, 1), (2, 2), (1, 3), (2, 2)\}$

$EN'_+(1, 3) = \{(1, 1), (1, 2), (1, 1), (2, 3), (1, 2), (2, 3)\}$

$EN'_+(2, 1) = \{(2, 2), (2, 2), (2, 2), (2, 3), (2, 2), (2, 3)\}$

$EN'_+(2, 2) = \{(2, 1), (2, 3)\}$

$EN'_+(2, 3) = \{(2, 1), (2, 2)\}$

Therefore, $E(CG^l(D_1 + D_2)) = E(CG^l(D'_1 + D_2))$ in spite of $E(CG^l(D_1) \neq E(CG^l(D'_1))$.

(c) First of all, it is clear that in case of $|V_1| = 1$ or $|V_2| = 1$ the reconstruction is trivial; the same holds for $|E'_1| = 0$. So let $|V_1|, |V_2| \geq 2$ and $|E'_1| \geq 1$. Since every arc $(a, a') \in A_1$ induces $|V_2|$ arcs in $D_1 + D_2$ (analogously for arcs $(b, b') \in A_2$ and $|V_1|$), we have always $|E'_1| \geq 2$.

An isolated vertex $(a, b)$ in $CG^l(D_1 + D_2)$ has the outdegree $d^+_l((a, b)) = 0$ in $D_1 + D_2$. Therefore, $a \in V_1$ and $b \in V_2$ have the outdegree $d^+_l(a) = 0$ and $d^+_l(b) = 0$, respectively. Consequently,

$E_1^l = \{\pi_1(e) \mid e \in E'_1 \land \pi_2(e) = \{a\}\}$

$E_2^l = \{\pi_2(e) \mid e \in E'_1 \land \pi_1(e) = \{a\}\}$

Note that the existence of a loop in $CG^l(D_1 + D_2)$ (as in Theorem 3(c) for the Cartesian product) is not sufficient for the reconstructibility of $CG^l(D_1)$ and $CG^l(D_2)$. As an example consider the case that $E'_1 = \{(a, b) \mid (a, b) \in V_1 \times V_2\}$. Then one of the arc sets $A_1$ or $A_2$ has to be empty, but it is undecidable which of these sets is the empty one.
3.3. The normal product $D_1 \ast D_2$.

Let $\{\alpha, \beta\} = \{1, 2\}$, $D_1 = (V_1, A_1)$, $D_2 = (V_2, A_2)$, $V_1 = \{1, \ldots, r_1\}$, $V_2 = \{1, \ldots, r_2\}$, $k \in \{1, \ldots, r_3\}$ and $R_k^\beta = \begin{cases} Z_k & \text{if } \beta = 1 \\ S_k & \text{if } \beta = 2. \end{cases}$

**Theorem 5.** Let $\{\alpha, \beta\} = \{1, 2\}$. For the normal product $D_1 \ast D_2$ it holds:

(a) If there is no edge $\bar{e} \in E^\alpha_\ast$ with $|\pi_1(\bar{e})| = |\pi_2(\bar{e})| = 2$ and $E^\alpha_\ast \neq \{(a, b) \mid (a, b) \in V_1 \times V_2\}$, then $CG^\alpha(D_1)$ and $CG^\alpha(D_2)$ can be reconstructed from $CG^\alpha(D_1 \ast D_2)$.

(b) If there exists an edge $\bar{e} \in E^\alpha_\ast$ with $|\pi_1(\bar{e})| = |\pi_2(\bar{e})| = 2$ and $\exists j \in \{1, \ldots, r_3\} \forall e \in E^\beta_j: R_j^\beta \cap e \neq \emptyset \implies e \subseteq R_j^\beta, \quad (2)$

then $CG^\beta(D_a)$ can be reconstructed from $CG^\beta(D_1 \ast D_2)$.

**Proof.**

(a) If there is no edge $e \in E^\alpha_\ast$ with $|\pi_1(e)| = |\pi_2(e)| = 2$, then either $E^\alpha_\ast = \emptyset$ (i.e. $A_1 = A_2 = \emptyset$ and – equivalently – $E^\alpha_\ast = E^\beta_2 = \emptyset$) or $E^\alpha_\ast \neq \emptyset$ (i.e. only one of the sets $A_1$ and $A_2$ is empty).

So let $E^\alpha_\ast \neq \emptyset$. If $e \in E^\alpha_\ast$ with $|e| = 2 = |\pi_\alpha(e)|$, then $A_\alpha \neq \emptyset$, $A_\beta = \emptyset$ and, consequently, $E^\alpha_\alpha = \{\pi_\alpha(e) \mid e \in E^\alpha_\ast\}$ as well as $E^\beta_\beta = \emptyset$.

If $E^\alpha_\ast$ contains only loops, because of $E^\alpha_\ast \neq \{(a, b) \mid (a, b) \in V_1 \times V_2\}$ there is an isolated vertex $(x, y) \in V_1 \times V_2$ in $CG^\alpha(D_1 \ast D_2)$. Hence $x \in V_1$ and $y \in V_2$ is an isolated vertex in $CG^\alpha(D_1)$ and $CG^\alpha(D_2)$, respectively.

Owing to $E^\alpha_\ast \neq \emptyset$, there has to be a row or a column $R_k^\alpha$ $(k \in \{1, \ldots, r_\alpha\})$ such that every vertex of $R_k^\alpha$ is contained in a loop of $CG^\alpha(D_1 \ast D_2)$, i.e. $\{(a, b) \mid (a, b) \in R_k^\alpha\} \subseteq E^\alpha_\ast$.

For simplicity, we consider the case $\alpha = 1$, i.e. $E^\alpha_\ast = Z_2 = Z_k$ $(\alpha = 2$ can be treated analogously). Then $Z_k = \{(k, j) \mid j \in \{1, \ldots, r_2\}\}$ includes the vertex $(k, y)$. Since $\{(k, y)\} \in E^\ast_\ast$ and $\{y\} \neq E^\ast_2$ (because $y$ is isolated in $CG^\ast_2(D_2)$, see above) we have $\{k\} \in E^\ast_1$. Therefore, $E^\ast_1 = \emptyset$ and $E^\ast_1 = \{\pi_1(e) \mid e \in E^\ast_1\}$.

Trivially, $E^\ast_1$ contains nothing else than loops.

(b) It suffices to verify the theorem for $\alpha = 1$ and $\beta = 2$, i.e. $R_j^\beta = S_j$. Then the case $\alpha = 2 \land \beta = 1$ follows from $D_1 \ast D_2 \simeq D_2 \ast D_1$.

The existence of an edge $\bar{e} \in E^\alpha_\ast$ with $|\pi_1(\bar{e})| = |\pi_2(\bar{e})| = 2$ implies $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. For $\bar{e} = \{(a, b), (a', b')\}$, this is a conclusion from $a \neq a', b \neq b'$ and the definition of $D_1 \ast D_2$, since

$\exists x \in V_1: \{a, a'\} \subseteq N^\ast_1(a) \land a \in N^\ast_1(a') \land a' \in N^\ast_1(a)$ and

$\exists y \in V_2: \{b, b'\} \subseteq N^\ast_2(b) \land b \in N^\ast_2(b') \land b' \in N^\ast_2(b)$.

Let $j \in \{1, \ldots, r_2\}$ fulfill the condition in (2) and, moreover, $(i, i') \in A_1$.

Assume there is a vertex $j' \in V_2 \setminus \{j\}$ with $(j, j') \in A_2$. Then $e' := \{(i, j), (i, j')\} \subseteq N^\ast((i', j'))$ is an edge in $E^\ast_\ast$ with $e' \cap S_j \neq \emptyset \land e' \not\subseteq S_j$, in contradiction to (2).
Consequently, \( N^+_2(j) = \emptyset \).

Therefore, all edges \( e \in E^i_1 \) with \( e \cap S_j \neq \emptyset \) (or, equivalently, \( e \subseteq S_j \)) are induced by the sets of predecessors \( N^*_i((i', j)) \) of vertices \((i', j) \in S_j\). Hence the subgraph \( (S_j \cap CG^{\beta}(D_1 \cdot D_2)) = (S_j, \{ e \in E^i_1 \land e \subseteq S_j \}) \) of \( CG^{\beta}(D_1 \cdot D_2) \) is isomorphic to \( CG^i(D_1) \) and

\[
E^i_1 = \{ \pi_1(e) \mid e \in E^i_1 \land e \cap S_j \neq \emptyset \} \\
= \{ \pi_1(e) \mid e \in E^i_1 \land e \subseteq S_j \}.
\]

\( \square \)

**Remark 5.** The condition (2) is equivalent to \( \exists j \in V_\beta : N^+_j(j) = \emptyset \).

We prefer (2) in Theorem 5 since (2) uses only the edge set \( E^i_1 \) of the \( \beta \)-competition graph of \( D_1 \cdot D_2 \), which is given by the assumptions of the Theorem, and not properties of the (unknown) digraph \( D_\beta \).

If (2) does not hold, then – in general – the computation of \( E^i_1 \) is not so simple as at the end of the proof of Theorem 5.

**Example 4.** Let \( D_1 = (V_1 = \{ 1, 2 \}, A_1 = \{(1, 2), (2, 1)\}) \) and \( D_2 = (V_2 = \{ 1, 2 \}, A_2 = \{(1, 2)\}) \). Then

\[
D_1 \cdot D_2 = (V_1 \times V_2, A_1 \times \{ (1, 1), (1, 2), (2, 1) \}, ((1, 1), (2, 1)), ((1, 2), (2, 2)), ((2, 1), (1, 1)), ((2, 1), (1, 2)), ((2, 1), (2, 2)), ((2, 2), (1, 2)), ((1, 1), (1, 1)), ((1, 1), (2, 1)), ((1, 1), (1, 2)), ((1, 1), (2, 2)), ((1, 2), (1, 2)), ((1, 2), (2, 1)), ((1, 2), (2, 1)), ((2, 1), (2, 1)), ((2, 1), (2, 2))\},
\]

\[
\{ \pi_1(e) \mid e \in E^i_1 \land e \cap S_1 \neq \emptyset \} = \{ \{1\}, \{2\}, \{1, 2\}\}
\]

\[
\neq \{ \{1\}, \{2\}\} = E^i_1.
\]

Moreover,

\[
\{ \pi_1(e) \mid e \in E^i_1 \land e \cap S_2 \neq \emptyset \} = \{ \{1\}, \{2\}, \{1, 2\}\}
\]

\[
\neq \emptyset = \{ \pi_1(e) \mid e \in E^i_1 \land e \subseteq S_2 \}
\]

\[
\neq E^i_1.
\]

3.4. The lexicographic product \( D_1 \cdot D_2 \).

Note that if \( (i, i') \in A_1 \), then \( \forall j, j' \in V_2 : ((i, j), (i', j')) \in A \). and, consequently,

\[
\mathcal{P}^2(Z_i) := \{ ((i, j), (i, j')) \mid j, j' \in V_2 \land j \neq j' \} \subseteq E.
\]

(Conversely, this means that \( \mathcal{P}^2(Z_i) \setminus E \neq \emptyset \) includes \( N^+_1(i) = \emptyset \).

Hence, the existence of a vertex \( i \in V_1 \) having the outdegree 0 in \( D_1 \) is necessary to get some information on \( E^i_2 \) from \( CG^i(D_1 \cdot D_2) \); this will be used in (a) of our next Theorem.

Concerning the reconstruction of \( CG(D_\beta) \) we discuss (2) from Theorem 5:

Replacing \( E^i_1 \) by \( E^i_2 \) in (2), for \( \beta = 2 \) (i.e. \( R^3_j \) is the column \( S_j \)) the condition (2) makes no sense for the lexicographic product, because in the case \( (i, i'') \in A_1 \) the vertex \( (i'', j) \in V_1 \times V_2 \) has predecessors \( (i, j') \) in all columns \( S_j \) \( (j' \in \{ 1, \ldots, r_2 \}) \). Hence, \( \forall j' \in \{ 1, \ldots, r_2 \} \setminus \{ j \} : ((i, j), (i, j')) \in E_\beta, \) i.e. there are a lot of edges \( e \in E \) fulfilling \( e \cap S_j \neq \emptyset \land e \subseteq S_j \).

Therefore, for the lexicographic product we have to make use of another condition ((4), see below), which implies that there are no "horizontal arcs" \( ((i, j), (i, j')) \in A \) in \( D_1 \cdot D_2 \) \( (j' \in \{ 1, \ldots, r_2 \} \setminus \{ j \}) \).
Theorem 6. For the lexicographic product $D_1 \cdot D_2$ it holds:

(a) If

\[ \exists i \in \{1, \ldots, r_1\} : P^2(Z_i) \setminus E \neq \emptyset, \]

then $CG(D_2)$ can be reconstructed from $CG(D_1 \cdot D_2)$.

(b) If $D_1$ contains an isolated vertex $i \in V_1$, then $CG^1(D_2)$ can be reconstructed from $CG^1(D_1 \cdot D_2)$.

(c) If

\[ \exists j \in \{1, \ldots, r_2\} : N^+_2(j) = \emptyset, \]

then $CG(D_1)$ can be reconstructed from $CG(D_1 \cdot D_2)$.

(d) In general, the $l$-competition graph $CG^l(D_1)$ cannot be reconstructed from $CG^l(D_1 \cdot D_2)$.

Proof.

(a) Let $i \in \{1, \ldots, r_1\}$ with $P^2(Z_i) \setminus E \neq \emptyset$, i.e. $N^+_1(i) = \emptyset$. Further let $e \in P^2(Z_i) \cap E$ be an edge in $CG(D_1 \cdot D_2)$ containing only vertices of the row $Z_i$.

Since $i \in V_1$ has no successor in $D_1$, there is a vertex $(i, j''') \in Z_i$ with $e \subseteq N^-(i, j''')$.

Therefore, for $e = \{(i, j), (i, j''')\}$ it follows $\{j, j''\} \subseteq N^-_2(j''')$, i.e. $\pi_2(e) \in E_2$. Consequently, we obtain

\[ E_2 = \{\pi_2(e) \mid e \in E \land e \subseteq Z_i\}. \]

(Note that in the case of the $l$-competition graphs $CG^l(D_1 \cdot D_2)$ and $CG^l(D_2)$ the following problem occurs: if $i' \in V_1$ is a predecessor of the vertex $i$ and $N^-_2(j''') = \{j\}$ is a loop in $CG^l(D_2)$, then $\{(i, j), (i', j''')\} \subseteq N^-_2((i, j'''))$ and therefore there is no edge $e \in E^l$ with $e \subseteq Z_i$ and $\pi_2(e) = \{j\}$. In other words: there is no loop $e = \{(i, j)\}$ in $E^l$.)

(b) Now let $i \in V_1$ be an isolated vertex in $D_1$ (i.e. $N^-_1(i) = N^+_1(i) = \emptyset$).

Further let $j, j'' \in V_2$ such that $N^-_2(j'') = \{j\}$ is a loop in $CG^l(D_2)$. Since $N^-_1(i) = \emptyset$, $e = \{(i, j), (i, j'')\} = N^-((i, j'')) \in E^l$ is a loop in $CG^l(D_1 \cdot D_2)$ with $e \subseteq Z_i$ and $\pi_2(e) = \{j\}$.

This way, all loops in $CG^l(D_2)$ can be reconstructed. Because of $N^+_1(i) = \emptyset$, the reconstruction of the edges $e \in E_2$ can be done analogously to part (a) of the proof and we have

\[ E'_2 = \{\pi_2(e) \mid e \in E^l \land e \subseteq Z_i\}. \]

(c) Let $j$ fulfill the condition in (4) and consider an edge $\{(i, j), (i', j')\} \in E$. Then, in $D_1 \cdot D_2$ for all successors $(i'', j'')$ of the vertices $(i, j), (i', j')$ the vertex $i''$ must be a common successor of the vertices $i$ and $i'$, since $j'' \notin N^+_2(j) = \emptyset$.

Consequently, $\{(i, j), (i', j')\} \subseteq N^-_2(i'')$ and $\{(i, j), (i', j')\} \in E_1$. On the other hand, for all $\{i, i'\} \in E_1$, even in the subgraph $\langle S_j \rangle_{CG(D_1 \cdot D_2)} = (S_j, \{e \in E \land e \subseteq S_j\})$ of $CG(D_1 \cdot D_2)$, we find the edge $e = \{(i, j), (i', j')\}$ with $\pi_1(e) = \{i, i'\}$. Therefore, we get

\[ E_1 = \{\pi_1(e) \mid e \in E \land e \subseteq S_j\}. \]

(d) We consider an example fulfilling the condition (4).

Example 5. Let $D_1 = (V_1 = \{1, 2, 3\}, A_1 = \{(1, 2), (1, 3), (3, 2)\})$, $D_1' = (V_1', A_1' = \{(1, 2), (3, 2)\})$ and $D_2 = (V_2 = \{1, 2\}, A_2 = \emptyset)$. Obviously, $A(D_1' \cdot D_2) \subseteq A(D_1 \cdot D_2)$ and $E(CG^l(D_1' \cdot D_2)) \subseteq E(CG^l(D_1 \cdot D_2))$. On the other hand, the only edge $e \in E(CG^l(D_1 \cdot D_2))$ which is induced by the arc $(1, 3) \in A_1' \setminus A_1$ is $e = N^-(CG^l(D_1 \cdot D_2))(3, 1) = N^-_{CG(D_1 \cdot D_2)}((3, 1)) = \{(1, 1), (1, 2)\}$. Because of $e \subseteq N^-_{CG(D_1 \cdot D_2)}((2, 1))$, it follows $e \in E(CG^l(D_1' \cdot D_2))$ and, therefore, $E(CG^l(D_1' \cdot D_2)) = E(CG^l(D_1 \cdot D_2))$.

Consequently, $CG^l(D_1) = \{\{1\}, \{1, 3\}\} \neq \{\{1, 3\}\} = CG^l(D_1')$ cannot be reconstructed from $CG^l(D_1 \cdot D_2) = CG^l(D_1' \cdot D_2)$. \[ \square \]
3.5. **The disjunction** $D_1 \lor D_2$.

In analogy with Theorem 6, (c) and (d), we obtain a corresponding result for the disjunction.

**Theorem 7.** Let $\{\alpha, \beta\} = \{1, 2\}$. For the disjunction $D_1 \lor D_2$ it holds:

(a) If
\[ \exists j \in \{1, \ldots, r_\beta\} : N_\beta^+(j) = \emptyset, \]
then $CG(D_\alpha)$ can be reconstructed from $CG(D_1 \lor D_2)$.

(b) In general, the $l$-competition graph $CG^l(D_\alpha)$ cannot be reconstructed from $CG^l(D_1 \lor D_2)$.

**Proof.** Owing to $D_1 \lor D_2 \simeq D_2 \lor D_1$ it is sufficient to consider the case $\alpha = 1$ and $\beta = 2$.

Then, replacing $D_1 \cdot D_2$ by $D_1 \lor D_2$, the argumentation from the proof of Theorem 6 (parts (c) and (d)) as well the counterexample (Example 5) can be used without any changes.

**References**


