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Neighborhood hypergraphs and products of undirected graphs

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Abstract

If $G = (V, E)$ is a simple undirected graph, its *neighborhood hypergraph* $\mathcal{N}(G) = (V, \mathcal{E}^{\mathcal{N}})$ has the vertex set V and $e \subseteq V$ is an edge of $\mathcal{N}(G)$ iff $|e| \geq 1$ and there is a vertex $v \in V$, such that e is the neighborhood of the vertex v in the graph G. For several products $G_1 \circ G_2$ of simple undirected graphs G_1 and G_2, we investigate the question whether G_1, G_2, $\mathcal{N}(G_1) = (V_1, \mathcal{E}_1^{\mathcal{N}})$ or $\mathcal{N}(G_2) = (V_2, \mathcal{E}_2^{\mathcal{N}})$ can be reconstructed from $\mathcal{N}(G_1 \circ G_2)$ or not. Vice versa, we solve the problem how $\mathcal{N}(G_1 \circ G_2)$ can be obtained from G_1, G_2, $\mathcal{N}(G_1) = (V_1, \mathcal{E}_1^{\mathcal{N}})$ and $\mathcal{N}(G_2) = (V_2, \mathcal{E}_2^{\mathcal{N}})$.

Keywords: neighborhood hypergraph, graph product
Mathematics Subject Classification (2010): 05C65, 05C76, 05C75

1 Introduction

All hypergraphs $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ and graphs $G = (V(G), E(G))$ considered here may have isolated vertices but no multiple edges. Moreover, in graphs loops are forbidden.
The *neighborhood hypergraph* $\mathcal{N}(G) = (V, \mathcal{E}^{\mathcal{N}})$ of the graph $G = (V, E)$ has the vertex set V and $e \subseteq V$ is an edge of $\mathcal{N}(G)$ iff $|e| \geq 1$ and there is a vertex $v \in V$, such that e is the *neighborhood* of the vertex v in the graph $G = (V, E)$, i.e. $e = N_G(v) := \{w \in V \mid \{w, v\} \in E\}$.
Moreover, $N(G) = (V, E^N)$ is the *neighborhood graph* of the graph $G = (V, E)$, if and only if $E^N = \{\{a, b\} \mid a \neq b \wedge \exists\, x \in V : \{x, a\} \in E \wedge \{x, b\} \in E\}$.

This is the usual definition of neighborhood graphs (cf. the references below), where an edge $\{a, b\} \in E^N(G)$ indicates that a and b have a common neighbor v in G. In general,

contrary to neighborhood hypergraphs, the edges of the neighborhood graph $N(G)$ do not contain the full information on the sets of neighbors of the vertices of G.
Our decision, to define $\mathcal{E}^{\mathcal{N}}(G)$ as the set of all nonempty neighborhoods of the vertices of G has the consequence that vertices $v \in V(G)$ of degree 1 correspond to loops in $\mathcal{N}(G)$, whereas they do not lead to any edge in $N(G)$. Of course, accepting to loose this information it would be also possible to forbid loops in neigborhood hypergraphs.
Note that – using our definitions – in the case of graphs G with maximum degree 2 containing at least one vertex of degree 1, $\mathcal{N}(G)$ is a graph (with loops) which is not isomorphic to $N(G)$ (which is loopless per definition).
In addition to the definition of the neighborhood $N_G(v)$, we refer to the set $N_G^+(v) := \{w \in V \mid \{w, v\} \in E\} \cup \{v\}$ as the *closed neighborhood* of the vertex $v \in V(G)$.
Definitions not explicitly given here can be found in Diestel [6] and Berge [2], respectively.

Several aspects of neighborhood graphs were investigated in the last thirty years (cf. Acharya and Vartak [1]; Boland, Brigham and Dutton [3], [4]; Brigham and Dutton [5]; Exoo and Harary [7]; Greenberg, Lundgren and Maybee [8]; Lundgren, Merz, Maybee and Rasmussen [11]; Lundgren, Merz and Rasmussen [12]; Lundgren and Rasmussen [13]; Lundgren, Rasmussen and Maybee [14]; Miller, Brigham and Dutton [15]; Schiermeyer, Sonntag and Teichert [18]; Sonntag and Teichert [20]). Some of these papers use the notation *2-step graph* or *competition graph* instead of neighborhood graph. As the latter name indicates, the neighborhood graph $N(G)$ of an undirected graph G is closely related to the competition graph $C(D)$ of a digraph D. Surveys about competition graphs can be found in Kim [9], Lundgren [10] and Roberts [17].

Motivated by the results in [20], in Section 2 for five products $G_1 \circ G_2$ (*Cartesian sum* $G_1 + G_2$, *Cartesian product* $G_1 \times G_2$, *normal product* $G_1 * G_2$, *lexicographic product* $G_1 \cdot G_2$ and *disjunction* $G_1 \vee G_2$) of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, G_2)$ we investigate the relations between the neighborhood hypergraph $\mathcal{N}(G_1 \circ G_2) = (V, \mathcal{E}_\circ^{\mathcal{N}})$ on the one hand and G_1, G_2, $\mathcal{N}(G_1) = (V_1, \mathcal{E}_1^{\mathcal{N}})$ as well as $\mathcal{N}(G_2) = (V_2, \mathcal{E}_2^{\mathcal{N}})$ on the other hand. In a certain sense this continues our investigations in [19], where we dealt with competition hypergraphs of products of digraphs. In the present paper we place the emphasis on the question whether G_1, G_2, $\mathcal{N}(G_1) = (V_1, \mathcal{E}_1^{\mathcal{N}})$ or $\mathcal{N}(G_2) = (V_2, \mathcal{E}_2^{\mathcal{N}})$ can be reconstructed from $\mathcal{N}(G_1 \circ G_2)$ or not (cf. Subsections 2.1–2.5). To obtain $\mathcal{N}(G_1 \circ G_2)$ from G_1, G_2, $\mathcal{N}(G_1) = (V_1, \mathcal{E}_1^{\mathcal{N}})$ and $\mathcal{N}(G_2) = (V_2, \mathcal{E}_2^{\mathcal{N}})$ is an easier problem (cf. Theorem 1 and Remark 1).
The products $G_1 \circ G_2$ considered here have always the vertex set $V := V_1 \times V_2$; using the notation $\widetilde{E} := \{\{(a, b), (a', b')\} \mid a, a' \in V_1 \wedge b, b' \in V_2\}$ their edge sets $E_\circ := E(G_1 \circ G_2)$ are defined as follows:

$$E_+ := \{\{(a, b), (a', b')\} \in \widetilde{E} \mid (\{a, a'\} \in E_1 \wedge b = b') \vee (a = a' \wedge \{b, b'\} \in E_2)\},$$

$$E_\times := \{\{(a, b), (a', b')\} \in \widetilde{E} \mid \{a, a'\} \in E_1 \wedge \{b, b'\} \in E_2\},$$

$$E_* := E_\times \cup E_+,$$

$$E_. := \{\{(a, b), (a', b')\} \in \widetilde{E} \mid \{a, a'\} \in E_1 \vee (a = a' \wedge \{b, b'\} \in E_2)\},$$

$$E_\vee := \{\{(a, b), (a', b')\} \in \widetilde{E} \mid \{a, a'\} \in E_1 \vee \{b, b'\} \in E_2\}.$$

We obtain immediately $E_+ \subseteq E_* \subseteq E_\cdot \subseteq E_\vee$ and $E_\times \subseteq E_*$. Except for the lexicographic product all these products are commutative in the sense that $G_1 \circ G_2 \simeq G_2 \circ G_1$, where $\circ \in \{+, \times, *, \vee\}$.
In the following, let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be graphs; $V_1 = \{1, 2, \ldots, r\}$, $V_2 = \{1, 2, \ldots, s\}$ and arrange the vertices of $V = V_1 \times V_2$ according to the places of an (r, s)-matrix. Then, for each $\circ \in \{+, *, \cdot, \vee\}$, the subgraph of $G_1 \circ G_2$ generated by the vertices of a column and a row of this matrix scheme is isomorphic to G_1 and G_2, respectively. We refer to these subgraphs as the *columns* and the *rows* of $G_1 \circ G_2$. Analogously, we define the *columns* and the *rows* of $\mathcal{N}(G_1 \circ G_2)$.
If $G = (V, E)$ and $G' = (V', E')$ are graphs we often use the notation $G \cup G' := (V \cup V', E \cup E')$. Moreover, if $e \subseteq V_1 \times V_2$ is a set of vertices of $G_1 \circ G_2$, then $\pi_1(e)$ and $\pi_2(e)$ denotes the set of the first and the second components of the vertices of e, respectively. In other words, π_1 and π_2 is the projection of the vertices of e onto their first and their second components, respectively.

2 Results

Per definition, the edges $e \in \mathcal{E}^{\mathcal{N}}$ of the neighborhood hypergraph $\mathcal{N}(G) = (V, \mathcal{E}^{\mathcal{N}})$ of a graph $G = (V, E)$ are the neighbor sets $N_G(v)$ of cardinality at least 1 of the vertices $v \in V(G)$. Therefore, the definition of the graph product $G_1 \circ G_2$ easily yields the edge set $\mathcal{E}_\circ^{\mathcal{N}}$ of the neighborhood hypergraph $\mathcal{N}(G_1 \circ G_2) = (V, \mathcal{E}_\circ^{\mathcal{N}})$.
In the following theorem we give the edge sets $\mathcal{E}_\circ^{\mathcal{N}}$ of the neighborhood hypergraphs $\mathcal{N}(G_1 \circ G_2) = (V, \mathcal{E}_\circ^{\mathcal{N}})$ of the five graph products $G_1 \circ G_2$, where $\circ \in \{+, \times, *, \cdot, \vee\}$.

Theorem 1 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs and $V := V_1 \times V_2$. Then:*

(1) $\mathcal{E}_+^{\mathcal{N}} = \{(\{i\} \times N_{G_2}(j)) \cup (N_{G_1}(i) \times \{j\}) \,|\, (i, j) \in V_1 \times V_2\} \setminus \{\emptyset\}$

(2) $\mathcal{E}_\times^{\mathcal{N}} = \{\, e_1 \times e_2 \,|\, e_1 \in \mathcal{E}_1^{\mathcal{N}} \wedge e_2 \in \mathcal{E}_2^{\mathcal{N}}\}$

(3) $\mathcal{E}_*^{\mathcal{N}} = \{(N_{G_1}^+(i) \times N_{G_2}^+(j)) \setminus \{(i, j)\} \,|\, (i, j) \in V_1 \times V_2\} \setminus \{\emptyset\}$

(4) $\mathcal{E}_\cdot^{\mathcal{N}} = \{(\{i\} \times N_{G_2}(j)) \cup (N_{G_1}(i) \times V_2) \,|\, (i, j) \in V_1 \times V_2\} \setminus \{\emptyset\}$

(5) $\mathcal{E}_\vee^{\mathcal{N}} = \{(V_1 \times e_2) \cup (e_1 \times V_2) \,|\, e_1 \in \mathcal{E}_1^{\mathcal{N}} \wedge e_2 \in \mathcal{E}_2^{\mathcal{N}}\}$

Note that – in order simplify the notation – we fix here that *(5)* is a short form for

$$(*) \qquad \mathcal{E}_\vee^{\mathcal{N}} = \begin{cases} \{(V_1 \times e_2) \cup (e_1 \times V_2) \,|\, e_1 \in \mathcal{E}_1^{\mathcal{N}} \wedge e_2 \in \mathcal{E}_2^{\mathcal{N}}\}, & \text{if } \mathcal{E}_1^{\mathcal{N}} \neq \emptyset \wedge \mathcal{E}_2^{\mathcal{N}} \neq \emptyset \\ \{e_1 \times V_2 \,|\, e_1 \in \mathcal{E}_1^{\mathcal{N}}\} & , \text{if } \mathcal{E}_1^{\mathcal{N}} \neq \emptyset \wedge \mathcal{E}_2^{\mathcal{N}} = \emptyset \\ \{V_1 \times e_2 \,|\, e_2 \in \mathcal{E}_2^{\mathcal{N}}\} & , \text{if } \mathcal{E}_1^{\mathcal{N}} = \emptyset \wedge \mathcal{E}_2^{\mathcal{N}} \neq \emptyset \\ \emptyset & , \text{if } \mathcal{E}_1^{\mathcal{N}} = \emptyset = \mathcal{E}_2^{\mathcal{N}}. \end{cases}$$

Proof.

(1) In $G_1 + G_2$, a vertex $(i', j') \in V_1 \times V_2$ is a neighbor of (i, j) if and only if
   (a) (i', j') is in the same row as (i, j) (i.e. $i' = i$) and j' is a neighbor of j in G_2 or
   (b) (i', j') is in the same column as (i, j) (i.e. $j' = j$) and i' is a neighbor of i in G_1.

(2) In the Cartesian product $G_1 \times G_2$, the neighborhood $N_{G_1 \times G_2}((i, j))$ of a vertex $(i, j) \in V(G_1 \times G_2)$ contains all vertices (i', j') with $i' \in N_{G_1}(i) \wedge j' \in N_{G_2}(j)$. Therefore, $\mathcal{E}_\times^{\mathcal{N}} = \{N_{G_1}(i) \times N_{G_2}(j) \,|\, (i, j) \in V_1 \times V_2\} \setminus \{\emptyset\}$. Moreover, $\mathcal{E}_1^{\mathcal{N}} = \{e_1 \,|\, \exists i \in V_1 : e_1 = N_{G_1}(i) \neq \emptyset\}$ and $\mathcal{E}_2^{\mathcal{N}} = \{e_2 \,|\, \exists j \in V_2 : e_2 = N_{G_2}(j) \neq \emptyset\}$.

(3) This follows from $E_* := E_\times \cup E_+$.

(4) A vertex (i', j') is a neighbor of the vertex (i, j) in the lexicographic product $G_1 \cdot G_2$ if and only if
   (a) (i', j') is in the same row as (i, j) (i.e. $i' = i$) and j' is a neighbor of j in G_2 or
   (b) i' is a neighbor of i in G_1 and $j' \in V_2$.

(5) (i', j') is a neighbor of (i, j) in the disjunction $G_1 \vee G_2$ if and only if $i' \in N_{G_1}(i)$ or $j' \in N_{G_2}(j)$. Analogously to the second part in (2) we obtain *(5)*. □

Now the question arises whether or not the neighborhood hypergraph $\mathcal{N}(G_1 \circ G_2)$ of the product graph $G_1 \circ G_2$ can be obtained from $\mathcal{N}(G_1)$ and $\mathcal{N}(G_2)$ using suitable hypergraph products.
In case of the Cartesian product $G_1 \times G_2$ and the disjunction $G_1 \vee G_2$ this is possible (see the following Remark 1). Generally, for the other three products the same is impossible: To construct the edges in $\mathcal{E}_+^{\mathcal{N}}$, $\mathcal{E}_*^{\mathcal{N}}$ and $\mathcal{E}_\cdot^{\mathcal{N}}$, for every $i \in \{1, 2\}$ and all edges $e \in \mathcal{E}_i^{\mathcal{N}}$ it would be necessary to know the vertices $v \in V_i$ with $e = N_{G_i}(v)$. But, in general, this information cannot be obtained from the neighborhood hypergraph $\mathcal{N}(G_i)$.
Let $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$ be hypergraphs and $V := V_1 \times V_2$. We define the edge sets of the *Cartesian product* $\mathcal{H}_1 \times_H \mathcal{H}_2 = (V, \mathcal{E}_\times)$ and the *disjunction* $\mathcal{H}_1 \vee_H \mathcal{H}_2 = (V, \mathcal{E}_\vee)$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ as follows:

$$\mathcal{E}_\times := \{e_1 \times e_2 \,|\, e_1 \in \mathcal{E}_1 \wedge e_2 \in \mathcal{E}_2\}$$

$$\mathcal{E}_\vee := \{(V_1 \times e_2) \cup (e_1 \times V_2) \,|\, e_1 \in \mathcal{E}_1 \wedge e_2 \in \mathcal{E}_2\}.$$

For the definition of $\mathcal{E}_\vee$ we use a convention analogously to $(*)$ (cf. Theorem 1). Note that the Cartesian product $\mathcal{H}_1 \times_H \mathcal{H}_2$ is defined in [16] as the *direct product* of $\mathcal{H}_1$ and $\mathcal{H}_2$. In general, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are graphs, both products provide hypergraphs but not graphs, i.e. $\mathcal{H}_1 \times_H \mathcal{H}_2 \not\cong \mathcal{H}_1 \times \mathcal{H}_2$ and $\mathcal{H}_1 \vee_H \mathcal{H}_2 \not\cong \mathcal{H}_1 \vee \mathcal{H}_2$.
Using these hypergraph products, by Theorem 1, *(2)* and *(5)*, we obtain

Remark 1 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs and $V := V_1 \times V_2$. Then:*

(1) $\mathcal{N}(G_1 \times G_2) = \mathcal{N}(G_1) \times_H \mathcal{N}(G_2)$.

(2) $\mathcal{N}(G_1 \vee G_2) = \mathcal{N}(G_1) \vee_H \mathcal{N}(G_2)$.

In the following subsections, we turn to the problem of reconstructing the neighborhood hypergraphs $\mathcal{N}(G_1) = (V_1, \mathcal{E}_1^{\mathcal{N}})$, $\mathcal{N}(G_2) = (V_2, \mathcal{E}_2^{\mathcal{N}})$ and the graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ from $\mathcal{N}(G_1 \circ G_2) = (V, \mathcal{E}_\circ^{\mathcal{N}})$, where $\circ \in \{+, \times, *, \cdot, \vee\}$.

2.1 The Cartesian sum $G_1 + G_2$

In this subsection, we will demonstrate that (excepting some special cases) the neighborhood hypergraphs $\mathcal{N}(G_1) = (V_1, \mathcal{E}_1^{\mathcal{N}})$ and $\mathcal{N}(G_2) = (V_2, \mathcal{E}_2^{\mathcal{N}})$ as well as the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ can be reconstructed from the neighborhood hypergraph $\mathcal{N}(G_1 + G_2) = (V, \mathcal{E}_+^{\mathcal{N}})$.
At first, we investigate the exceptional cases.

Claim 1 *Let* $|V(\mathcal{N}(G_1 + G_2))| = |V_2|$, *i.e.* $|V_1| = 1$.
Then $G_1 + G_2 \simeq G_2$ *and therefore* $\mathcal{N}(G_1 + G_2) \simeq \mathcal{N}(G_2)$. *Obviously both,* G_1 *and* $\mathcal{N}(G_1)$, *have only one vertex and no edge.*
Apart from very special cases (e.g. $E_2 = \emptyset$ *or* $|E_2| = 1$*) the graph* G_2 *is not reconstructable from* $\mathcal{N}(G_1 + G_2) \simeq \mathcal{N}(G_2)$.

Obviously, because of $G_1 + G_2 \simeq G_2 + G_1$ analog conclusions follow from $|V(\mathcal{N}(G_1 + G_2))| = |V_1|$.

Claim 2 *Let* $|V_1| > 1$, $|V_2| > 2$ *and* $\mathcal{N}(G_1 + G_2)$ *be 1-uniform, i.e.* $\mathcal{E}_+^{\mathcal{N}}$ *solely consists of loops.*
Then either $E_1 = \emptyset$ *or* $E_2 = \emptyset$. *Which of the sets* E_1 *and* E_2 *is empty can be found out if and only if* $|\mathcal{E}_+^{\mathcal{N}}| < |V_1| \cdot |V_2|$. *In exactly this case* $\mathcal{N}(G_1)$ *and* $\mathcal{N}(G_2)$ *can be reconstructed from* $\mathcal{N}(G_1 + G_2)$.

Proof. First, assume $\{i, i'\} \in E_1$ and $\{j, j'\} \in E_2$. Then $(i, j'), (i', j) \in N_{G_1+G_2}((i, j)) \in \mathcal{E}_+^{\mathcal{N}}$ in contradiction to the 1-uniformity of $\mathcal{N}(G_1 + G_2)$. Hence, either $E_1 = \emptyset$ or $E_2 = \emptyset$.
Now consider an edge $\{(i, j)\} \in \mathcal{E}_+^{\mathcal{N}}$. If $i' \in V_1$ is a neighbor of the vertex i, then for each $j' \in V_2$ we have $\{(i, j')\}, \{(i', j')\} \in \mathcal{E}_+^{\mathcal{N}}$, since $\{(i, j'), (i', j')\} \in E_+$.
Consequently,

(a) $\mathcal{E}_+^{\mathcal{N}}$ includes all singletons of the vertices of the rows i and i' of $\mathcal{N}(G_1 + G_2)$, i.e.

$$\forall j' \in V_2 : \{(i, j')\} \in \mathcal{E}_+^{\mathcal{N}} \wedge \{(i', j')\} \in \mathcal{E}_+^{\mathcal{N}}.$$

(b) $E_2 = \emptyset$ as well as $\mathcal{E}_2^{\mathcal{N}} = \emptyset$.

(c) The set of the non-isolated vertices of G_1 is $V_1^+ := \{i \,|\, i \in \bigcup_{e \in \mathcal{E}_+^{\mathcal{N}}} \pi_1(e)\}$.
Moreover, $\mathcal{E}_1^{\mathcal{N}} = \{\, \{i\} \,|\, i \in V_1^+\}$.

For each $k \in V_1$ property (a) implies

- all singletons of the vertices of the row k are contained in $\mathcal{E}_+^{\mathcal{N}}$ if and only if $k \in V_1^+$

and

- no singleton of any vertex of the row k is contained in $\mathcal{E}_+^{\mathcal{N}}$ if and only if $k \in V_1 \setminus V_1^+$.

As a consequence, in the case $|\mathcal{E}_+^{\mathcal{N}}| < |V_1| \cdot |V_2|$ we have $E_2 = \emptyset$ if and only if $\mathcal{E}_+^{\mathcal{N}}$ is the union of systems of singletons of vertices of complete rows of $\mathcal{N}(G_1 + G_2)$. Trivially, an analogous statement holds for $E_1 = \emptyset$ and the columns of $\mathcal{N}(G_1 + G_2)$. On the contrary, if $|\mathcal{E}_+^{\mathcal{N}}| = |V_1| \cdot |V_2|$ is valid, then $\mathcal{E}_+^{\mathcal{N}} = \{\{(i,j)\} \,|\, (i,j) \in V_1 \times V_2\}$ is the union of the systems of the singletons of the vertices of all (obviously complete) rows as well as the union of the systems of the singletons of the vertices of all (obviously complete) columns of $\mathcal{N}(G_1 + G_2)$. In this case, both $\mathcal{E}_1^{\mathcal{N}} = \{\,\{i\} \,|\, i \in V_1\} \;\wedge\; E_2 = \emptyset$ and $\mathcal{E}_2^{\mathcal{N}} = \{\,\{j\} \,|\, j \in V_2\} \;\wedge\; E_1 = \emptyset$ is possible. □

Note that under the suppositions of Claim 2, where $\mathcal{E}_+^{\mathcal{N}}$ includes only singletons, for the graphs G_1 and G_2 we have the two alternatives $\forall i \in V_1 : d_{G_1}(i) \leq 1 \;\wedge\; d_{G_2} \equiv 0$ and $\forall j \in V_2 : d_{G_2}(j) \leq 1 \;\wedge\; d_{G_1} \equiv 0$.

Claim 3 *Let $|V_1| > 1$, $|V_2| > 1$ and $\mathcal{N}(G_1 + G_2)$ be not 1-uniform.*
If (i) $\forall e \in \mathcal{E}_+^{\mathcal{N}} : |\pi_1(e)| = 1$ and (ii) $\forall e \in \mathcal{E}_+^{\mathcal{N}} : |\pi_2(e)| = 1$,
then (i') $E_1 = \emptyset$ and (ii') $E_2 = \emptyset$, respectively.
In case (i) each row of $\mathcal{N}(G_1 + G_2)$ is isomorphic to $\mathcal{N}(G_2)$ and in case (ii) each column of $\mathcal{N}(G_1 + G_2)$ is isomorphic to $\mathcal{N}(G_1)$. In general, G_2 and G_1, respectively, are reconstructable only in special situations (cf. Claim 1).

Proof. We consider only case (i), since (ii) can be investigated analogously. If $\mathcal{E}_+^{\mathcal{N}} = \emptyset$, then $E_1 = \emptyset$ is trivial. Now let $(i,j), (i,j') \in e \in \mathcal{E}_+^{\mathcal{N}} \neq \emptyset$ with $j \neq j'$ and $\{j,l\}, \{j',l\} \in E_2$. Assume $\{k,k'\} \in E_1$. Then $(k,j), (k',l) \in e' := N_{G_1+G_2}((k,l)) \in \mathcal{E}_+^{\mathcal{N}}$ and $|\pi_1(e')| > 1$ contradicting (i). Hence E_1 is empty, there are no edges between different rows in $G_1 + G_2$ and the isomorphism of each row of $G_1 + G_2$ to G_2 implies the isomorphism of each row of $\mathcal{N}(G_1 + G_2)$ to $\mathcal{N}(G_2)$. □

Claims 1–3 include all cases where at least one of E_1 and E_2 is empty. The only situation where $\mathcal{N}(G_1)$ or $\mathcal{N}(G_2)$ cannot be reconstructed from $\mathcal{N}(G_1 + G_2)$ is the case

$$|V_1| > 1 \;\wedge\; |V_2| > 1 \;\wedge\; \mathcal{E}_+^{\mathcal{N}} = \{\{(i,j)\} \,|\, (i,j) \in V_1 \times V_2\} \qquad \text{(cf. Claim 2).}$$

In the following theorem we investigate the cases not being covered by the above Claims, i.e. the situation $E_1 \neq \emptyset \;\wedge\; E_2 \neq \emptyset$.
Obviously, the case $E_1 \neq \emptyset \;\wedge\; E_2 \neq \emptyset$ is equivalent to the existence of an edge $e \in \mathcal{E}_+^{\mathcal{N}}$ with $|\pi_1(e)| \geq 2 \wedge |\pi_2(e)| \geq 2$. This becomes clear if we consider edges $\{x,x'\} \in E_1$ and $\{y,y'\} \in E_2$ and the neighborhood of the vertex $(x,y) \in V$ in $G_1 + G_2$, since $(x,y'), (x',y) \in N_{G_1+G_2}((x,y)) \in \mathcal{E}_+^{\mathcal{N}}$.

Theorem 2 *Let $\mathcal{N}(G_1 + G_2) = (V, \mathcal{E}_+^{\mathcal{N}})$ be the neighborhood hypergraph of the Cartesian sum of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that there exists an edge $e \in \mathcal{E}_+^{\mathcal{N}}$ with $|\pi_1(e)| \geq 2 \wedge |\pi_2(e)| \geq 2$.*
Then $\mathcal{N}(G_1)$ and $\mathcal{N}(G_2)$ as well as G_1 and G_2 can be reconstructed from $\mathcal{N}(G_1 + G_2)$.

Proof. Obviously, $\mathcal{E}_+^{\mathcal{N}} = \{(\{i\} \times N_{G_2}(j)) \cup (N_{G_1}(i) \times \{j\}) \,|\, (i,j) \in V_1 \times V_2\} \setminus \{\emptyset\}$ (cf. Theorem 1(1)).
In the following, occasionally we will make use of the existence of two edges $\{x, x'\} \in E_1$ and $\{y, y'\} \in E_2$ in G_1 and G_2, respectively.

Let $k \in V_1$ be an arbitrary vertex.
Because of $(k, y) \in N_{G_1+G_2}((k, y')) \in \mathcal{E}_+^{\mathcal{N}}$ the vertex k appears as the first component of a vertex of an edge $e \in \mathcal{E}_+^{\mathcal{N}}$: $k \in \pi_1(e)$. If for each $e \in \mathcal{E}_+^{\mathcal{N}}$ from $k \in \pi_1(e)$ it follows $|\pi_1(e)| = 1$, then k is isolated, since the existence of a $k' \in N_{G_1}(k)$ would imply $(k, y'), (k', y) \in N_{G_1+G_2}((k', y')) \in \mathcal{E}_+^{\mathcal{N}}$ and, trivially, $|\pi_1(N_{G_1+G_2}((k', y')))| \geq 2$.
On the other hand, if $k \in \pi_1(e)$, where $e \in \mathcal{E}_+^{\mathcal{N}}$ and $|\pi_1(e)| \geq 2$, then k is non-isolated. This way the isolated vertices in G_1 and the set V_1^+ of the non-isolated vertices of G_1 (and, analogously, of G_2) can be determined. As a consequence, for the determination of G_1, G_2, $\mathcal{N}(G_1)$ and $\mathcal{N}(G_2)$ it suffices to consider the edges $e \in \mathcal{E}_+^{\mathcal{N}}$ with $|e| \geq 2$.

Now let $k \in V_1^+$.

Then there exists an edge $e \in \mathcal{E}_+^{\mathcal{N}}$ with $k \in \pi_1(e)$ and $|\pi_1(e)| \geq 2$. In the following, for such edges e we will write $e = \{(k, j_1), \ldots, (k, j_p), (i_1, j), \ldots, (i_q, j)\}$, where $k, i_1, \ldots, i_q$ are pairwise distinct, $j_1, \ldots, j_p, j$ are pairwise distinct and $p \geq 1$, $q \geq 1$.

Case 1: $\exists e \in \mathcal{E}_+^{\mathcal{N}} : k \in \pi_1(e) \wedge |\pi_1(e)| \geq 2 \wedge p > 1$.

Since each vertex (i', j') in $G_1 + G_2$ has only neighbors in the union of row i' and column j' of $G_1 + G_2$, and k appears at least twice as the first component of vertices in $e = \{(k, j_1), \ldots, (k, j_p), (i_1, j), \ldots, (i_q, j)\}$, it follows $e = N_{G_1+G_2}((k, j))$.
Therefore, $N_{G_1}(k) = \{i_1, \ldots, i_q\}$ (and, obviously, $N_{G_2}(j) = \{j_1, \ldots, j_p\}$).

Case 2: $\forall e \in \mathcal{E}_+^{\mathcal{N}} : k \in \pi_1(e) \wedge |\pi_1(e)| \geq 2 \Rightarrow p = 1$.

Let $e \in \mathcal{E}_+^{\mathcal{N}}$ be an edge of maximum cardinality with the property $k \in \pi_1(e) \wedge |\pi_1(e)| \geq 2$. In detail, let $e = \{(k, j_1), (i_1, j), \ldots, (i_q, j)\}$.
If $q > 1$ then it is clear that $e = N_{G_1+G_2}((k, j))$ and, therefore, $N_{G_1}(k) = \{i_1, \ldots, i_q\}$ (as well as $N_{G_2}(j) = \{j_1\}$).
If $q = 1$, we obtain $e = \{(k, j_1), (i_1, j)\}$.
Owing to $|\pi_1(e)| \geq 2$ we get $k \neq i_1$. The assumption $j_1 = j$ would contradict our supposition that $j, j_1, \ldots, j_p$ are pairwise distinct. Hence we have $k \neq i_1 \wedge j_1 \neq j$ and this leads to $e = N_{G_1+G_2}((k, j))$ or $e = N_{G_1+G_2}((i_1, j_1))$.
In both cases $\{k, i_1\} \in E_1$ and $\{j_1, j\} \in E_2$ are valid. Assume, there is a vertex $i_2 \in N_{G_1}(k) \setminus \{i_1\}$. Then $\{(k, j_1), (i_1, j), (i_2, j)\} \subseteq N_{G_1+G_2}((k, j))$ in contradiction to the maximality of the edge e. Therefore $N_{G_1}(k) = \{i_1\}$.

Using only the neighborhood hypergraph $\mathcal{N}(G_1 + G_2)$, the above considerations provide for each vertex $k \in V_1$ the neighborhood $N_{G_1}(k)$. This way we obtain G_1 as well as $\mathcal{N}(G_1)$ from $\mathcal{N}(G_1 + G_2)$.
Analogously, G_2 and $\mathcal{N}(G_2)$ can be constructed from $\mathcal{N}(G_1 + G_2)$ and the Theorem is proved. $\square$

2.2 The Cartesian product $G_1 \times G_2$

The edge set $\mathcal{E}_\times^\mathcal{N}$ of the neighborhood hypergraph $\mathcal{N}(G_1 \times G_2)$ includes all cartesian products of edges $e_1 \in \mathcal{E}_1^\mathcal{N}$ and $e_2 \in \mathcal{E}_2^\mathcal{N}$ of the neighborhood hypergraphs of G_1 and G_2: $\mathcal{E}_\times^\mathcal{N} = \{\, e_1 \times e_2 \mid e_1 \in \mathcal{E}_1^\mathcal{N} \wedge e_2 \in \mathcal{E}_2^\mathcal{N} \,\} \setminus \{\emptyset\}$. Therefore, to reconstruct $\mathcal{N}(G_1)$ and $\mathcal{N}(G_2)$ from

$\mathcal{N}(G_1 \times G_2)$ it is necessary to have $\mathcal{E}_\times^\mathcal{N} \neq \emptyset$ which is equivalent to $\mathcal{E}_1^\mathcal{N} \neq \emptyset \;\wedge\; \mathcal{E}_2^\mathcal{N} \neq \emptyset$ as well as to $E_1 \neq \emptyset \;\wedge\; E_2 \neq \emptyset$.

Theorem 3 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs and $\mathcal{E}_\times^\mathcal{N} \neq \emptyset$. Then $\mathcal{N}(G_1)$ and $\mathcal{N}(G_2)$ can be reconstructed from $\mathcal{N}(G_1 \times G_2)$.*

Proof. Obviously, from $\mathcal{E}_\times^\mathcal{N} \neq \emptyset$ we obtain $\mathcal{E}_i^\mathcal{N} = \{\pi_i(e) \mid e \in \mathcal{E}_\times^\mathcal{N}\}$, for $i = 1, 2$. □

Considering $\mathcal{E}_\times^\mathcal{N} = \{\, e_1 \times e_2 \mid e_1 \in \mathcal{E}_1^\mathcal{N} \wedge e_2 \in \mathcal{E}_2^\mathcal{N} \,\}$ it is clear that for graphs $G_2 \neq G_2'$ having the same neighborhood hypergraph $\mathcal{N}(G_2) = \mathcal{N}(G_2')$ and for any graph G_1 the resulting neighborhood hypergraphs of their Cartesian products coincide:
$\mathcal{N}(G_1 \times G_2) = \mathcal{N}(G_1 \times G_2')$.
As an example we choose graphs $G_2 = (V_2, E_2)$ and $G_2' = (V_2, E_2')$ with $V_2 = \{1, 2, 3, 4\}$, $E_2 = \{\{1, 2\}, \{3, 4\}\}$ and $E_2' = \{\{1, 4\}, \{2, 3\}\}$. Because of $\mathcal{E}(\mathcal{N}(G_2)) = \{\{1\}, \{2\}, \{3\}, \{4\}\}$ $= \mathcal{E}(\mathcal{N}(G_2'))$, Theorem 1(2) implies $\mathcal{N}(G_1 \times G_2) = \mathcal{N}(G_1 \times G_2')$ for arbitrarily chosen G_1. Therefore we obtain:

Remark 2 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Then, in general, G_1 and G_2 cannot be reconstructed from $\mathcal{N}(G_1 \times G_2)$.*

2.3 The normal product $G_1 * G_2$

Considering the normal product $G_1 * G_2$ of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ we will find several similarities to the Cartesian sum $G_1 + G_2$.

Remark 3 *Claims 1–3 remain valid for the normal product $G_1 * G_2$ instead of the Cartesian sum $G_1 + G_2$.*

Proof. In Claims 1–3 at least one of E_1 and E_2 is empty. Therefore, under the assumptions of the claims $G_1 + G_2 = G_1 * G_2$ is valid. □

We obtain an analog to Theorem 2:

Theorem 4 *Let $\mathcal{N}(G_1 * G_2) = (V, \mathcal{E}_*^\mathcal{N})$ be the neighborhood hypergraph of the normal product of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that there exists an edge $e \in \mathcal{E}_*^\mathcal{N}$ with $|\pi_1(e)| \geq 2 \;\wedge\; |\pi_2(e)| \geq 2$. Then $\mathcal{N}(G_1)$ and $\mathcal{N}(G_2)$ as well as G_1 and G_2 can be reconstructed from $\mathcal{N}(G_1 * G_2)$.*

Proof. Note that $\mathcal{E}_*^{\mathcal{N}} = \{\ (N_{G_1}^+(i) \times N_{G_2}^+(j)) \setminus \{(i,j)\}\ \mid\ (i,j) \in V_1 \times V_2\ \} \setminus \{\emptyset\}$ (cf. Theorem 1(3)). Additionally, the existence of an edge $e \in \mathcal{E}_*^{\mathcal{N}}$ with $|\pi_1(e)| \geq 2\ \wedge$ $|\pi_2(e)| \geq 2$ implies $E_1 \neq \emptyset$ and $E_2 \neq \emptyset$, i.e. there are edges $\{x, x'\} \in E_1$ and $\{y, y'\} \in E_2$.

As a preliminary consideration we investigate an edge $e \in \mathcal{E}_*^{\mathcal{N}}$ with $|\pi_1(e)| \geq 2\ \wedge$ $|\pi_2(e)| \geq 2$ (we will see that it suffices to use such edges in order to obtain G_1 and G_2). Then there is a vertex $(i,j) \in V$ with $e = N_{G_1 * G_2}((i,j))$, i.e. $\pi_1(e) = N_{G_1}^+(i)$ and $\pi_2(e) = N_{G_2}^+(j)$.
For each $i' \in N_{G_1}(i) = N_{G_1}^+(i) \setminus \{i\} = \pi_1(e) \setminus \{i\}$ in e there are exactly $n_2 := |N_{G_2}^+(j)| = |\pi_2(e)| \geq 2$ vertices having the first component i'. On the other hand, since the vertex (i,j) have to be deleted in $N_{G_1}^+(i) \times N_{G_2}^+(j)$ in order to obtain e, in the edge e there are exactly $n_2 - 1 \geq 1$ vertices having the first component i. Analogously, for every $j' \in N_{G_2}(j) = \pi_2(e) \setminus \{j\}$ in e there are exactly $n_1 := |N_{G_1}^+(i)| = |\pi_1(e)|$ vertices with the second component j' and exactly $n_1 - 1 \geq 1$ vertices having the second component j.
That way, for every $e \in \mathcal{E}_*^{\mathcal{N}}$ with $|\pi_1(e)| \geq 2\ \wedge\ |\pi_2(e)| \geq 2$ the vertices $i_e \in V_1$ and $i_e \in V_2$ with $e = N_{G_1 * G_2}((i_e, j_e))$ and, therefore, also their neighborhoods $N_{G_1}(i_e) = N_{G_1}^+(i_e) \setminus \{i_e\} = \pi_1(e) \setminus \{i_e\} \neq \emptyset$ as well as $N_{G_2}(j_e) = N_{G_2}^+(j_e) \setminus \{j_e\} = \pi_2(e) \setminus \{j_e\} \neq \emptyset$ are uniquely determined.

On the other hand, if $i \in V_1$ is a non-isolated vertex in G_1, then there exists an edge $e \in \mathcal{E}_*^{\mathcal{N}}$ with $|\pi_1(e)| \geq 2\ \wedge\ |\pi_2(e)| \geq 2$ and a vertex $j \in V_2$ such that $e = N_{G_1 * G_2}((i,j))$: the vertex y from above will be a suitable vertex having the required property. Of course, the analog holds for an arbitrarily chosen non-isolated vertex $j \in V_2$.
Hence the investigation of the edges $e \in \mathcal{E}_*^{\mathcal{N}}$ with $|\pi_1(e)| \geq 2\ \wedge\ |\pi_2(e)| \geq 2$ including the corresponding vertices $i_e \in V_1$, $j_e \in V_2$ enables us to determine all non-isolated vertices of G_1 and G_2 as well as their neighborhoods:
Let $\mathcal{E}^2 := \{e \,|\, e \in \mathcal{E}_*^{\mathcal{N}}\ \wedge\ |\pi_1(e)| \geq 2\ \wedge\ |\pi_2(e)| \geq 2\}$. Then the set of the non-isolated vertices in G_1 is $V_1^+ = \bigcup\limits_{e \in \mathcal{E}^2} \pi_1(e) = \{i \,|\, \exists e \in \mathcal{E}^2 : i = i_e\}$. Moreover, for each $i \in V_1^+$ there exists a unique $e \in \mathcal{E}^2$ with $i = i_e$ and we obtain $N_{G_1}(i) = \pi_1(e) \setminus \{i\}$. Consequently, $\mathcal{E}_1^{\mathcal{N}} = \{\pi_1(e) \setminus \{i_e\} \,|\, e \in \mathcal{E}^2\}$.
Analogously, in G_2 we have $V_2^+ = \bigcup\limits_{e \in \mathcal{E}^2} \pi_2(e) = \{j \,|\, \exists e \in \mathcal{E}^2 : j = j_e\}$ and for every $j \in V_2^+$ there is a unique $e \in \mathcal{E}^2$ with $j = j_e$ and $N_{G_2}(j) = \pi_2(e) \setminus \{j\}$. Therefore, $\mathcal{E}_2^{\mathcal{N}} = \{\pi_2(e) \setminus \{j_e\} \,|\, e \in \mathcal{E}^2\}$.
Obviously, $V_1 \setminus V_1^+$ and $V_2 \setminus V_2^+$ is the set of the isolated vertices vertices in G_1 and G_2, respectively.
This way we obtain G_1, G_2, $\mathcal{N}(G_1)$ and $\mathcal{N}(G_2)$ and the proof is complete. □

Additionally, we mention that the isolated vertices in G_1 are the vertices which occur in at least one edge $e \in \mathcal{E}_*^{\mathcal{N}}$ with $|\pi_1(e)| = 1$ (and, of course, in no edge $e' \in \mathcal{E}^2$).
Therefore, $V_1 \setminus V_1^+ = \bigcup\{e \,|\, e \in \mathcal{E}_*^{\mathcal{N}} \wedge |\pi_1(e)| = 1\} = \bigcup\limits_{e \in \mathcal{E} \setminus \mathcal{E}^2} \pi_1(e)$. This follows from the existence of at least one edge $\{y, y'\} \in E_2$, since for any isolate $i \in V_1$ trivially $(i, y') \in N_{G_1 * G_2}((i,y)) \in \mathcal{E}_*^{\mathcal{N}}\ \wedge\ N_{G_1 * G_2}((i,y)) \subseteq \{i\} \times V_2$ holds.
The analog is true for the isolates in G_2.

In general, it is not possible to reconstruct an arbitrary graph G from its neighborhood hypergraph $\mathcal{N}(G)$. For the Cartesian sum and the normal product, Theorems 2 and 4 show that the neighborhood hypergraphs $\mathcal{N}(G_1+G_2)$ and $\mathcal{N}(G_1*G_2)$ contain the complete information on G_1+G_2 and G_1*G_2, respectively:

Remark 4 *Since the graphs G_i $(i=1,2)$ can be reconstructed from $\mathcal{N}(G_1+G_2)$ and $\mathcal{N}(G_1*G_2)$ under the assumptions of Theorem 2/4, trivially the same holds for G_1+G_2 and G_1*G_2, respectively.*

2.4 The lexicographic product $G_1 \cdot G_2$

As known from Theorem 1 we have
$\mathcal{E}_{\cdot}^{\mathcal{N}} = \{(\{i\} \times N_{G_2}(j)) \cup (N_{G_1}(i) \times V_2) \,|\, (i,j) \in V_1 \times V_2\} \ \setminus \ \{\emptyset\}$.

Theorem 5 *Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$.*
Then $\mathcal{N}(G_1)$ and $\mathcal{N}(G_2)$ can be reconstructed from $\mathcal{N}(G_1 \cdot G_2)$.

Proof. If $\mathcal{E}_{\cdot}^{\mathcal{N}} = \emptyset$ then, obviously, all vertices in G_1 and all vertices in G_2 are isolated, i.e. $E_1 = \emptyset = E_2$.
So we assume $\mathcal{E}_{\cdot}^{\mathcal{N}} \neq \emptyset$ and consider first the reconstruction of $\mathcal{N}(G_1)$.

(A) $\mathcal{E}_1^{\mathcal{N}} = \{\{i \,|\, \{i\} \times V_2 \subseteq e\} \mid e \in \mathcal{E}_{\cdot}^{\mathcal{N}}\} \setminus \{\emptyset\}$.

To see this, as a first step note that for an arbitrary $e \in \mathcal{E}_{\cdot}^{\mathcal{N}}$ it holds
$\emptyset = \{i \,|\, \{i\} \times V_2 \subseteq e\}$ if and only if $e = N_{G_1 \cdot G_2}((i_e,j_e))$ with $N_{G_1}(i_e) = \emptyset$.
In this case we have $e = \{i_e\} \times N_{G_2}(j_e)$, where j_e is not isolated, i.e.
$\emptyset \neq N_{G_2}(j_e) \subset V_2$. (If both i_e and j_e would be isolated we had the contradiction $e = N_{G_1 \cdot G_2}((i_e,j_e)) = \emptyset \notin \mathcal{E}_{\cdot}^{\mathcal{N}}$.)

On the other hand, if $\emptyset \neq \{i \,|\, \{i\} \times V_2 \subseteq e\}$, then – for $e = N_{G_1 \cdot G_2}((i_e,j_e))$ – trivially $i_e \notin \{i \,|\, \{i\} \times V_2 \subseteq e\}$, since $(i_e,j_e) \notin N_{G_1 \cdot G_2}((i_e,j_e))$. Therefore, every $i \in V_1$ with $\{i\} \times V_2 \subseteq e$ has to be a neighbor of i_e and even $\{i \,|\, \{i\} \times V_2 \subseteq e\} = N_{G_1}(i_e) \in \mathcal{E}_1^{\mathcal{N}}$.

Clearly, that way we obtain all elements of $\mathcal{E}_1^{\mathcal{N}}$.

(B) $\mathcal{E}_2^{\mathcal{N}} = \{\{j \,|\, \exists i \in V_1 : (i,j) \in e \,\wedge\, \{i\} \times V_2 \not\subseteq e\} \mid e \in \mathcal{E}_{\cdot}^{\mathcal{N}}\} \setminus \{\emptyset\}$.

Let $e \in \mathcal{E}_{\cdot}^{\mathcal{N}}$. It follows $\{j \,|\, \exists i \in V_1 : (i,j) \in e \,\wedge\, \{i\} \times V_2 \not\subseteq e\} = \emptyset$ if and only if e is the union of nothing but complete rows of $G_1 \cdot G_2$.

Such a complete row $\{i\} \times V_2$ is a subset of $e = N_{G_1 \cdot G_2}((i_e,j_e))$ if and only if $i \in N_{G_1}(i_e)$. Trivially, $\{i_e\} \times V_2 \not\subseteq e$ and $e_2 := \{j \,|\, j \in V_2 \,\wedge\, (i_e,j) \in e\} \subset V_2$. Moreover, $e_2 = N_{G_2}(j_e)$ and $e_2 \in \mathcal{E}_2^{\mathcal{N}} \iff j_e$ is not isolated in $G_2 \iff e_2 \neq \emptyset$.

That way the edges $e \in \mathcal{E}_{\cdot}^{\mathcal{N}}$, which do not consist of nothing but complete rows of $G_1 \cdot G_2$, provide all edges of $\mathcal{E}_2^{\mathcal{N}}$. □

Remark 5 *If there exists an edge $e \in \mathcal{E}_{\cdot}^{\mathcal{N}}$, which does not consist of nothing but complete rows of $G_1 \cdot G_2$, then $G_1 = (V_1, E_1)$ can be reconstructed from $\mathcal{N}(G_1 \cdot G_2)$.*

Proof. Using part (A) of the above proof, we obtain $\mathcal{E}_1^{\mathcal{N}}$, i.e. all non-empty neighborhoods $N_{G_1}(i')$ of vertices $i' \in V_1$.
In the following we will describe how to find the (unique) vertex i_{e_1} with $N_{G_1}(i_{e_1})$ for an arbitrarily chosen $e_1 \in \mathcal{E}_1^{\mathcal{N}}$. So let $e_1 \in \mathcal{E}_1^{\mathcal{N}}$ and $e \in \mathcal{E}_{\cdot}^{\mathcal{N}}$ such that $e_1 = \{i \,|\, \{i\} \times V_2 \subseteq e\}$. Further let $e' := e \setminus (e_1 \times V_2)$. Then there exists a unique $i_e \in V_1$ with $e' \subseteq \{i_e\} \times V_2$ and, obviously, $N_{G_1}(i_e) = e_1$. (Equivalently, i_e can be also determined by $\{i_e\} := \pi_1(e')$.)
Each vertex $i \in V_1$, which cannot be obtained in this manner, is an isolate. □

Remark 6 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.*
Then, in general, G_2 cannot be reconstructed from $\mathcal{N}(G_1 \cdot G_2)$.

Proof. We re-use the example from the end of subsection 2.2. So let $G_2 = (V_2, E_2)$ and $G_2' = (V_2, E_2')$ be the graphs with $V_2 = \{1, 2, 3, 4\}$, $E_2 = \{\{1, 2\}, \{3, 4\}\}$ and $E_2' = \{\{1, 4\}, \{2, 3\}\}$. In case of the lexicographic product it is recommended to choose special graphs for the graph G_1 in order to demonstrate $\mathcal{E}(\mathcal{N}(G_1 \cdot G_2)) = \mathcal{E}(\mathcal{N}(G_1 \cdot G_2'))$. An appropriate family of graphs may contain all graphs $G_1 = (V_1, E_1)$ with $V_1\{1, \ldots, n\}$ and $E_1 = \{\{i, i'\} \,|\, i, i' \in V_1 \,\wedge\, |i - i'| = 1\}$, where $n \geq 1$.
For $i \in V_1$ we use the notation $z_i := \bigcup \{\{i'\} \times V_2 \,|\, i' \in V_1 \,\wedge\, |i - i'| = 1\}$. Then, for all $i \in V_1$ and $j \in V_2$, the set z_i consists of all vertices contained in the row $i - 1$ (if $i > 1$) and in the row $i + 1$ (if $i < n$) of $G_1 \cdot G_2$, respectively. Therefore, z_i includes all neighbors of the vertex (i, j) in $G_1 \cdot G_2$ being not contained in row i. It follows

$$\begin{aligned}
\mathcal{E}(\mathcal{N}(G_1 \cdot G_2)) &= \bigcup_{i \in V_1} \{N_{G_1 \cdot G_2}((i,1)), N_{G_1 \cdot G_2}((i,2)), N_{G_1 \cdot G_2}((i,3)), N_{G_1 \cdot G_2}((i,4))\} \\
&= \bigcup_{i \in V_1} \{\ z_i \cup \{(i,2)\}\ ,\ \ z_i \cup \{(i,1)\}\ ,\ \ z_i \cup \{(i,4)\}\ ,\ \ z_i \cup \{(i,3)\}\ \} \\
&= \bigcup_{i \in V_1} \{N_{G_1 \cdot G_2'}((i,3)), N_{G_1 \cdot G_2'}((i,4)), N_{G_1 \cdot G_2'}((i,1)), N_{G_1 \cdot G_2'}((i,2))\} \\
&= \mathcal{E}(\mathcal{N}(G_1 \cdot G_2')).
\end{aligned}$$

Hence we have $\mathcal{N}(G_1 \cdot G_2) = \mathcal{N}(G_1 \cdot G_2')$, where $G_2 \neq G_2'$ and the proof is complete. □

2.5 The disjunction $G_1 \vee G_2$

For given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ Theorem 1 gives the edge set of $\mathcal{N}(G_1 \vee G_2)$ as $\mathcal{E}_{\vee}^{\mathcal{N}} = \{(V_1 \times e_2) \cup (e_1 \times V_2) \,|\, e_1 \in \mathcal{E}_1^{\mathcal{N}} \,\wedge\, e_2 \in \mathcal{E}_2^{\mathcal{N}}\}$.
Since $\mathcal{E}_{\vee}^{\mathcal{N}}$ can be completely described without using G_1 and G_2 (only their neighborhood hypergraphs $\mathcal{N}(G_1)$ are $\mathcal{N}(G_2)$ are needed), analogously to the Cartesian product in subsection 2.2 also in the case of the disjunction it is impossible to reconstruct the graphs G_1 and G_2 from $\mathcal{N}(G_1 \vee G_2)$. This can be seen considering the same example used for Remark 2. Thus we obtain

Remark 7 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.*
Then, in general, G_1 and G_2 cannot be reconstructed from $\mathcal{N}(G_1 \vee G_2)$.

On the other hand, concerning the neighborhood hypergraphs of G_1 and G_2 we have

Theorem 6 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.*
Then $\mathcal{N}(G_1)$ and $\mathcal{N}(G_2)$ can be reconstructed from $\mathcal{N}(G_1 \vee G_2)$.

Proof. If $\mathcal{E}_\vee^\mathcal{N} = \emptyset$, then we get $\mathcal{E}_1^\mathcal{N} = \emptyset = \mathcal{E}_2^\mathcal{N}$ as well as $E_1 = \emptyset = E_2$, i.e. $\mathcal{N}(G_1)$ and $\mathcal{N}(G_2)$ as well as G_1 and G_2 can be reconstructed.
Now let $\mathcal{E}_\vee^\mathcal{N} \neq \emptyset$ and $e \in \mathcal{E}_\vee^\mathcal{N}$. Per definition, e has to be the neighborhood of a vertex in $G_1 \vee G_2$, i.e. there exist vertices $i_e \in V_1$ and $j_e \in V_2$ such that $e = N_{G_1 \vee G_2}((i_e, j_e))$. Thus e is the union of several complete rows $\{i\} \times V_2$ and complete columns $V_1 \times \{j\}$ of $G_1 \vee G_2$, where $i \in \{i_1, \ldots, i_p\} \subseteq V_1$ $(0 \leq p < |V_1|)$ and $j \in \{j_1, \ldots, j_q\} \subseteq V_2$ $(0 \leq q < |V_2|)$.
From Remark 7 it follows that i_e as well as j_e can be determined only in special cases (e.g. i_e is unique if it has the degree $|V_1| - 1$). But we will see that the set $\{i_1, \ldots, i_p\}$ of all numbers i with $\{i\} \times V_2 \subseteq e$ is the neighborhood of i_e in G_1, i.e. $\{i_1, \ldots, i_p\} = N_{G_1}(i_e) \in \mathcal{E}_1^\mathcal{N}$. This way we can obtain the neighborhoods of all vertices in G_1, i.e. we get $\mathcal{E}_1^\mathcal{N}$.
Therefore, we have to show $\{i \,|\, \{i\} \times V_2 \subseteq e\} = N_{G_1}(i_e)$.
The first inclusion $N_{G_1}(i_e) \subseteq \{i \,|\, \{i\} \times V_2 \subseteq e\}$ follows immediately from the definition of the disjunction $G_1 \vee G_2$ since for every neighbor $i \in N_{G_1}(i_e)$ clearly $\{i\} \times V_2 \subseteq e$ is valid.
Conversely, let $i \in V_1$ such that $\{i\} \times V_2 \subseteq e = N_{G_1 \vee G_2}((i_e, j_e))$ is fulfilled. For each $j \in V_2$ this implies $(i, j) \in N_{G_1 \vee G_2}((i_e, j_e))$. Hence, especially $(i, j_e) \in N_{G_1 \vee G_2}((i_e, j_e))$ is true. From $j_e \notin N_{G_2}(j_e)$ the definition of the disjunction provides $i \in N_{G_1}(i_e)$.
Consequently, the neighborhood hypergraph of G_1 is given by
$\mathcal{E}_1^\mathcal{N} = \{\{i \,|\, \{i\} \times V_2 \subseteq e\} \mid e \in \mathcal{E}_\vee^\mathcal{N}\} \setminus \emptyset$.
Analogously, we obtain $\mathcal{E}_2^\mathcal{N} = \{\{j \,|\, V_1 \times \{j\} \subseteq e\} \mid e \in \mathcal{E}_\vee^\mathcal{N}\} \setminus \emptyset$. □

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