Four-Colored Rectangle-Free Grids: Solution of an Extremely Complex Boolean Problem

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1 The Problem

A two-dimensional *grid* is a set $G_{m,n} = [m] \times [n]$. A grid $G_{m,n}$ is *4-colorable* if there is a function $\chi_{m,n} : G_{m,n} \to [4]$ such that there are no rectangles with all four corners of the same color [1]. A rectangle-free grid can be mapped to a bipartite graph that does not contain a complete subgraph $K_{2,2}$. Figure 1 shows the mapping between grid and bipartite graph. Such graphs without short cycles are needed for instance for cryptographic applications. It is known that there are four-colorable rectangle-Free grids of the size 16×16 and no four-colorable rectangle-Free grids of the size 19×19. It was unknown whether there are four-colorable rectangle-Free grids of the sizes 17×17, 17×18, 18×17, and 18×18.

2 The Complexity

The number of different color patterns of the grid $G_{m,n}$ is

$$n_{cp}(m,n) = 4^{m*n} \ .$$

For the grid $G_{18,18}$ the number of different color patterns is

$$n_{cp}(18,18) = 4^{18*18} = 4^{324} = 2^{648} \approx 1,16798 * 10^{195} \ .$$

The number of all possible rectangles depends on the number of rows m and the number of columns n of a grid $G_{m,n}$. Each pair of rows and each pair of columns generates one possible rectangle. Hence,

$$n_r(m,n) = \binom{m}{2} * \binom{n}{2}$$

rectangles must be checked for each color pattern. The number of all rectangles n_{ar} for all grids of a certain size is equal to

$$n_{ar}(m,n) = \binom{m}{2} * \binom{n}{2} * 4^{m*n} \ .$$

The number $n_{ar}(18,18)$ of all rectangles of all color pattern is equal to $\mathbf{2.7341 * 10^{199}}$ for the grid $G_{18,18}$.

The extreme complexity of the problem to solve becomes imaginable in comparison to the number of all electrons and protons in the whole universe which is *only* $1.57 * 10^{79}$.

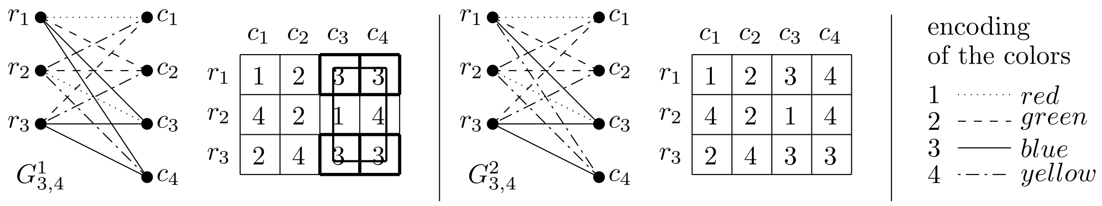

Figure 1: Edge colorings of two complete bipartite graphs $G^1_{3,4}$ and $G^2_{3,4}$ using 4 colors: incorrect graph $G^1_{3,4}$ that violates the rectangle-free condition and rectangle-free graph $G^2_{3,4}$

2	4	1	1	4	2	1	3	3	4	4	2	2	2	1	3	1	3
4	2	4	3	3	4	1	3	2	2	1	1	2	4	2	1	3	1
2	4	2	2	1	4	3	4	2	1	3	1	3	1	4	3	1	2
4	1	3	1	3	3	2	4	1	3	4	1	4	2	2	3	4	2
1	3	4	1	4	1	2	3	2	3	2	2	3	1	4	2	4	1
1	1	2	3	2	3	3	1	4	2	2	4	4	2	4	1	1	3
1	4	4	4	1	3	1	3	1	2	3	2	4	3	3	4	2	2
3	4	2	3	1	1	2	2	3	3	3	4	2	4	1	1	4	4
3	1	4	2	2	1	1	2	4	1	4	2	1	3	2	3	3	4
2	1	1	4	1	3	4	2	3	2	4	3	3	4	4	2	3	1
2	2	3	3	2	4	2	1	1	1	4	4	3	3	1	4	2	1
4	4	2	1	1	2	4	1	4	3	1	3	1	3	2	2	2	3
1	3	3	2	4	2	2	4	4	2	3	1	1	4	1	4	3	3
3	2	4	2	3	1	4	4	1	4	1	4	3	2	3	2	1	3
4	2	1	4	4	2	3	2	1	3	2	4	1	1	3	1	3	2
4	3	1	2	3	1	3	1	3	4	2	1	2	3	4	4	2	4
3	1	3	4	2	4	3	3	4	4	1	3	2	1	1	2	4	2
1	3	1	3	4	2	2	4	4	1	1	3	4	2	3	3	2	4

Figure 2: Cyclic 4-colored grid $G_{18,18}$

3 The Solution

After a very extensive research [4], [5], [7], [9], [10], [11], [12], [13] and [14] Christian Posthoff and myself modeled the problem by a special cyclic SAT-formula which depends on 324 variables and contains 23,976 clauses for the grid $G_{18,18}$ and tried to find a solution using the improved version SAT-solver *clasp* [2]. This SAT-solver *clasp-2.0.0-st-win32* found the first cyclic reusable solution for the grid $G_{18,18}$ after 2 days 10 hours 58 minute 21.503 seconds. Using this solution for the first color we have constructed the 4-colored grid $G_{18,18}$ of Figure 2 by three times rotating around the grid center by 90 degrees each and assigning the next color [8].

With this solution we contribute to the extension of the knowledge within the theory of bipartite Ramsey numbers BR [3]. Instead of the uncertain relation $17 \leq BR(2,4) \leq 19$ we have now the unique solution $BR(2,4) = 19$. Removing a single row, a single column or both of them from the rectangle-free 4-colored grid $G_{18,18}$ correct 4-colored grids $G_{17,18}$, $G_{18,17}$, and $G_{17,17}$ are built. Hence, we solved these four so far open rectangle-free grid coloring problems for 4 colors.

Many researchers all over the world tried during several years to solve this problem, but all of them failed due to the extreme complexity of the problem. Hence, our solution found international approval and was honored on *Spiegel-Online*.

`http://www.spiegel.de/wissenschaft/mensch/`
`loesung-des-vierfarbenraetsels-drei-ecken-duerft-ihr-bilden-a-816497.html`

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