

Hyperelastic UMAT

This routine assumes a constant stiffness in the material configuration

$$U = \frac{1}{2} \underline{\underline{E}} : \underline{\underline{C}} : \underline{\underline{E}} \quad (1)$$

with the Green Lagrange strain

$$\underline{\underline{E}} = \frac{1}{2} [\underline{\underline{F}}^T \underline{\underline{F}} - \underline{\underline{I}}] \quad (2)$$

The 2. Piola Kirchhoff stress then reads

$$\underline{\underline{S}} = \underline{\underline{C}} : \underline{\underline{E}} \quad (3)$$

It is transformed to the Kirchhoff stress by

$$\underline{\underline{\tau}} = \underline{\underline{F}} \underline{\underline{S}} \underline{\underline{F}}^T \quad (4)$$

The cauchy stress $\underline{\underline{\sigma}}$ is in relation to the Kirchhoff stress $\underline{\underline{\tau}}$ via

$$\underline{\underline{\tau}} = \frac{1}{J} \underline{\underline{\sigma}} \quad (5)$$

For the UMAT needs the consistent tangent in the spatial configuration a incremental form is adopted

$$\underline{\underline{\dot{E}}} = \frac{1}{2} [\underline{\underline{F}}^T \underline{\underline{\dot{F}}} + \underline{\underline{F}} \underline{\underline{\dot{F}}}] = \underline{\underline{F}}^T \underline{\underline{d}} \underline{\underline{F}} \quad (6)$$

Thereby $\underline{\underline{d}}$ is

$$\underline{\underline{d}} = \text{sym}(\underline{\underline{l}}), \quad \underline{\underline{w}} = \text{skw}(\underline{\underline{l}}), \quad \underline{\underline{l}} = \underline{\underline{\dot{F}}} \underline{\underline{F}}^{-1} \quad (7)$$

At first equation 3 is differentiated after time

$$\underline{\underline{\dot{S}}} = \underline{\underline{C}} : \underline{\underline{\dot{E}}} \quad (8)$$

Now equation 4 is differentiated after time

$$\underline{\underline{\dot{\tau}}} = \underline{\underline{\dot{F}}} \underline{\underline{S}} \underline{\underline{F}}^T + \underline{\underline{F}} \underline{\underline{S}} \underline{\underline{\dot{F}}}^T + \underline{\underline{F}} \underline{\underline{\dot{S}}} \underline{\underline{F}}^T = \underline{\underline{l}} \underline{\underline{\tau}} + \underline{\underline{\tau}} \underline{\underline{l}}^T + \underline{\underline{F}} [\underline{\underline{C}} : \underline{\underline{\dot{E}}}] \underline{\underline{F}}^T \quad (9)$$

The rate of Green Lagrange strain is substituted with 6

$$\underline{\underline{\dot{\tau}}} = \underline{\underline{l}} \underline{\underline{\tau}} + \underline{\underline{\tau}} \underline{\underline{l}}^T + \underline{\underline{F}} \underline{\underline{C}} : [\underline{\underline{F}}^T \underline{\underline{d}} \underline{\underline{F}}] \underline{\underline{F}}^T = [\underline{\underline{d}} + \underline{\underline{w}}] \underline{\underline{\tau}} + \underline{\underline{\tau}} [\underline{\underline{d}} - \underline{\underline{w}}] + \underline{\underline{F}} \underline{\underline{C}} : [\underline{\underline{F}}^T \underline{\underline{d}} \underline{\underline{F}}] \underline{\underline{F}}^T \quad (10)$$

The Jaumann rate of the Kirchhoff stress reads

$$\underline{\underline{\overset{\nabla}{\tau}}} = \underline{\underline{\dot{\tau}}} + \underline{\underline{\tau}} \underline{\underline{w}} - \underline{\underline{w}} \underline{\underline{\tau}} \quad (11)$$

Now equation (10) is inserted into (11)

$$\underline{\underline{\overset{\nabla}{\tau}}} = \underline{\underline{\tau}} \underline{\underline{d}} + \underline{\underline{d}} \underline{\underline{\tau}} + \underline{\underline{F}} \underline{\underline{C}} : [\underline{\underline{F}}^T \underline{\underline{d}} \underline{\underline{F}}] \underline{\underline{F}}^T \quad (12)$$

The for the hypoelastic/update Lagrange scheme required tangent reads

$$\text{DDSDDE}_{ijkl} = \frac{1}{J} \frac{d[J\sigma_{ij}^{m+1}]}{d\Delta d_{kl}} = \frac{1}{J} \frac{d\tau_{ij}^{m+1}}{d\Delta d_{kl}} = \frac{1}{J} \frac{d\Delta \overset{\nabla}{\tau}_{ij}}{d\Delta d_{kl}} \quad (13)$$

$$\frac{d\Delta \overset{\nabla}{\tau}_{ij}}{d\Delta d_{kl}} = \frac{1}{2} [\tau_{jl} \delta_{ik} + \tau_{il} \delta_{jk} + \tau_{ik} \delta_{jl} + \tau_{jk} \delta_{il}] + C_{IJKL} F_{iI} F_{jJ} F_{kK} F_{lL} \quad (14)$$