# St. Venant Kirchhoff UMAT

This routine assumes a constant stiffness in the material configuration

$$\Psi = \frac{1}{2}\underset{\sim}{E} : \underset{\approx}{C} : \underset{\sim}{E} \tag{1}$$

with the stiffness tensor

$$C_{IJKL} = \lambda \delta_{ij}\delta_{kl} + \mu \left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right] \tag{2}$$

and the Green Lagrange strain

$$\underset{\sim}{E} = \frac{1}{2}[\underset{\sim}{F}^{\mathrm{T}}\underset{\sim}{F} - \underset{\sim}{I}] \tag{3}$$

The 2. Piola Kirchhoff stress then reads

$$\underset{\sim}{S} = \frac{\partial \Psi}{\partial \underset{\sim}{E}} = \underset{\approx}{C} : \underset{\sim}{E} \tag{4}$$

It is transformed to the Kirchhoff stress by

$$\underset{\sim}{\tau} = \underset{\sim}{F}\underset{\sim}{S}\underset{\sim}{F}^{\mathrm{T}} \tag{5}$$

The cauchy stress $\underset{\sim}{\sigma}$ is in relation to the Kirchhoff stress $\underset{\sim}{\tau}$ via

$$\underset{\sim}{\tau} = \frac{1}{J}\underset{\sim}{\sigma} \tag{6}$$

For the UMAT needs the consistent tangent in the spatial configuration a incremental form is adopted

$$\dot{\underset{\sim}{E}} = \frac{1}{2}[\underset{\sim}{F}^{\mathrm{T}}\dot{\underset{\sim}{F}} + \underset{\sim}{F}^{\mathrm{T}}\dot{\underset{\sim}{F}}] = \underset{\sim}{F}^{\mathrm{T}}\underset{\sim}{d}\underset{\sim}{F} \tag{7}$$

Thereby $\underset{\sim}{d}$ is

$$\underset{\sim}{d} = \mathrm{sym}(\underset{\sim}{l}),\ \underset{\sim}{w} = \mathrm{skw}(\underset{\sim}{l}),\ l = \dot{\underset{\sim}{F}}\underset{\sim}{F}^{-1} \tag{8}$$

At first equation 4 is differentiated after time

$$\dot{\underset{\sim}{S}} = \underset{\approx}{C} : \dot{\underset{\sim}{E}} \tag{9}$$

Now equation 5 is differentiated after time

$$\dot{\underset{\sim}{\tau}} = \dot{\underset{\sim}{F}}\underset{\sim}{S}\underset{\sim}{F}^{\mathrm{T}} + \underset{\sim}{F}\underset{\sim}{S}\dot{\underset{\sim}{F}}^{\mathrm{T}} + \underset{\sim}{F}\dot{\underset{\sim}{S}}\underset{\sim}{F}^{\mathrm{T}} = \underset{\sim}{l}\underset{\sim}{\tau} + \underset{\sim}{\tau}\underset{\sim}{l}^{\mathrm{T}} + \underset{\sim}{F}[\underset{\approx}{C} : \dot{\underset{\sim}{E}}]\underset{\sim}{F}^{\mathrm{T}} \tag{10}$$

The rate of Green Lagrange strain is substitued with 7

$$\dot{\underset{\sim}{\tau}} = \underset{\sim}{l}\underset{\sim}{\tau} + \underset{\sim}{\tau}\underset{\sim}{l}^{\mathrm{T}} + \underset{\sim}{F}\underset{\approx}{C} : [\underset{\sim}{F}^{\mathrm{T}}\underset{\sim}{d}\underset{\sim}{F}]\underset{\sim}{F}^{\mathrm{T}} = [\underset{\sim}{d} + \underset{\sim}{w}]\underset{\sim}{\tau} + \underset{\sim}{\tau}[\underset{\sim}{d} - \underset{\sim}{w}] + \underset{\sim}{F}\underset{\approx}{C} : [\underset{\sim}{F}^{\mathrm{T}}\underset{\sim}{d}\underset{\sim}{F}]\underset{\sim}{F}^{\mathrm{T}} \tag{11}$$

The Jaumann rate of the Kirchhoff stress reads

$$\overset{\nabla}{\underset{\sim}{\tau}} = \dot{\underset{\sim}{\tau}} + \underset{\sim}{\tau}\underset{\sim}{w} - \underset{\sim}{w}\underset{\sim}{\tau} \tag{12}$$

Now equation (11) is inserted into (12)

$$\overset{\nabla}{\underset{\sim}{\tau}} = \underset{\sim}{\tau}\,\underset{\sim}{d} + \underset{\sim}{d}\,\underset{\sim}{\tau} + \underset{\sim}{F}\underset{\approx}{C} : [\underset{\sim}{F}^{\mathrm{T}}\underset{\sim}{d}\underset{\sim}{F}]\underset{\sim}{F}^{\mathrm{T}} \tag{13}$$

The for the hypoelastic/update Lagrange scheme required tangent reads

$$\mathrm{DDSDDE}_{ijkl} = \frac{1}{J}\frac{\mathrm{d}[J\sigma_{ij}^{m+1}]}{\mathrm{d}\Delta d_{kl}} = \frac{1}{J}\frac{\mathrm{d}\tau_{ij}^{m+1}}{\mathrm{d}\Delta d_{kl}} = \frac{1}{J}\frac{\mathrm{d}\Delta\overset{\nabla}{\tau}_{ij}}{\mathrm{d}\Delta d_{kl}} \tag{14}$$

$$\frac{\mathrm{d}\Delta\overset{\nabla}{\tau}_{ij}}{\mathrm{d}\Delta d_{kl}} = \frac{1}{2}[\tau_{jl}\delta_{ik} + \tau_{il}\delta_{jk} + \tau_{ik}\delta_{jl} + \tau_{jk}\delta_{il}] + C_{IJKL}F_{iI}F_{jJ}F_{kK}F_{lL} \tag{15}$$